PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion on Lecture 30 -- Chap. 9 of F&W

Wave equation for sound in linear approximation

1. **Wave equations for sound**
2. **Plane wave solutions**
3. **Standing wave solutions**

In this lecture, we will consider some solutions to the linear sound wave equations.

Tentative schedule --

28	Mon, 11/01/2021	Chap. 9	Mechanics of 3 dimensional fluids	#19	11/03/2021
29	Wed, 11/03/2021	Chap. 9	Linearized hydrodynamics equations	#20	11/05/2021
30	Fri, 11/05/2021	Chap. 9	Linear sound waves	#21	11/08/2021
31	Mon, 11/08/2021	Chap. 9	Sound sources and scattering; Nonlinear effects		
32	Wed, 11/10/2021	Chap. 9	Non linear effects in sound waves and shocks		
33	Fri, 11/12/2021	Chap. 10	Surface waves in fluids		
34	Mon, 11/15/2021	Chap. 10	Surface waves in fluids; soliton solutions		
35	Wed, 11/17/2021	Chap. 11	Heat conduction		
	Fri, 11/19/2021		Presentations I		
	Mon, 11/22/2021		Presentations II		
	Wed, 11/24/2021		Thanksgiving		
	Fri, 11/26/2021		Thanksgiving		
36	Mon, 11/29/2021	Chap. 12	Viscous effects on hydrodynamics		
37	Wed, 12/01/2021	Chap. 1-12	Review		
38	Fri, 12/03/2021	Chap. 1-12	Review		

No HW; work on "mini-projects".

Here is a tentative schedule for the next several weeks.

PHY 711 -- Assignment #21

Nov. 05, 2021

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Consider a cylindrical pipe of length 0.5 m and radius 0.05 m, open at both ends. For air at 300 K and atmospheric pressure in this pipe, find several of the lowest frequency resonances.

Homework problem based on today's lecture.

Your questions –
From Owen –
I have one question for class today. In a previous class, we saw a demonstration of a plate vibrating with sand on it. At different frequencies, the arrangement of the sand revealed the vibrational mode of the plate. These effects are present in many musical instruments, particularly stringed instruments which have their characteristic shape to take advantage of that effect. Must the vibrations of the solid container and the vibrations of the air within it (like in the slides) couple somehow?

Comment – Great question; we will try to have some demos about this a little later in the lecture.

Review –

Hydrodynamic equations for isentropic air + linearization about equilibrium → wave equation for air (sound waves)

Which of the following things correctly describe the wave equation for sound in air and the wave equation for elastic media?

a. The wave velocity is different for sound in air and waves in elastic media.
b. The wave motion in elastic media can be either transverse or longitudinal.
c. The wave motion for sound in air can be either transverse or longitudinal.

Equations to lowest order in perturbation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} \qquad \Rightarrow \qquad \frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \Rightarrow \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0$$

In terms of the velocity potential:

$$\delta\mathbf{v} = -\nabla\Phi$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \qquad \Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0 \qquad \Rightarrow \frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2\Phi = 0$$

Expressing pressure in terms of the density assuming constant entropy:

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_{s,\rho_0} \delta\rho \equiv c_0^2\ \delta\rho$ (strictly keeping to the linear approximation)

$$\nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \qquad \Rightarrow -\frac{\partial \Phi}{\partial t} + c_0^2 \frac{\delta\rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2 \Phi}{\partial t^2} + \frac{c_0^2}{\rho_0}\frac{\partial \delta\rho}{\partial t} = 0$$ (assuming we can adjust Φ accordingly)

$$\frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \qquad \Rightarrow \frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi = 0$$

$$\Rightarrow \delta\rho = \frac{\rho_0}{c_0^2}\frac{\partial \Phi}{\partial t} \qquad \delta p = \rho_0 \frac{\partial \Phi}{\partial t}$$

Wave equation for air:

$$\frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi = 0$$

Here, $c_0^2 = \left(\frac{\partial p}{\partial \rho} \right)_{s.\rho_0}$

$\mathbf{v} = -\nabla \Phi$

Note that, we also have:

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c_0^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c_0^2 \nabla^2 \delta p = 0$$

Review of the equations we derived last time.

Solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution:

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

$$\delta\mathbf{v} = -\nabla\Phi = -i\mathbf{k}Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$$

$\Leftarrow$ Note this is a pure longitudinal wave

$$\delta\rho = \frac{\rho_0}{c_0^2}\frac{\partial\Phi}{\partial t} = -i\omega\frac{\rho_0}{c_0^2}Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\delta p = \rho_0\frac{\partial\Phi}{\partial t} = -i\omega\rho_0 Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

In general, we will be interested in time harmonic solutions to the wave equation, where omega denotes the pure frequency of the wave.

