PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion notes for Lecture 3

Scattering theory – Coordinate frames
Center of mass and laboratory frames

Your questions –

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 AM	**OPL 103**	**http://www.wfu.edu/~natalie/f21phy711/**

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/23/2021	Chap. 1	Introduction	#1	8/27/2021
2	Wed, 8/25/2021	Chap. 1	Scattering theory	#2	8/30/2021
3	Fri, 8/27/2021	Chap. 1	Scattering theory		
4	Mon, 8/30/2021	Chap. 1	Scattering theory		
5	Wed, 9/01/2021	Chap. 1	Scattering theory		
6	Fri, 9/03/2021	Chap. 2	Non-inertial coordinate systems		

Clarification on homework #2

The total energy of the system is a given constant and V_0 happens to a particular energy value (that hopefully makes the algebra of the problem somewhat convenient.).

PHY 711 – Assignment #2

08/25/2021

Assume particle M is at rest.

1. Consider a particle of mass m moving in the vicinity of another particle of mass M where $m << M$. The particles interact with a conservative central potential of the form

$$V(r) = V_0\left(\left(\frac{r_0}{r}\right)^2 - \left(\frac{r_0}{r}\right)\right),$$

where r denotes the magnitude of the particle separation and V_0 and r_0 denote energy and length constants, respectively. The total energy of the system is V_0.

   (a) First consider the case where the impact parameter $b = 0$. Find the distance of closest approach of the particles.

   (b) Now consider the case where the impact parameter $b = r_0$. Find the distance of closest approach of the particles.

Scattering theory:

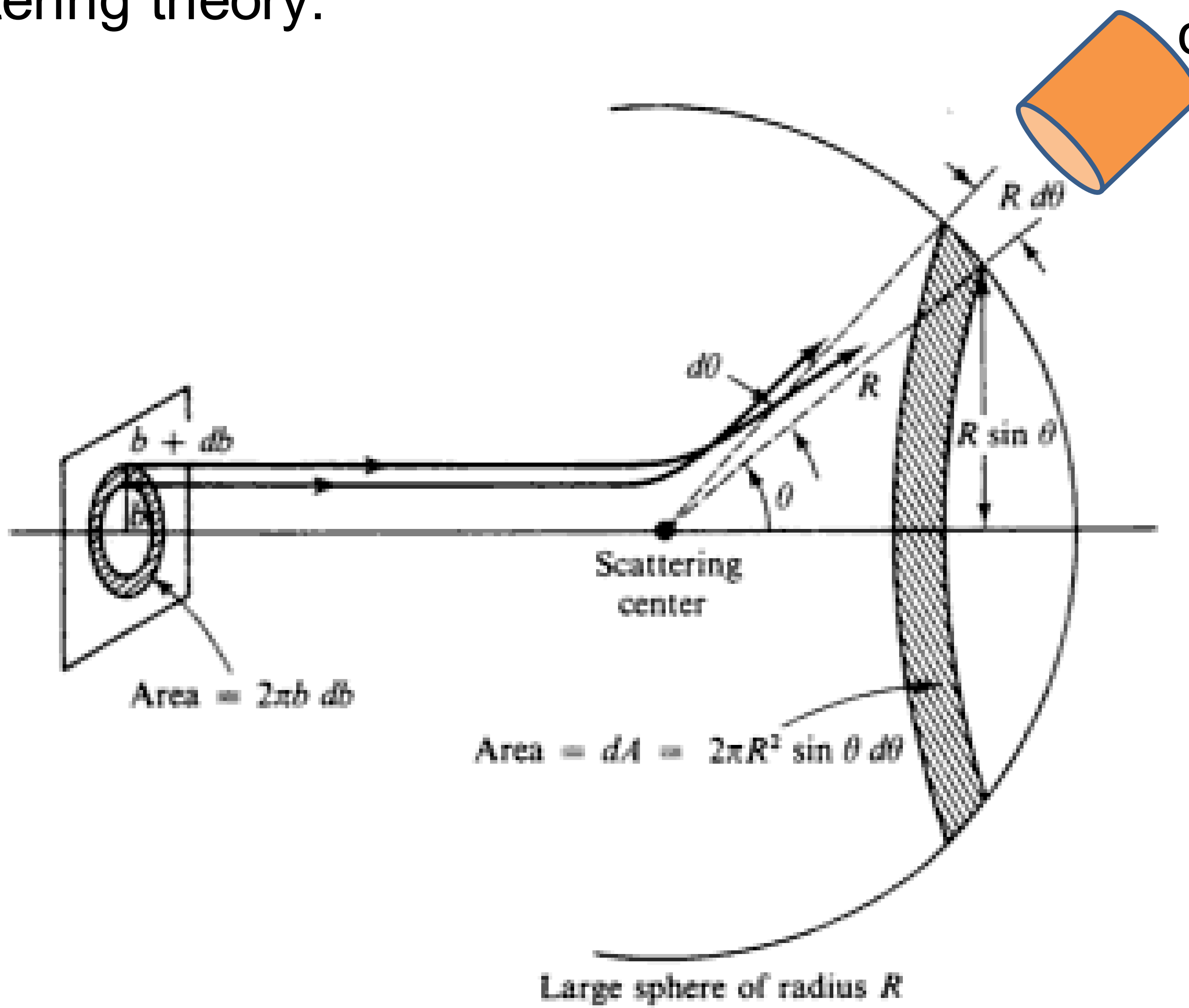


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Can you think of examples of such an experimental setup?

Other experimental designs –

At CERN https://home.cern/science/experiments/totem

the study of highly energetic proton-proton scattering is designed in the center of mass frame of reference by accelerating two proton beams focused to collide head on in the Large Hadron Collider LHC facility.

