PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion for Lecture 29 -- Chap. 9 in F & W

Introduction to hydrodynamics

1. **Newton's laws for fluids and the continuity equation**
2. **Approximate solutions in the linear limit**

In this lecture, we will continue our discussion of hydrodynamics which is presented in Chapter 9 of your textbook. The focus will be on treating the equations in the linear regime.

PHYSICS COLLOQUIUM

THURSDAY

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NOVEMBER 4, 2021

4 PM Olin 101

"First Principles Investigations of Electrolyte Materials in All-Solid-State Batteries"

Research into solid electrolytes has recently attracted significant interest along with the emerging demands of developing all-solid-state batteries with major benefits of superior safety, high energy density, and long lifespan. The general purpose of our work is to apply first principles calculations and other computational techniques to reliably explain and predict the detailed structural and ionic transport properties of known and designed solid electrolyte materials. The contents of this colloquium have a few overlaps with my Ph.D. defense presentation on Thursday, October 21, 2021. In this talk, I will discuss two

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Tentative schedule --

27	Fri, 10/29/2021	Chap. 9	Mechanics of 3 dimensional fluids		
28	Mon, 11/01/2021	Chap. 9	Mechanics of 3 dimensional fluids	#19	11/03/2021
29	Wed, 11/03/2021	Chap. 9	Linearized hydrodynamics equations	#20	11/05/2021
30	Fri, 11/05/2021	Chap. 9	Linear sound waves		
31	Mon, 11/08/2021	Chap. 9	Sound sources and scattering; Nonlinear effects		
32	Wed, 11/10/2021	Chap. 9	Non linear effects in sound waves and shocks		
33	Fri, 11/12/2021	Chap. 10	Surface waves in fluids		
34	Mon, 11/15/2021	Chap. 10	Surface waves in fluids; soliton solutions		
35	Wed, 11/17/2021	Chap. 11	Heat conduction		
	Fri, 11/19/2021		Presentations I		
	Mon, 11/22/2021		Presentations II		
	Wed, 11/24/2021		Thanksgiving		
	Fri, 11/26/2021		Thanksgiving		
36	Mon, 11/29/2021	Chap. 12	Viscous effects on hydrodynamics		
37	Wed, 12/01/2021	Chap. 1-12	Review		
38	Fri, 12/03/2021	Chap. 1-12	Review		

Updated schedule

PHY 711 -- Assignment #20

Nov. 03, 2021

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Using the analysis covered in class, estimate the speed of sound in the fluid of He gas at 1 atmosphere of pressure and at 300K temperature.

Question about Spring 2022 schedule –

Current schedule –

PHY 712	Electrodynamics	10-10:50 MWF
PHY 742	Quantum II	12-12:50 MWF

Proposed new schedule –

PHY 712	Electrodynamics	11- 11:50 MWF
PHY 742	Quantum II	12-12:50 MWF

Recall the basic equations of hydrodynamics

Basic variables: Density $\rho(\mathbf{r},t)$

Velocity $\mathbf{v}(\mathbf{r},t)$

Pressure $p(\mathbf{r},t)$

Newton-Euler equation of motion:

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\right) = 0$$

+ relationships among the variables due to principles of thermodynamics due to the particular fluid (In fact, we will focus on an ideal gas.)

Review of the basic equations of hydrodynamics.

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times \left(\nabla \times \mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Additional relationships among the variables apply, depending on the fluid material and on thermodynamics

At the moment we are interested in the case where there is no heat exchange.

A little thermodynamics

First law of thermodynamics: $dE_{\text{int}} = dQ - dW$

For isentropic conditions: $dQ = 0$

$dE_{\text{int}} = -dW = -pdV$ Here W == work
V == volume

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}} = -dW = -pdV$$

In terms of mass density: $\rho = \frac{M}{V}$

For fixed M and variable V: $d\rho = -\frac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$ ← Internal energy per unit mass

$$dE_{\text{int}} = Md\varepsilon = -dW = -pdV = M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Note: Under conditions of constant entropy, we assume e can be expressed in terms of the density alone.

$$\text{Consider:} \quad \nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$$

$$\text{Rearranging:} \quad \nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

if $\nabla\times\mathbf{v} = 0$ ➔ $\mathbf{v} = -\nabla\Phi$ $\qquad \mathbf{f}_{applied} = -\nabla U$

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

For isentropic irrotation fluid.

Some details --

$$\left(\nabla\times\mathbf{v}\right)=0 \qquad \text{"irrotational flow"} \qquad \Rightarrow \mathbf{v}=-\nabla\Phi$$

Check: $\left(\nabla\times\mathbf{v}\right)=-\left(\nabla\times\nabla\Phi\right)=?$

$$\left.\left(\nabla\times\nabla\Phi\right)\right|_x=\frac{\partial^2\Phi}{\partial y\partial z}-\frac{\partial^2\Phi}{\partial z\partial x}$$

Summary: For isentropic and irrotational fluid with internal energy per unit mass ε:

$$\nabla\left(\varepsilon+\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Here ε is the internal energy of the fluid per unit mass. For an ideal gas fluid, it has a relatively simple form.