Boundary values of wave equation

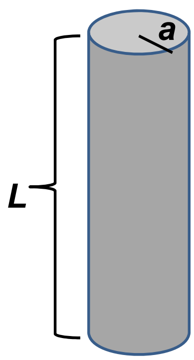
Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta\mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla\Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

Time harmonic standing waves in a pipe



$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Boundary values:

At fixed surface: $\hat{\mathbf{n}} \cdot \nabla\Phi = 0$

At free surface: $\dfrac{\partial \Phi}{\partial t} = 0$

For example, consider a pipe of length L and radius a. In this pipe, we are interested in the behavior of the air. Should you have such a piper at home, put your ear close to one end. What do you hear?

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \qquad \text{Define : } \quad k \equiv \frac{\omega}{c}$$

In cylindrical coordinates:

$$\Phi(r,\phi,z,t) = R(r)F(\phi)Z(z)e^{-i\omega t} \equiv R(r)F(\phi)Z(z)e^{-ikct}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Phi(r,\phi,z,t) = 0$$

Here we consider the equations of linear air within the paper. Cylindrical coordinates are the natural analysis tools for this case.

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\Phi(r,\varphi,z,t)=0$$

$$\Phi(r,\varphi,z,t)=R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$F(\varphi)=e^{im\varphi};\ F(\varphi)=F(\varphi+2\pi N)\Rightarrow m=\text{integer}$$

$$Z(z)=e^{i\alpha z};\ \alpha=\text{real}\ \left(+\text{ other restrictions}\right)$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

The equation is separable in the radial, angular, z, and time variables. Because of the cylindrical geometry, the angular part takes the form of exp(I m phi), where m has to be an integer. We also are motivated to assume that the Z(z) function has a sinusoidal form with an unknown constant alpha. Finally, the equation for the radial equation now takes a familiar form.

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

For $k^2 \geq \alpha^2$ define $\kappa^2 \equiv k^2-\alpha^2$

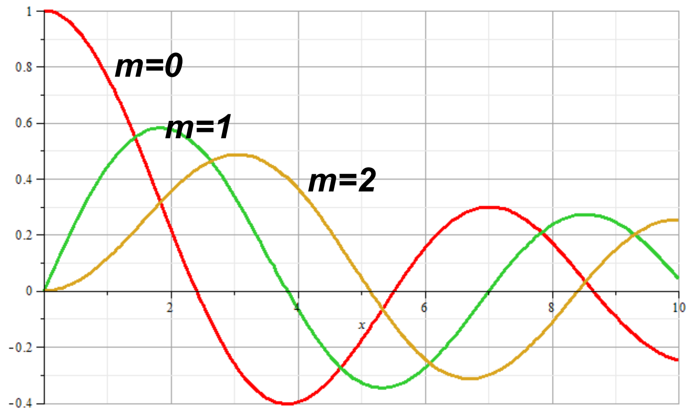
$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+\kappa^2\right)R(r)=0$$

Cylinder surface boundary conditions: $\left.\frac{dR}{dr}\right|_{r=a}=0$

$$\Rightarrow R(r)=J_m(\kappa r) \quad \text{where for} \quad \frac{dJ_m(x'_{mn})}{dx}=0, \quad \kappa_{mn}=\frac{x'_{mn}}{a}$$

With certain assumptions, we can show that the radial solutions for the air motion, are Bessel functions of order m.

Bessel functions : $J_m(x)$



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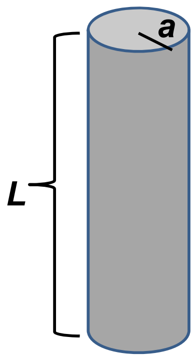
Some plots of Bessel functions.

Now recall the boundary conditions



Boundary values:

At fixed surface: $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$

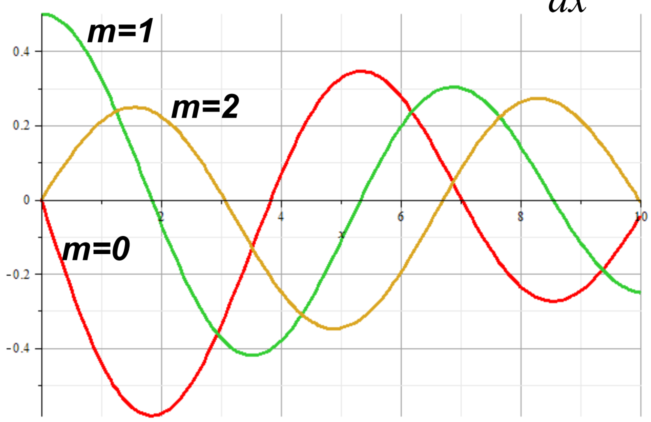
At free surface: $\dfrac{\partial \Phi}{\partial t} = 0$

$$\Phi(r, \varphi, z, t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$
$$= J_m(\kappa r)e^{im\varphi}e^{i\alpha z}e^{-i\omega t}$$

$$\text{For} \quad r = a, \quad \left.\frac{\partial \Phi(r, \varphi, z, t)}{\partial r}\right|_{r=a} = 0$$

Now consider the boundary conditions for the sound wave within the pipe, focusing on the radial direction.