Figure from CERN website

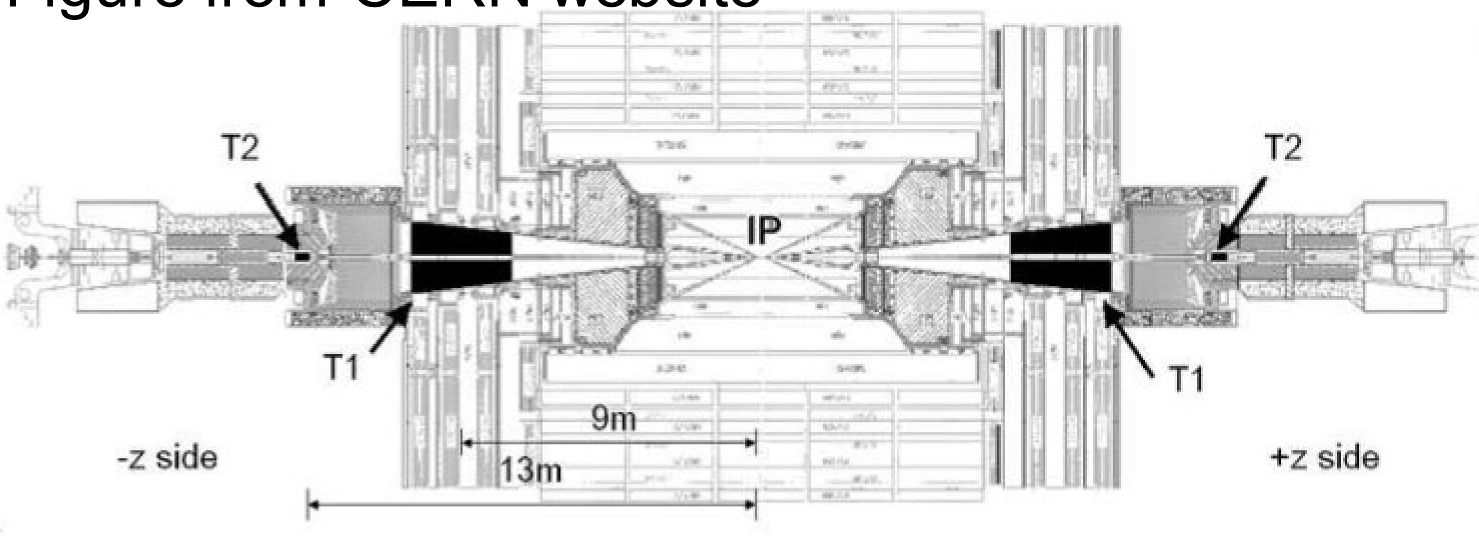


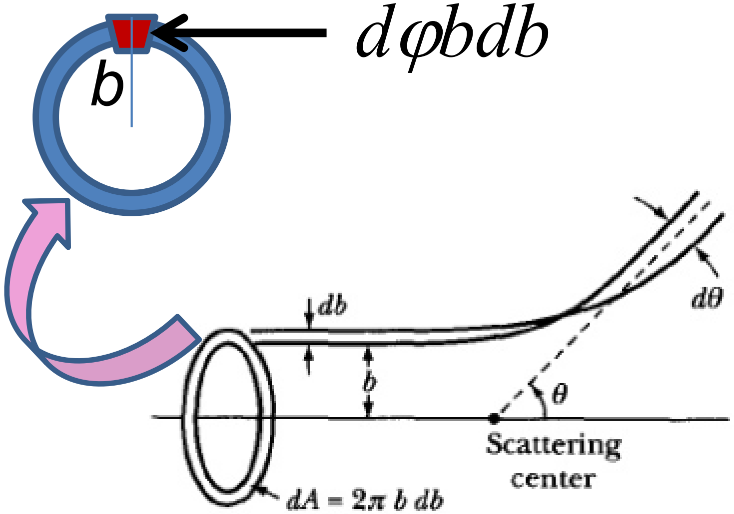
Figure 1.17: View of the inelastic forward trackers T1 and T2 inside the CMS detector.

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector at angle } \theta$$

$$d\varphi b db$$

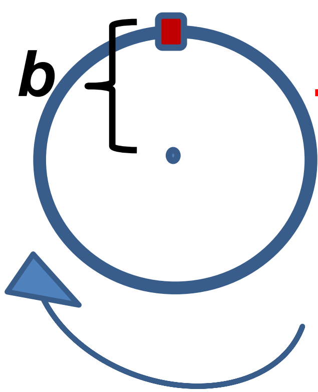


$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

More details --

We imagine that the beam of particles has a cylindrical geometry and that the physics is totally uniform in the azimuthal direction. The cross section of the beam is a circle.

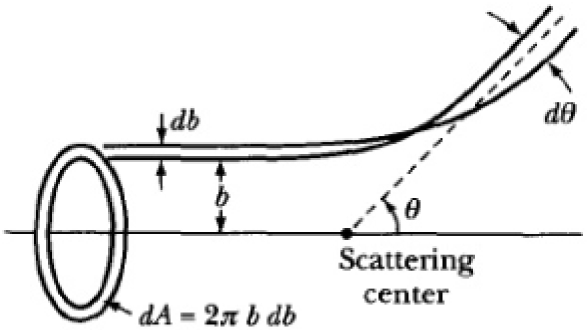


←This piece of the beam scatters into the detector at angle θ

This logic leads to the notion that b is a function of theta and we will try to find b(θ) for various cases.

$$\varphi \equiv \text{azimuthal angle}$$

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



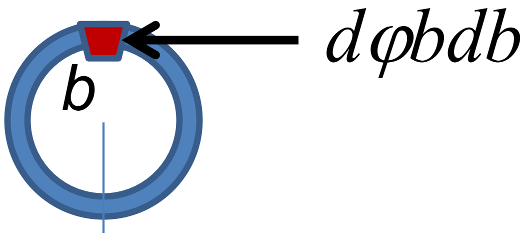
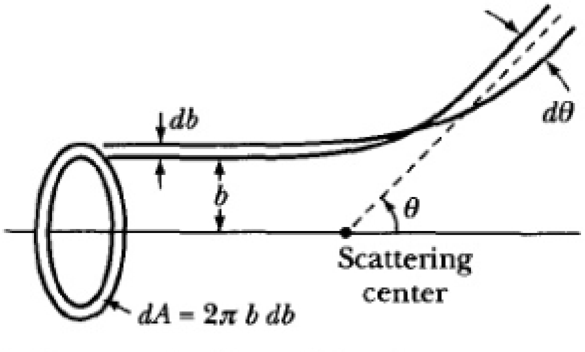


