PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 31: Chap. 9 of F&W

Wave equation for sound

1. Linear approximation: sound generation
2. Linear approximation: sound scattering

In this lecture, we will consider traveling wave solutions to the linear sound wave equations and also consider non-linear effects.

28	Mon, 11/01/2021	Chap. 9	Mechanics of 3 dimensional fluids	#19	11/03/2021
29	Wed, 11/03/2021	Chap. 9	Linearized hydrodynamics equations	#20	11/05/2021
30	Fri, 11/05/2021	Chap. 9	Linear sound waves	#21	11/08/2021
31	Mon, 11/08/2021	Chap. 9	Sound sources and scattering; Nonlinear effects	#22	11/10/2021
32	Wed, 11/10/2021	Chap. 9	Non linear effects in sound waves and shocks	Topic due	11/12/2021
33	Fri, 11/12/2021	Chap. 10	Surface waves in fluids		
34	Mon, 11/15/2021	Chap. 10	Surface waves in fluids; soliton solutions		
35	Wed, 11/17/2021	Chap. 11	Heat conduction		
	Fri, 11/19/2021		Presentations I		
	Mon, 11/22/2021		Presentations II		
	Wed, 11/24/2021		Thanksgiving		
	Fri, 11/26/2021		Thanksgiving		
36	Mon, 11/29/2021	Chap. 12	Viscous effects on hydrodynamics		
37	Wed, 12/01/2021	Chap. 1-12	Review		
38	Fri, 12/03/2021	Chap. 1-12	Review		

No HW; work on presentations

Schedule.

PHY 711 -- Assignment #22

Nov. 08, 2021

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Equation 51.69 of **F & W** gives the expansion of a plane wave in cylindrical coordinates in terms of an infinite summation of Bessel functions of order m. Using the asymptotic form of the Bessel function (given in the notes and in the appendix D3.24), show the validity of this identity in the limit $kr \rightarrow \infty$.

Comment about the units of sound frequency

Harmonic time dependence of a wave:

$$\Phi(\mathbf{r},t) = f(\mathbf{r})e^{-i\omega t} = f(\mathbf{r})e^{-2\pi i\nu t}$$

Note that ω has units of radians/sec

ν has units of cycles/sec (Hz)

$$\nu = \frac{\omega}{2\pi}$$

Solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution:

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

Note that these sound waves are "longitudinal"

-- the velocity wave direction is along the propagation direction: $\delta\mathbf{v} = -\nabla\Phi = -iA\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$

Review wave equation and plane wave solutions.

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Solution in terms of Green's function :

$$\Phi(\mathbf{r},t) = \int d^3r' \int dt'\, G(\mathbf{r}-\mathbf{r}', t-t')\, f(\mathbf{r}', t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Now think of wave equation with a source. The Green's function is a very powerful tool for solving these problems. We will use similar techniques in solving the wave equation for electromagnetic waves.

Where does this force term come from?

Equations to lowest order in perturbation -- keeping applied force:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} \qquad \Rightarrow \qquad \frac{\partial \delta\mathbf{v}}{\partial t} = \mathbf{f}_{applied} - \frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \Rightarrow \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0$$

Assuming $\delta\mathbf{v} = -\nabla\Phi$ and $\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2\delta\rho$ and $\mathbf{f}_{\text{applied}} = -\nabla U_{\text{applied}}$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = \mathbf{f}_{applied} - \frac{\nabla \delta p}{\rho_0} \qquad \Rightarrow \qquad -\nabla\left(\frac{\partial \Phi}{\partial t} - \frac{c^2\delta\rho}{\rho_0} - U_{\text{applied}}\right) = 0$$

When the dust clears -- $\frac{\partial^2 \Phi}{\partial t^2} - c^2\nabla^2\Phi = \frac{\partial U_{\text{applied}}}{\partial t}$

In fact, in our example, the forcing term occurs at a boundary of our system and can be treated in terms of a boundary value.

Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Result that we will derive.

Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Recall that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega$$

First step of derivation using Fourier transform in the time domain.