Bessel function derviatives : $\dfrac{dJ_m(x)}{dx}$



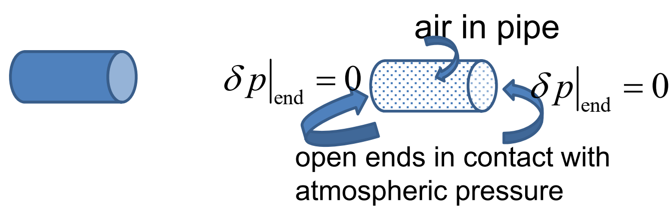
Zeros of derivatives: m=0: 0.00000, 3.83171, 7.01559
m=1: 1.84118, 5.33144, 8.53632
m=2: 3.05424, 6.70613, 9.96947

Zeroes of the derivatives of Bessel functions.

Some details on the open pipe boundary conditions --

Comment --

1. Open pipe boundary condition



Boundary condition for z=0, z=L:

For open - open pipe:

$$Z(0) = Z(L) = 0 \qquad \Rightarrow Z(z) = \sin\left(\frac{p\pi z}{L}\right)$$

$$\Rightarrow \alpha_p = \frac{p\pi}{L}, \qquad p = 1,2,3...$$

Resonant frequencies:

$$\frac{\omega^2}{c^2} = k^2 = \kappa_{mn}^2 + \alpha_p^2$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2$$

We also need to consider the boundary conditions for the air motion in the z direction where the paper can be either open or closed. For the open, open pipe, we then find the resonant wavevectors.

Example

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2 = \left(\frac{\pi p}{L}\right)^2 \left(1 + \left(\frac{L}{a}\right)^2 \left(\frac{x'_{mn}}{\pi p}\right)^2\right)$$

$$\pi p = 3.14, 6.28, 9.42....$$

$$x'_{mn} = 0.00, 1.84, 3.05.... \quad \text{for } x'_{00}, x'_{10}, x'_{20}$$

$$\Phi(r,\varphi,z,t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$= J_m\left(\frac{x'_{mn}}{a} r\right) e^{im\varphi} \sin\left(\frac{p\pi}{L} z\right) e^{-i\omega t}$$

More details.

Alternate boundary condition for z=0, z=L:

For open - closed pipe :

$$\frac{dZ(0)}{dz} = Z(L) = 0 \qquad \Rightarrow \; Z(z) = \cos\left(\frac{(2p+1)\pi z}{2L}\right)$$

$$\Rightarrow \alpha_p = \frac{(2p+1)\pi}{2L}, \qquad p = 0,1,2,3\ldots$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi(2p+1)}{2L}\right)^2$$

Now consider other boundary conditions and their resonances.

The above analysis pertains to resonant air waves within a cylindrical pipe. As previously mentioned, you can hear these resonances if you put your ear close to such a pipe. The same phenomenon is the basis of several musical instruments such as organ pipes, recorders, flutes, clarinets, oboes, etc.

Question – what about a trumpet, trombone, French horn, etc?

a. Same idea?
b. Totally different?

But for musical instruments, you do not want to put your ear next to the device – additional considerations must apply. Basically, you want to couple these standing waves to produce traveling waves.

Modifications needed for the pandemic --

Image from the Winston-Salem Journal 11/1/2020

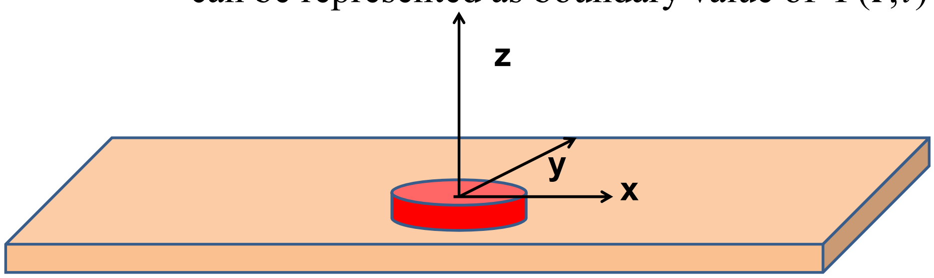
For other instruments, the resonance is initiated by another resonant device which couples to air --

Simple model of a sound amplifier --

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$ Wave equation with source:

Example:

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$ can be represented as boundary value of $\Phi(\mathbf{r},t)$



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This is what we will consider on Monday.