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in ϕ

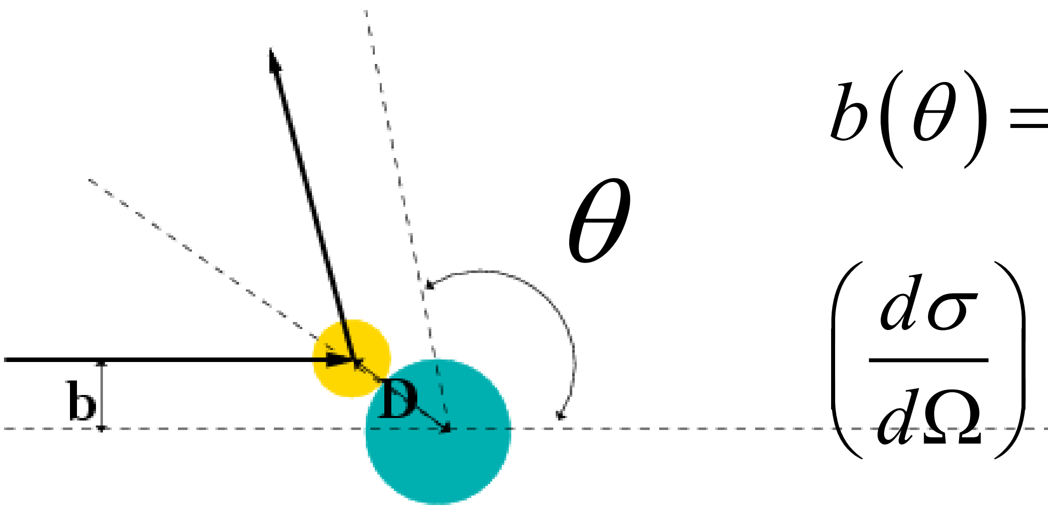
Simple example – collision of hard spheres



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Microscopic view:

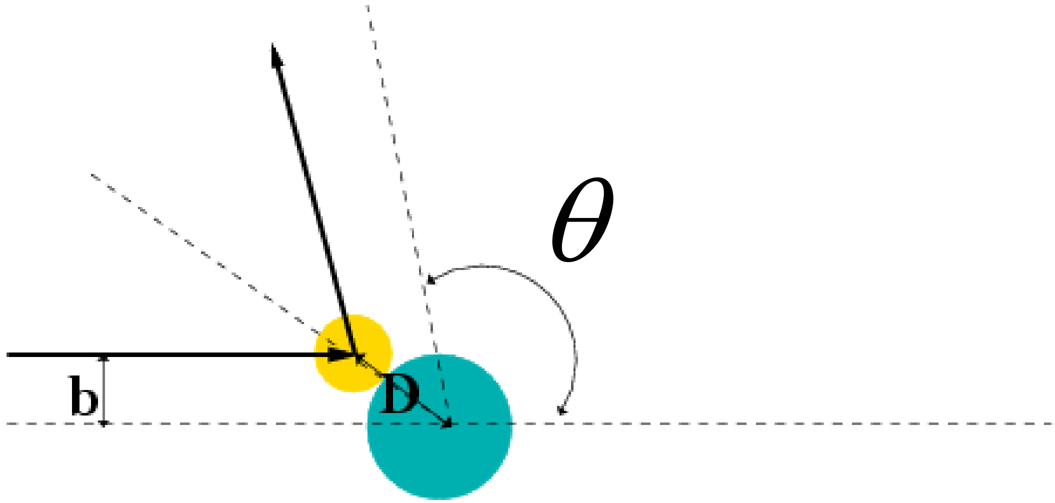
$$b(\theta) = ?$$

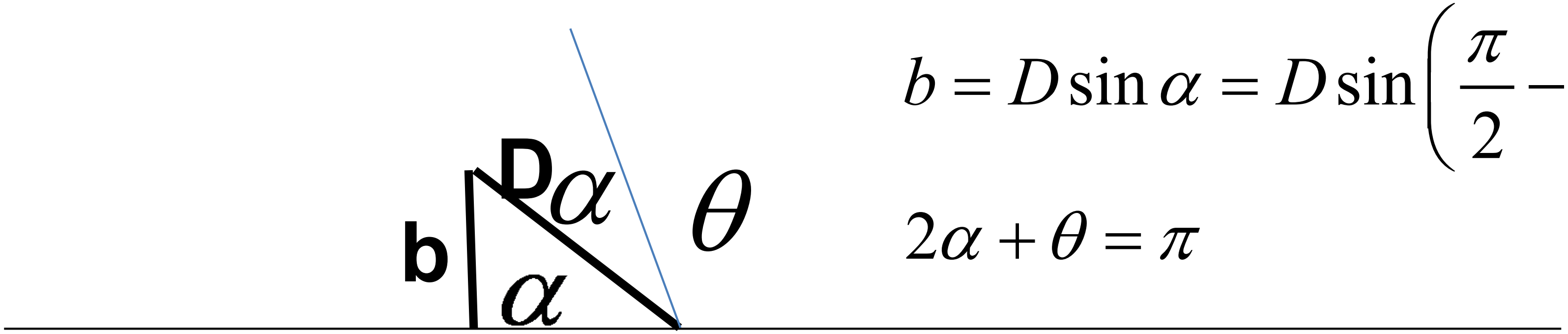


$$b(\theta) = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$

Some more details of form of *b(θ)*

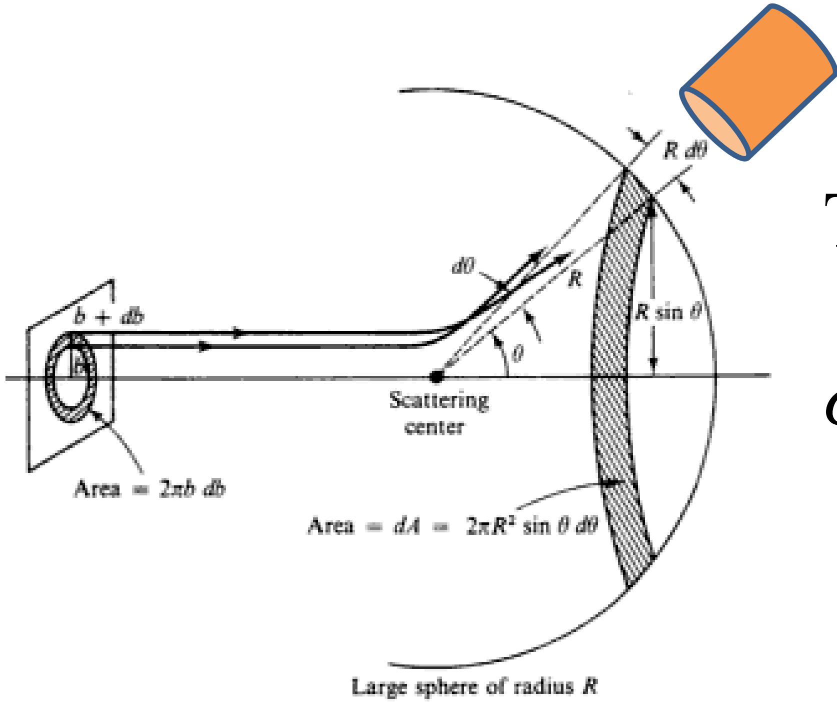




$$b = D\sin\alpha = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$2\alpha + \theta = \pi$$

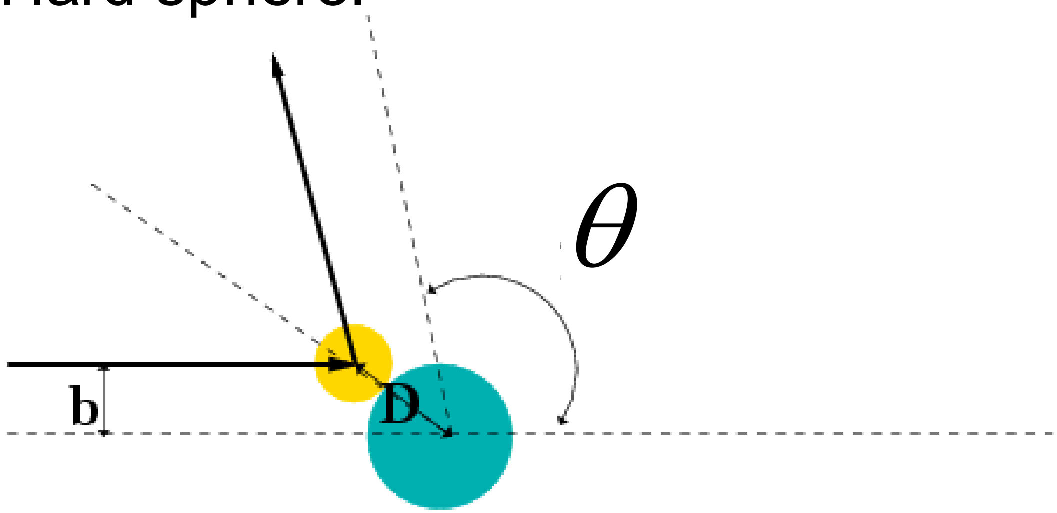
Simple example – collision of hard spheres -- continued



Total scattering cross section:

$$\sigma = \int \left(\frac{d\sigma}{d\Omega} \right) d\Omega$$

Hard sphere:



$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{D^2}{4}$$

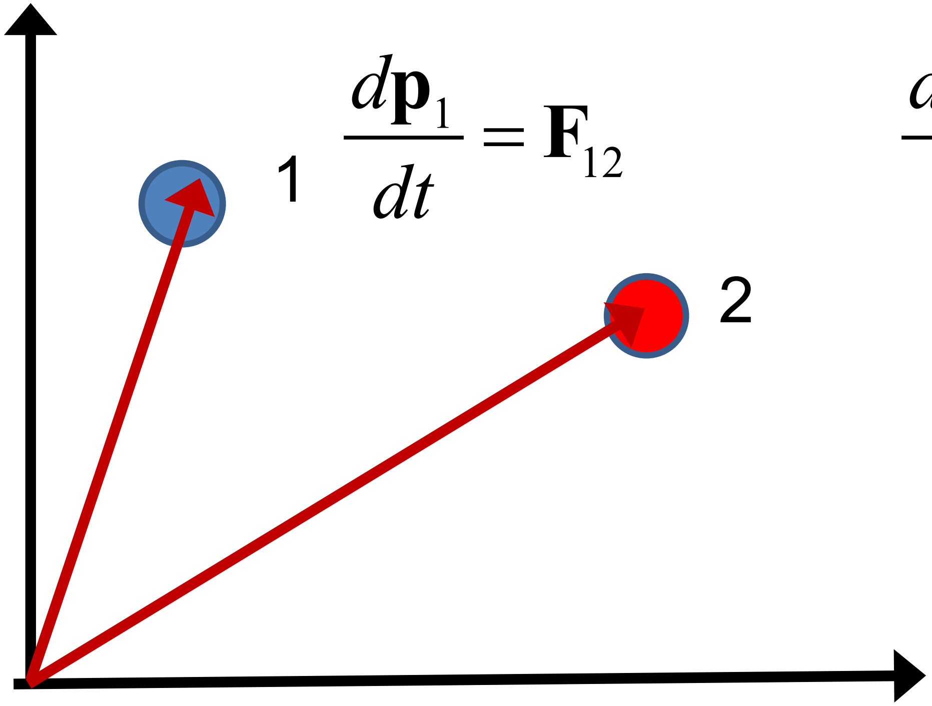
$$\sigma = \pi D^2$$

Now consider the more general case of particle interactions and the corresponding scattering analysis.