Derivation of Green's function for wave equation -- continued

$$\text{Define}: \quad \tilde{G}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} G(\mathbf{r},t) e^{i\omega t} dt$$

$$G(\mathbf{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\mathbf{r},\omega) e^{-i\omega t} d\omega$$

$\tilde{G}(\mathbf{r},\omega)$ must satisfy :

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}') \quad \text{where} \quad k^2 = \frac{\omega^2}{c^2}$$

Spatial equation for Fourier amplitudes.

Derivation of Green's function for wave equation -- continued

$$\left(\nabla^2 + k^2\right)\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

Solution assuming isotropy in $\mathbf{r}-\mathbf{r}'$:

$$\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega) = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Check -- Define $R \equiv |\mathbf{r}-\mathbf{r}'|$ and for $R > 0$:

$$\left(\nabla^2 + k^2\right)\widetilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\widetilde{G}(R,\omega)\right) + k^2\widetilde{G}(R,\omega) = 0$$

Solution for isotropic system

Derivation of Green's function for wave equation -- continued

For $R > 0$:

$$\left(\nabla^2 + k^2\right)\tilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\tilde{G}(R,\omega) = 0$$

$$\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\left(R\tilde{G}(R,\omega)\right) = 0$$

$$\left(R\tilde{G}(R,\omega)\right) = A\ e^{ikR} + Be^{-ikR}$$

$$\Rightarrow\ \tilde{G}(R,\omega) = A\frac{e^{ikR}}{R} + B\frac{e^{-ikR}}{R}$$

Isotropic solutions continued.

Derivation of Green's function for wave equation – continued
need to find *A* and *B*.

$$\text{Note that :}\quad \nabla^2 \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\delta(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow A = B = \frac{1}{4\pi}$$

$$\tilde{G}(R,\omega) = \frac{e^{\pm ikR}}{4\pi R}$$

A special property of the Laplace operator.

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{G}(\mathbf{r}-\mathbf{r}',\omega) e^{-i\omega(t-t')} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

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Taking the inverse Fourier transform.

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

Noting that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u} d\omega = \delta(u)$

$$\Rightarrow G(\mathbf{r}-\mathbf{r}',t-t') = \frac{\delta\left(t-\left(t' \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Details and final result.

➔In order to solve an inhomogenous wave equation with a time harmonic forcing or boundary term, we can use the corresponding Green's function:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

In fact, this Green's function is appropriate for solving equations with boundary conditions at infinity. For solving problems with surface boundary conditions where we know the boundary values or their gradients, the Green's function must be modified.

It is convenient/important to use the Green's function consistent with the boundary values of the particular system of interest.

Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that : $\int_V \left(h\nabla^2 g - g\nabla^2 h\right) d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot \hat{\mathbf{n}} d^2r$

$$\nabla^2\tilde{\Phi} + k^2\tilde{\Phi} = -\tilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

$$h \leftrightarrow \tilde{\Phi}; \qquad g \leftrightarrow \tilde{G}$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot \hat{\mathbf{n}} d^2r$$

In order to motivate the use of Green's functions, we consider the famous Green's theorem. Note that these details/derivations will also be discussed when we consider mathematically similar situations for electrodynamic systems.

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}}d^2r$$

Exchanging $\mathbf{r} \leftrightarrow \mathbf{r}'$:

$$\int_V \left(\tilde{\Phi}(\mathbf{r}',\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega)\right) d^3r' =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

If the integration volume V includes the point $\mathbf{r} = \mathbf{r}'$:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega) d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

➔extra contributions from boundary

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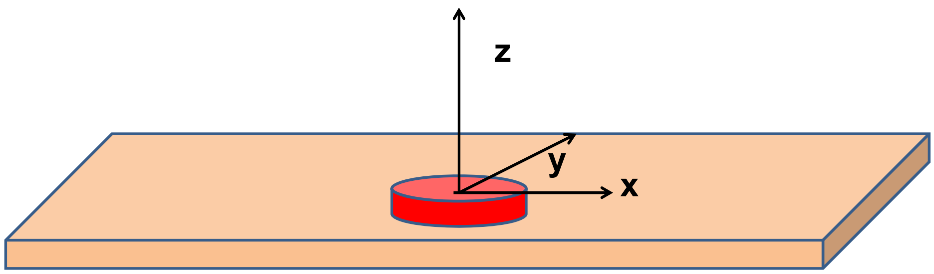
Derivation continued.