Now consider the fluid to be air near equilibrium

Near equilibrium:

$\rho = \rho_0 + \delta\rho$

$p = p_0 + \delta p$

$\mathbf{v} = 0 + \delta\mathbf{v}$

$\mathbf{f}_{applied} = 0$

ρ_0 represents the average air density

p_0 represents the average air pressure (usually $\approx$ 1 atmosphere)

$\mathbf{v}_0 = 0$ average velocity

Now consider air as the fluid near equilibrium with small fluctuations represented by the delta notation

Equations to lowest order in perturbation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} \qquad \Rightarrow \qquad \frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{v}) = 0 \qquad \Rightarrow \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0$$

In terms of the velocity potential :

$$\delta\mathbf{v} = -\nabla\Phi$$

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \qquad \Rightarrow \nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0 \qquad \Rightarrow \frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2\Phi = 0$$

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Coupled equations.

Expressing pressure in terms of the density assuming constant entropy:

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2 \delta\rho \qquad \text{Here} \quad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

$$\nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \qquad \Rightarrow -\frac{\partial \Phi}{\partial t} + c^2 \frac{\delta\rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho_0}\frac{\partial \delta\rho}{\partial t} = 0$$

$$\frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \quad \Rightarrow \frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Decoupling the equations in the velocity potential and the density fluctuation.

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$

$$\mathbf{v} = -\nabla \Phi$$

Note that, we also have :

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta \mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

The decoupled equation is a wave equation in the velocity potential, density fluctuation, and pressure fluctuation variables.

Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Equation of state for ideal gas:

$$pV = NkT \qquad N = \frac{M}{M_0}$$

$$p = \frac{M}{V}\frac{k}{M_0}T = \rho\frac{k}{M_0}T$$

$$k = 1.38\times 10^{-23} J/K$$

$M_0 =$ average mass of each molecule

Estimating the wave velocity for air assuming that it is an ideal gas.

Internal energy for ideal gas :

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

In terms of specific heat ratio : $\gamma \equiv \frac{C_p}{C_V}$

$$dE = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2}\frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2}\frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1}$$

Using the ideal gas law with f representing the degrees of freedom. It is convenient to replace the f with the gamma factor which can be measured experimentally.

Digression

Internal energy for ideal gas: $f \equiv$ "degrees of freedom"

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

$$\frac{f}{2} = \frac{1}{\gamma - 1} \quad \Rightarrow \quad E = \frac{1}{\gamma-1} NkT \qquad \varepsilon = \frac{1}{\gamma-1}\frac{k}{M_0}T = \frac{1}{\gamma-1}\frac{p}{\rho}$$

	f	γ
Spherical atom	3	1.66667
Diatomic molecule	5	1.40000

Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma - 1}\frac{k}{M_0}T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions:

$$d\varepsilon = -\frac{p}{M}dV = \frac{p}{\rho^2}d\rho$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)_s = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{1}{(\gamma - 1)\rho} - \frac{p}{(\gamma - 1)\rho^2}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Using the ideal gas law under isentropic conditions to derive the speed of sound.

Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Linearized speed of sound

$$c_0{}^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,p_0,\rho_0} = \frac{\gamma\, p_0}{\rho_0}$$

$$c_0{}^2 \approx \frac{1.5 \cdot 1.013 \times 10^5\, Pa}{1.3 kg / m^3} \qquad c_0 \approx 340 \text{ m/s}$$

Another analysis of the speed of sound.

More general case -- Isentropic or adiabatic equation of state:

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \qquad \Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

Density dependence of speed of sound for ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$c^2 = \frac{p_0 \gamma}{\rho_0} \frac{p / p_0}{\rho / \rho_0} = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1}$$

Some details of the analysis reveal that beyond the linear approximation, the velocity of sound is highly non-linear.

Summary of linearized hydrodynamic equations for isentropic fluid

In terms of the velocity potential:

$$\delta\mathbf{v} = -\nabla\Phi \qquad \frac{\partial^2\Phi}{\partial t^2} - c_0^2\nabla^2\Phi = 0 \qquad c_0^2 = \left(\frac{\partial p}{\partial\rho}\right)_{s,\rho_0}$$

In term of density fluctuation: $$\frac{\partial^2\delta\rho}{\partial t^2} - c_0^2\nabla^2\delta\rho = 0$$

In term of pressure fluctuation: $$\frac{\partial^2\delta p}{\partial t^2} - c_0^2\nabla^2\delta p = 0$$