Scattering theory can help us analyze the interaction potential $V(r)$. First we need to simply the number of variables.

Relationship of scattering cross-section to particle interactions --

Classical mechanics of a conservative 2-particle system.



$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21}$$

$$\mathbf{F}_{12} = -\nabla_1 V(\mathbf{r}_1 - \mathbf{r}_2) \qquad \Rightarrow E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

Relationship between center of mass and laboratory frames of reference. At and time *t*, the following relationships apply --

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \left(m_1 + m_2\right)\mathbf{R}_{CM}$$

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = \left(m_1 + m_2\right)\dot{\mathbf{R}}_{CM} = \left(m_1 + m_2\right)\mathbf{V}_{CM}$$

Note that $\dot{\mathbf{r}} \equiv \frac{d\mathbf{r}}{dt}$

$$E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

$$= \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu\left|\mathbf{v}_1 - \mathbf{v}_2\right|^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

where: $\mu \equiv \frac{m_1m_2}{m_1 + m_2}$

Classical mechanics of a conservative 2-particle system -- continued

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu|\mathbf{v}_1 - \mathbf{v}_2|^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

For central potentials: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r_{12})$

Relative angular momentum is also conserved:

$$\mathbf{L}_{12} \equiv \mathbf{r}_{12} \times \mu\mathbf{v}_{12}$$

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu v_{12}^2 + \frac{L_{12}^2}{2\mu r_{12}^2} + V(r_{12})$$

Simpler notation:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Simpler notation:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

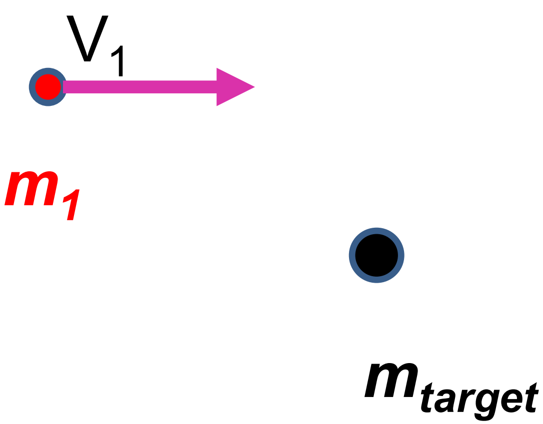
constants

vary in time

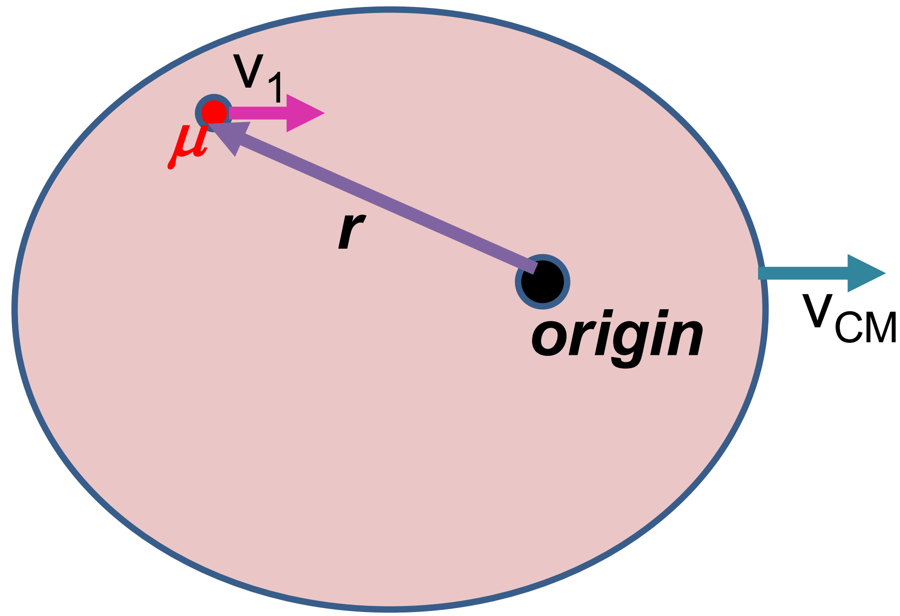
For scattering analysis only need to know trajectory **before** and **after** the collision.

Note: The following analysis will be carried out in the center of mass frame of reference.

In laboratory frame:



In center-of-mass frame:



Also note: We are assuming that the interaction between particle and target *V(r)* conserves energy and angular momentum.

$$\mu = \frac{m_1 m_{\text{target}}}{m_1 + m_{\text{target}}}$$

$$\ell = \left| \mathbf{r} \times \mu \mathbf{v}_1 \right|$$

Typically, the laboratory frame is where the data is taken, but the center of mass frame is where the analysis is most straightforward.

Previous equations --

$$E = \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V\left(r\right)$$

constant

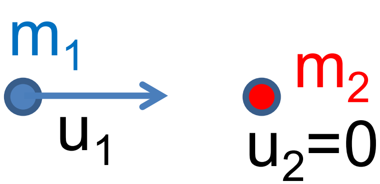
relative coordinate system; visualize as “in” CM frame

It is often convenient to analyze the scattering cross section in the center of mass reference frame.