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example:

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$ can be represented as boundary value of $\Phi(\mathbf{r},t)$



Now consider a simplified model of a sound amplifier where the red cylinder moves up and down in the z direction at a particular frequency omega.

Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\tilde{f}(\mathbf{r}',\omega)d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

Boundary values for our example:

$$\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)_{z=0} = \begin{cases} 0 & \text{for } x^2+y^2 > a^2 \\ i\omega\varepsilon a & \text{for } x^2+y^2 < a^2 \end{cases}$$

Note: Need Green's function with vanishing gradient at $z = 0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}' = -z'; \quad z > 0$$

In this case, we need to use a modified Green's function to satisfy the boundary condition at z=0.

$$\widetilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\widetilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

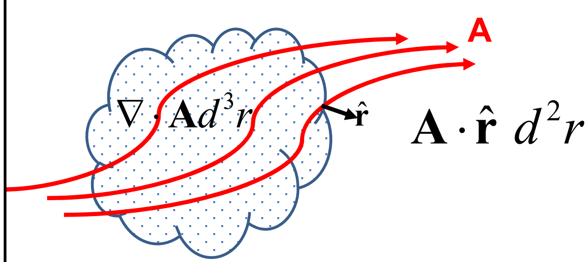
$$\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)_{z'=0} = \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}; \quad z > 0$$

Some details.

Some details about the boundary values --

$$\int_{\text{Volume}} \nabla \cdot \mathbf{A}\, d^3r = \int_{\text{Surface}} \mathbf{A} \cdot \hat{\mathbf{r}}\, d^2r$$



Note: The surface term is important when the analysis is performed in a confining volume. When the analysis involves an infinite volume and **A** vanishes at infinity, the surface term does not contribute.

Some more details --

Note: Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}\left(\left|\mathbf{r}-\mathbf{r}'\right|,\omega\right)=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}+\frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi\left|\mathbf{r}-\bar{\mathbf{r}}'\right|} \qquad \text{where } \bar{z}'=-z'; \quad z>0$$

Note that $\left|\mathbf{r}-\mathbf{r}'\right| \equiv \sqrt{\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2}$

$$\left|\mathbf{r}-\bar{\mathbf{r}}'\right| \equiv \sqrt{\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z+z'\right)^2}$$

Fourier transform of velocity potential:

$$\tilde{\Phi}(\mathbf{r},\omega)=\int_V \tilde{G}\left(\left|\mathbf{r}-\mathbf{r}'\right|,\omega\right) f\left(\mathbf{r}',\omega\right) d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}\left(\left|\mathbf{r}-\mathbf{r}'\right|,\omega\right)-\tilde{G}\left(\left|\mathbf{r}-\mathbf{r}'\right|,\omega\right)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}' d^2r'$$

Need this term to vanish at z'=0

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'dy'$$

$$= -i\omega\varepsilon a\int_0^a r'dr'\int_0^{2\pi} d\phi' \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$

Integration domain: $x' = r'\cos\varphi'$

$y' = r'\sin\varphi'$

For $r >> a$; $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\varphi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

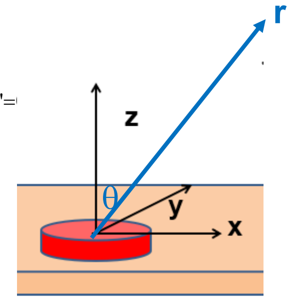
$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\varphi'$$

Changing to more convenient coordinates. Preparing to evaluate the expression far from the moving piston.