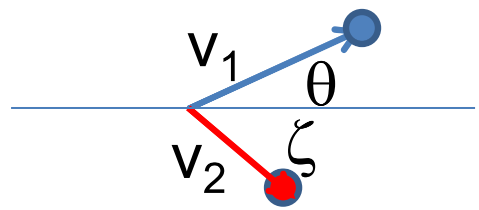
Relationship between normal laboratory reference and center of mass:

Laboratory reference frame:

Before

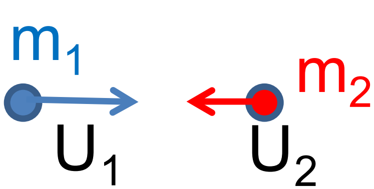


After

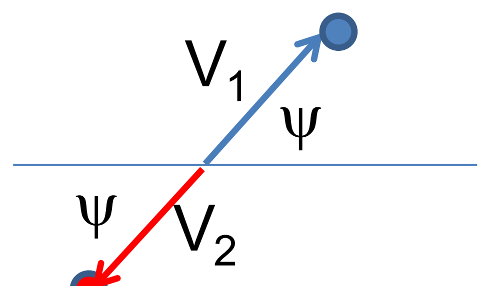


Center of mass reference frame:

Before



After



Relationship between center of mass and laboratory frames of reference -- continued

Since m_2 is initially at rest :

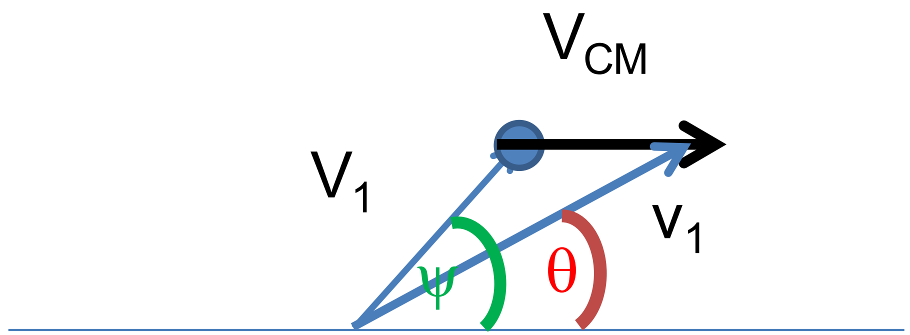
$$\mathbf{V}_{CM} = \frac{m_1}{m_1 + m_2}\mathbf{u}_1 \qquad \mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_1 = \frac{m_2}{m_1 + m_2}\mathbf{u}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM}$$

$$\mathbf{u}_2 = \mathbf{U}_2 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{u}_1 = -\mathbf{V}_{CM}$$

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$\mathbf{v}_2 = \mathbf{V}_2 + \mathbf{V}_{CM}$$

Relationship between center of mass and laboratory frames of reference for the scattering particle 1



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM}/V_1} = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$

For elastic scattering

Digression – elastic scattering

$$\tfrac{1}{2}m_1U_1^2+\tfrac{1}{2}m_2U_2^2+\tfrac{1}{2}(m_1+m_2)V_{CM}^2$$
$$=\tfrac{1}{2}m_1V_1^2+\tfrac{1}{2}m_2V_2^2+\tfrac{1}{2}(m_1+m_2)V_{CM}^2$$

Also note:

$$m_1\mathbf{U}_1+m_2\mathbf{U}_2=0 \qquad\qquad m_1\mathbf{V}_1+m_2\mathbf{V}_2=0$$

$$\mathbf{U}_1=\frac{m_2}{m_1}\mathbf{V}_{CM} \qquad\qquad \mathbf{U}_2=-\mathbf{V}_{CM}$$

$$\Rightarrow|\mathbf{U_1}|=|\mathbf{V_1}| \quad \text{and} \quad |\mathbf{U_2}|=|\mathbf{V_2}|=|\mathbf{V_{CM}}|$$

$$\text{Also note that :}\quad m_1|\mathbf{U_1}|=m_2|\mathbf{U_2}|$$

$$\text{So that :}\qquad V_{CM}/V_1=V_{CM}/U_1=m_1/m_2$$

Summary of results --

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM}/V_1} = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$

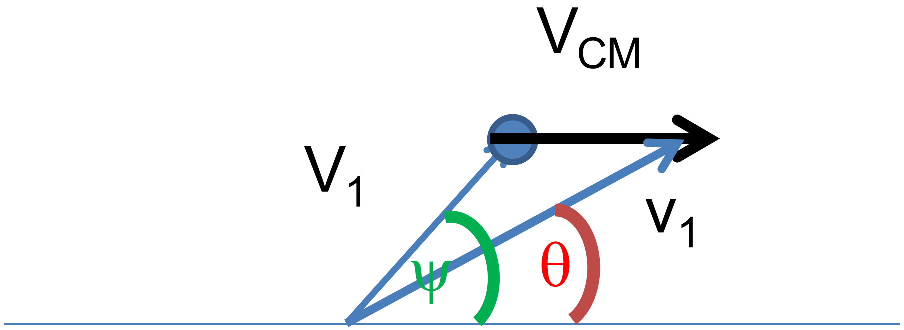
General case

Special case of elastic scattering

For elastic scattering

$$V_{CM}/V_1 = m_1/m_2$$

Relationship between center of mass and laboratory frames of reference – continued (elastic scattering)



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$

$$\tan\theta = \frac{\sin\psi}{\cos\psi + V_{CM} / V_1} = \frac{\sin\psi}{\cos\psi + m_1 / m_2}$$

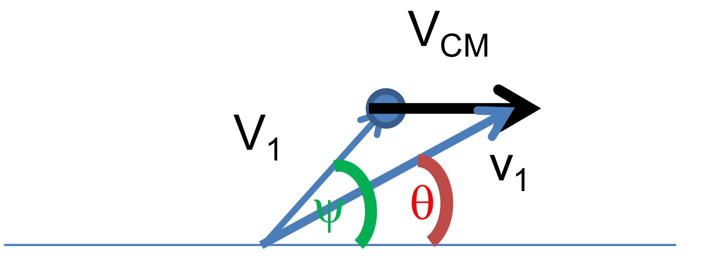
Also: $$\cos\theta = \frac{\cos\psi + m_1 / m_2}{\sqrt{1 + 2m_1 / m_2 \cos\psi + \left(m_1 / m_2\right)^2}}$$

More details -- from the diagram and equations --

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$v_1 \sin\theta = V_1 \sin\psi$$

$$v_1 \cos\theta = V_1 \cos\psi + V_{CM}$$



Take the dot product of the first equation with itself

$$v_1^2 = V_1^2 + 2V_1V_{CM}\cos\psi + V_{CM}^2$$

$$\text{or}\ \frac{v_1}{V_1} = \sqrt{1 + 2\frac{V_{CM}}{V_1}\cos\psi + \frac{V_{CM}^2}{V_1^2}} = \sqrt{1 + 2\frac{m_1}{m_2}\cos\psi + \left(\frac{m_1}{m_2}\right)^2}$$

$$\Rightarrow \quad \cos\theta = \frac{\cos\psi + m_1/m_2}{\sqrt{1 + 2m_1/m_2\cos\psi + (m_1/m_2)^2}}$$

Differential cross sections in different reference frames

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}}=\left|\frac{\sin\psi}{\sin\theta}\frac{d\psi}{d\theta}\right|=\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

Using:

$$\cos\theta=\frac{\cos\psi+m_1/m_2}{\sqrt{1+2(m_1/m_2)\cos\psi+(m_1/m_2)^2}}$$

$$\left|\frac{d\cos\theta}{d\cos\psi}\right|=\frac{(m_1/m_2)\cos\psi+1}{\left(1+2(m_1/m_2)\cos\psi+(m_1/m_2)^2\right)^{3/2}}$$

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+\left(m_1/m_2\right)^2\right)^{3/2}}{\left(m_1/m_2\right)\cos\psi+1}$$

where: $$\tan\theta = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

where: $\tan\theta=\dfrac{\sin\psi}{\cos\psi+m_1/m_2}$

Example: suppose $m_1 = m_2$

In this case: $\tan\theta=\dfrac{\sin\psi}{\cos\psi+1}\quad\Rightarrow\theta=\dfrac{\psi}{2}$

note that $0\leq\theta\leq\dfrac{\pi}{2}$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(2\theta)}{d\Omega_{CM}}\right)\cdot 4\cos\theta$$

Summary --

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+\left(m_1/m_2\right)^2\right)^{3/2}}{\left(m_1/m_2\right)\cos\psi+1}$$

For elastic scattering

where: $$\tan\theta = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$

Hard sphere example – continued

$m_1 = m_2$

Center of mass frame

$$\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right) = \frac{D^2}{4}$$

$$\int \frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}} d\Omega_{CM} =$$

$$\frac{D^2}{4} 4\pi = \pi D^2$$

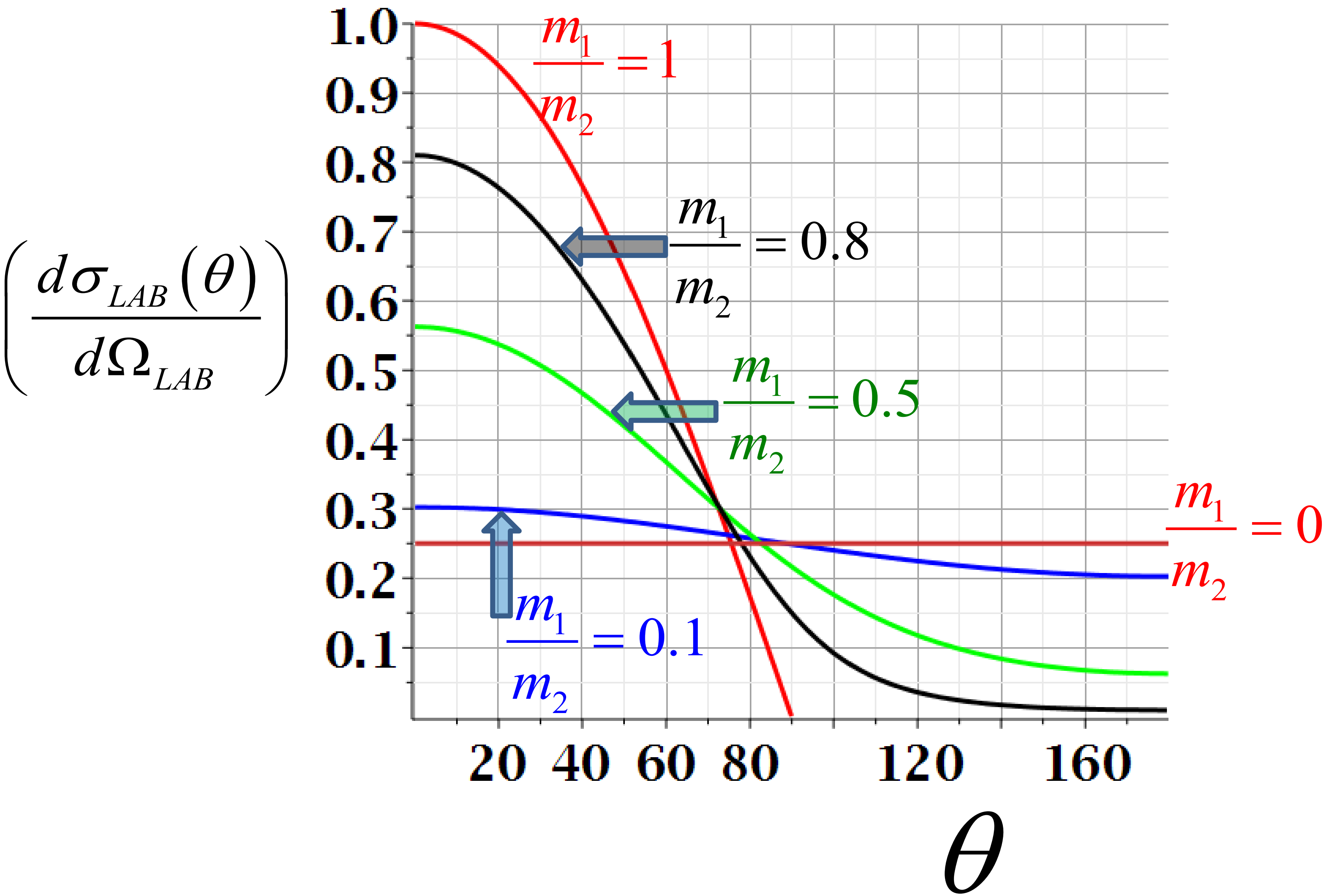
Lab frame

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = D^2 \cos\theta \quad \theta = \frac{\psi}{2}$$