More details

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:z'=0} \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z'}dx'dy'$$

$$= -i\omega\varepsilon a\int_0^a r'dr'\int_0^{2\pi} d\varphi' \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$



Integration domain: $x' = r'\cos\varphi'$

$y' = r'\sin\varphi'$

For $r >> a$; $\quad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\quad \varphi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\varphi'$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon a}{2\pi}\frac{e^{ikr}}{r}\int_0^a r'dr'\int_0^{2\pi} d\phi' e^{-ikr'\sin\theta\sin\phi'}$$

Note that : $$\frac{1}{2\pi}\int_0^{2\pi} d\phi' e^{-iu\sin\phi'} = J_0(u)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a\frac{e^{ikr}}{r}\int_0^a r'dr' J_0(kr'\sin\theta)$$

$$\int_0^w udu J_0(u) = wJ_1(w)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a^3\frac{e^{ikr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Approximate solution continued. In this approximation, the integral can be evaluated in terms of Bessel functions.

Energy flux : $\mathbf{j}_e = \delta\mathbf{v}p$

Taking time average : $\langle \mathbf{j}_e \rangle = \frac{1}{2}\Re\left(\delta\mathbf{v}p^*\right)$

$$= \tfrac{1}{2}\rho_0\Re\left((-\nabla\Phi)(-i\omega\Phi)^*\right)$$

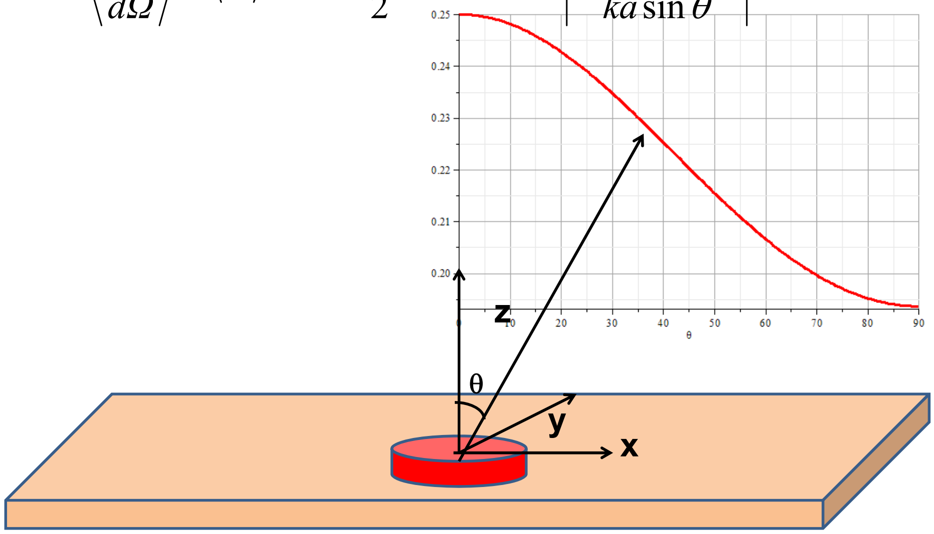
Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2}\rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$

Estimating the power of the sound wave in this asymptotic regime.

Time averaged power per solid angle:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$



Graph of the power as a function of the polar angle theta.

Scattering of sound waves –
for example, from a rigid cylinder

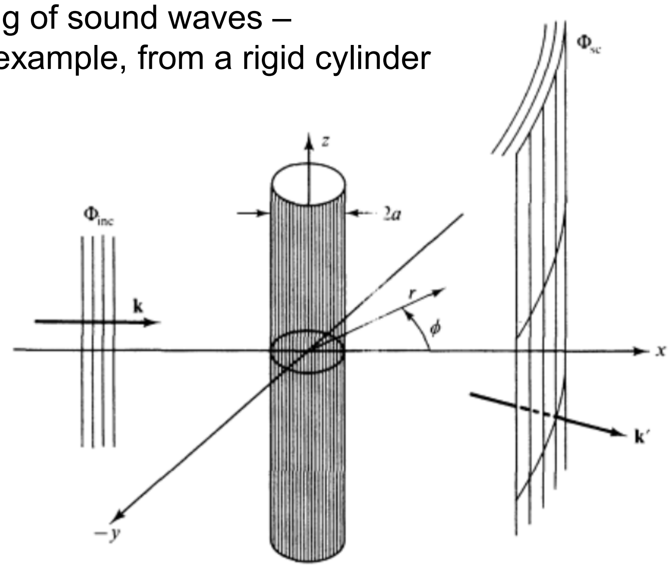


Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

Now consider the case of a plane wave of sound, scattering off of a cylindrical object. Can you think of a physical situation for this model?

Example of cylindrical scattering objects --

Suppose a trumpeter is playing near the columns. Maximal scattering occurs when

a. Facing toward the column b. Facing away from the column.

Scattering of sound waves –

for example, from a rigid cylinder

Velocity potential --

$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \qquad \Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$$

Helmholz equation in cylindrical coordinates:

$$\left(\nabla^2 + k^2\right)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\phi^2} + \frac{\partial}{\partial z^2} + k^2\right)\Phi(\mathbf{r})$$

Assume: $\Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi}R_m(r)$

where $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right)R_m(r) = 0$

Analysis of the scattering wave using cylindrical coordinates.

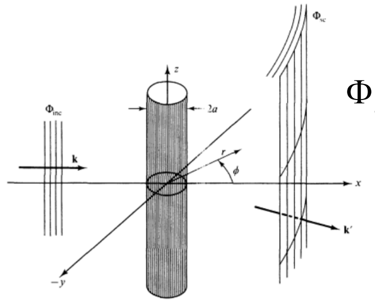


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

$$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr) \quad \text{where Hankel function}$$

represents an outgoing wave: $H_m(kr) = J_m(kr) + iN_m(kr)$

Boundary condition at $r = a$: $\left.\dfrac{\partial \Phi}{\partial r}\right|_{r=a} = 0$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \qquad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

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In this case we expect a cylindrical wave that can be represented in terms of Bessel and Neumann functions, or more conveniently in terms of Hankel functions H. Satisfying the boundary values on the surface of the scattering cylinder, we find the coefficients of the expression.

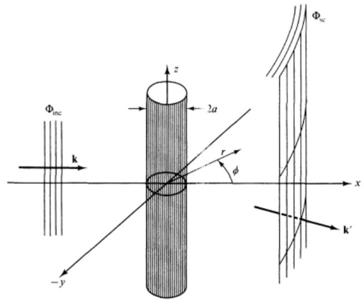


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = -\sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \underset{kr\to\infty}{\approx} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \underset{kr\to\infty}{\approx} f(\phi)\sqrt{\frac{1}{r}} e^{ikr} = -\sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Rightarrow f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi-\pi/4)}$$

Using the asymptotic form of the Hankel functions we can analyze the results further.

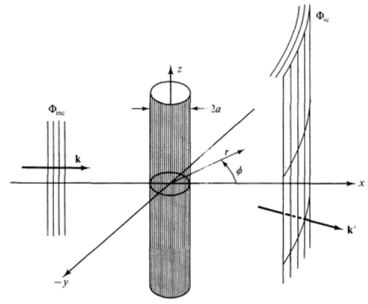
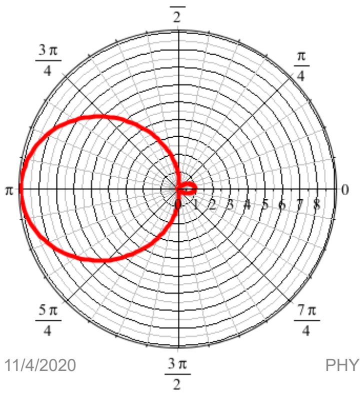


Figure 51.8 Scattering from a rigid cylinder.



$$\frac{d\sigma}{d\phi} = \left| f(\phi) \right|^2$$

$$f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi - \pi/4)}$$

For *ka << 1*

$$\frac{d\sigma}{d\phi} = \left| f(\phi) \right|^2 \approx \frac{1}{8}\pi k^3 a^4 \left(1 - 2\cos\phi\right)^2$$

Defining the appropriate scattering cross section, we can analyze the results further. For ka<<1 (long wavelengths, low frequencies) we find that most of the sound is scattered backwards from the propagation direction.