$$\int \frac{d\sigma_{lab}(\theta)}{d\Omega_{lab}} d\Omega_{lab} =$$

$$2\pi D^2 \int_0^{\pi/2} \cos\theta \sin\theta d\theta = \pi D^2$$

Scattering cross section for hard sphere in lab frame for various mass ratios:



For visualization, is convenient to make a "parametric" plot of

$$\left(\frac{d\sigma_{LAB}}{d\Omega}(\theta)\right) \text{ vs } \theta(\psi)$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

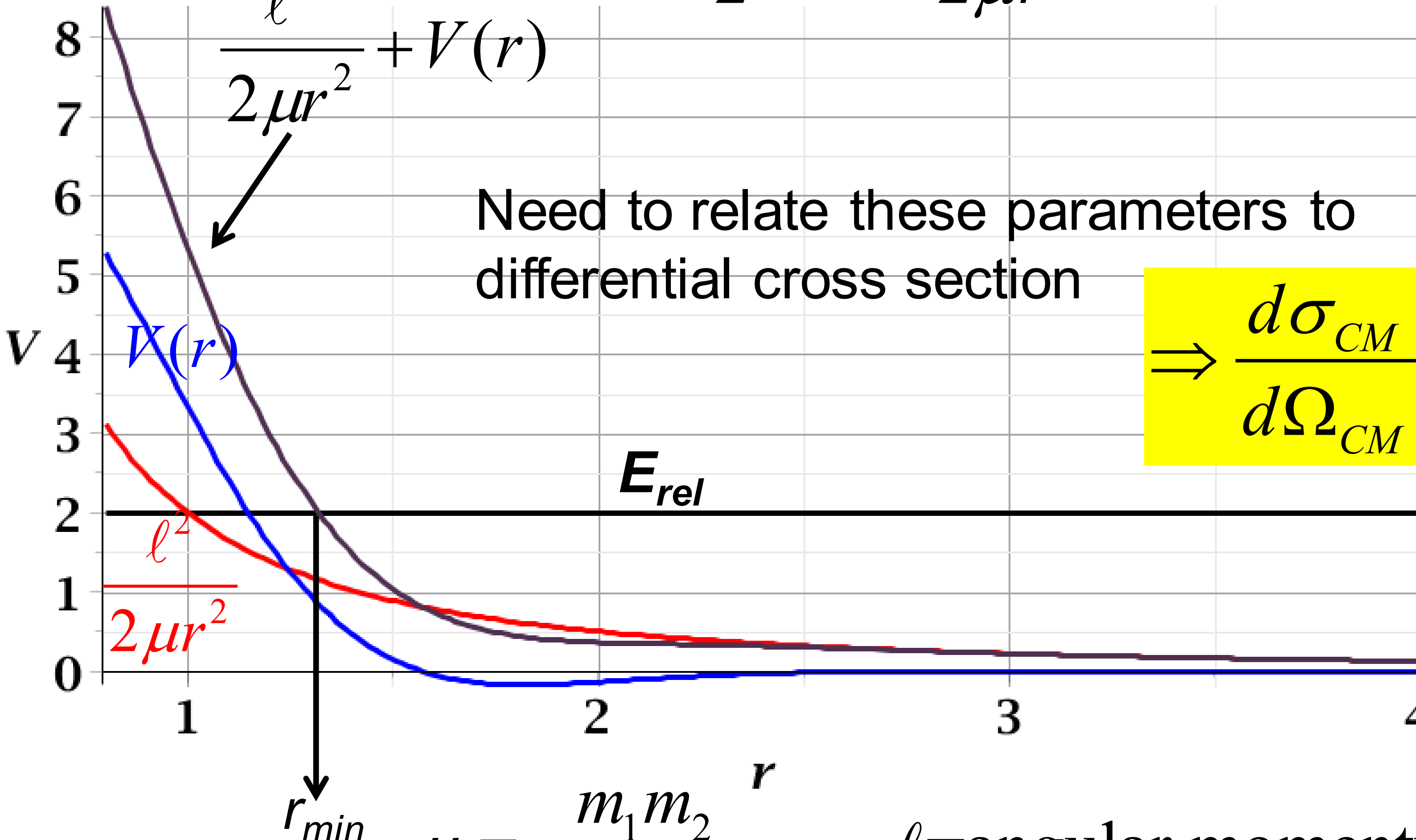
where: $\tan\theta = \dfrac{\sin\psi}{\cos\psi + m_1/m_2}$

Maple syntax:

```
> plot({[psi(theta, 0), sigma(theta, 0), theta = 0.001 ..3.14], [psi(theta, .1), sigma(theta, .1), theta
   = 0.001 ..3.14], [psi(theta, .5), sigma(theta, .5), theta = 0.001 ..3.14], [psi(theta, .8),
   sigma(theta, .8), theta = 0.001 ..3.14], [psi(theta, 1), sigma(theta, 1), theta = 0.001 ..3.14]},
   thickness = 3, font = ['Times','bold', 24], gridlines = true, color = [red, blue, green, black,
   orange])
```

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section

$$\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum