# PHY 711 Classical Mechanics and Mathematical Methods
## 10-10:50 AM MWF in Olin 103

## Notes on Lecture 33:

**Chapter 10 in F & W: Surface waves**

1. Water waves in a channel
2. Wave-like solutions; wave speed



In today's lecture we will investigate transverse waves at the surface of a channel of water.

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|---|---|---|---|---|---|
| **28** | Mon, 11/01/2021 | Chap. 9 | Mechanics of 3 dimensional fluids | #19 | 11/03/2021 |
| **29** | Wed, 11/03/2021 | Chap. 9 | Linearized hydrodynamics equations | #20 | 11/05/2021 |
| **30** | Fri, 11/05/2021 | Chap. 9 | Linear sound waves | #21 | 11/08/2021 |
| **31** | Mon, 11/08/2021 | Chap. 9 | Sound sources and scattering | #22 | 11/10/2021 |
| **32** | Wed, 11/10/2021 | Chap. 9 | Non linear effects in sound waves and shocks | Topic due | 11/12/2021 |
| **33** | Fri, 11/12/2021 | Chap. 10 | Surface waves in fluids | | |
| **34** | Mon, 11/15/2021 | Chap. 10 | Surface waves in fluids; soliton solutions | | |
| **35** | Wed, 11/17/2021 | Chap. 11 | Heat conduction | | |
| | Fri, 11/19/2021 | | Presentations I | | |
| | Mon, 11/22/2021 | | Presentations II | | |
| | Wed, 11/24/2021 | | Thanksgiving | | |
| | Fri, 11/26/2021 | | Thanksgiving | | |
| **36** | Mon, 11/29/2021 | Chap. 12 | Viscous effects on hydrodynamics | | |
| **37** | Wed, 12/01/2021 | Chap. 1-12 | Review | | |
| **38** | Fri, 12/03/2021 | Chap. 1-12 | Review | | |



Update to schedule including a homework dealing with today's topic.

Comment on defunct HW 22 --

A "proof" for identity $e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$

For integer $m$, $J_{-m}(kr) = (-1)^m J_m(kr) = e^{im\pi} J_m(kr)$

$$\sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr) = 1 + 2\sum_{m=1}^{\infty} i^m \cos(m\phi) J_m(kr)$$

Integral form for Bessel function

$$i^m J_m(z) = \frac{1}{\pi}\int_0^{\pi} e^{iz\cos\phi} \cos(m\phi) d\phi$$

$$\frac{1}{\pi}\int_0^{\pi} \cos(m\phi) d\phi = \delta_{0,m} \qquad \frac{1}{\pi}\int_0^{\pi} \cos(m'\phi)\cos(m\phi) d\phi = \frac{1}{2}\delta_{m',m}$$

$$\Rightarrow e^{ikr\cos\phi} = 1 + 2\sum_{m=1}^{\infty} i^m \cos(m\phi) J_m(kr)$$

Your question –
From Owen –
How does the viscosity of the fluid tie into all of this? Is there some term or constant in the equations that represents it, or is it more complicated to model viscosity?

Comment – The generalization of the hydrodynamics equations that we have used so far, is the Navier-Stokes equation which is covered in Chapter 12 of your text book. This scheme inserts two viscosity parameters into the equations of motion with additional constraints to account for heat flow.

Reference: Chapter 10 of Fetter and Walecka

Physics of incompressible fluids and their surfaces

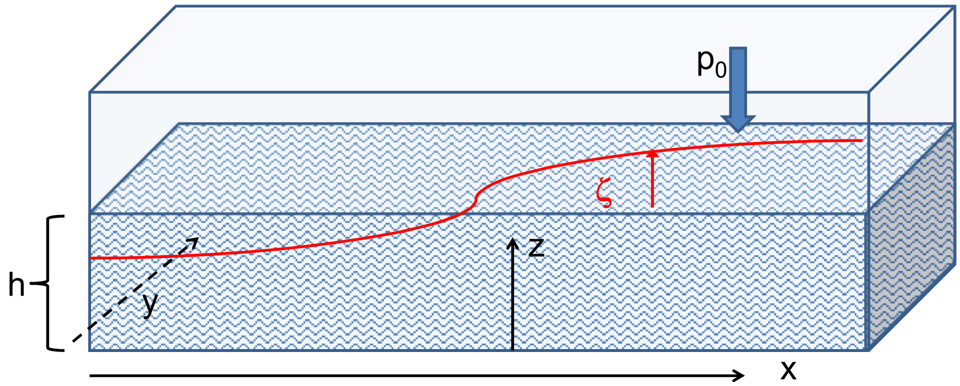
Consider a container of water with average height h and surface h+ζ(x,y,t); (h ←→ $z_0$ on some of the slides)

Atmospheric pressure is in equilibrium with the surface of water

Pressure at a height $z$ above the bottom where the surface is at a height $h+\zeta$:

$$p(z)=\begin{cases} p_0+\rho g\left(h+\zeta-z\right) & \text{For } z\leq h+\zeta \\ p_0 & \text{For } z>h+\zeta \end{cases}$$

Here $\rho$ represents density of water

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Defining the system and the notation.

Why do we not consider $\rho_{air}$ in this analysis?

a. Because it is a reasonable approximation
b. Because it simplifies the analysis
c. Both of the above

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -g\hat{\mathbf{z}} - \frac{\nabla p}{\rho}$$

Assume that $v_z << v_x, v_y$ $\Rightarrow -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \approx 0$

$$\Rightarrow p(x,y,z,t) = p_0 + \rho g\left(\zeta(x,y,t) + h - z\right)$$ within the water

Horizontal fluid motions (keeping leading terms):

$$\frac{dv_x}{dt} \approx \frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \zeta}{\partial x}$$

$$\frac{dv_y}{dt} \approx \frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial \zeta}{\partial y}$$



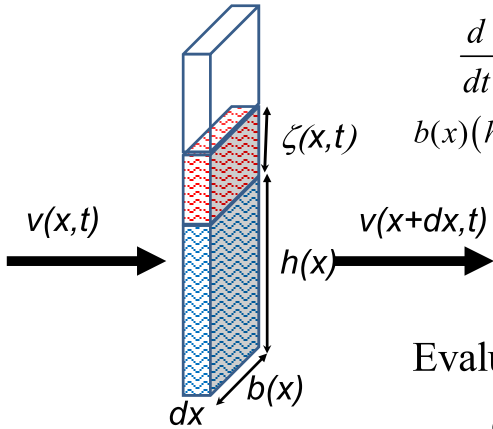
Hydrodynamic equations for this case.

Consider a surface $\zeta(x,t)$ wave moving in the $x$-direction in a channel of width $b(x)$ and height $h(x)$:

Continuity condition in integral form:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v} \cdot d\mathbf{A} = 0$$

$$b(x)\big(h(x)+\zeta(x,t)\big)dx \qquad b(x)\big(h(x)+\zeta(x,t)\big)\hat{\mathbf{x}}$$




Evaluating continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

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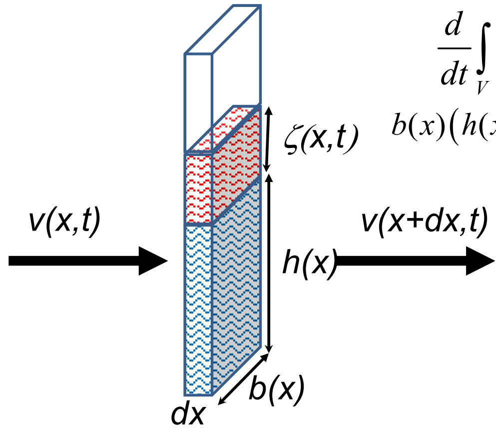
Considering an increment along the propagation direction including the effects of the continuity equation.

Some details

Continuity condition in integral form:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v}\cdot d\mathbf{A} = 0$$

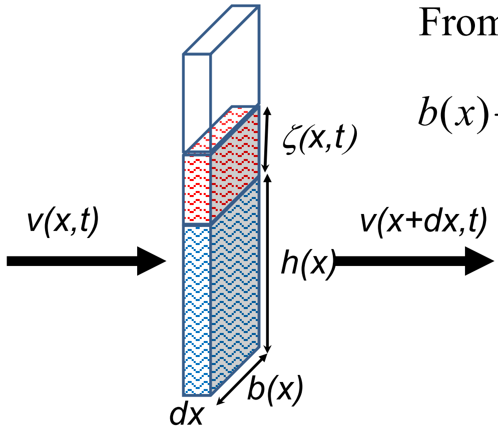
$b(x)\left(h(x)+\zeta(x,t)\right)dx \qquad b(x)\left(h(x)+\zeta(x,t)\right)\hat{\mathbf{x}}$




Here, we are assuming that ρ is constant

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{v}\cdot d\mathbf{A} = \rho\int b(x)\frac{\partial \zeta}{\partial t}dx + \rho\int \frac{\partial}{\partial x}\left(b(x)(h(x)+\zeta(x,t))v(x,t)\right)dx = 0$$

$$\Rightarrow b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$




From continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Example (Problem 10.3):

$$b(x) = b_0 \qquad h(x) = \kappa x$$

$$b_0\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big((\kappa x)b_0 v(x,t)\big)$$

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right)$$

From Newton-Euler equation:

$$\frac{dv}{dt} \approx \frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x}$$



Some details for the homework problem which is a special case.

## Example continued

$$\frac{\partial \zeta}{\partial t} = -\kappa\left(v + x\frac{\partial v}{\partial x}\right) \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = -\kappa\left(\frac{\partial v}{\partial t} + x\frac{\partial^2 v}{\partial x \partial t}\right)$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 \zeta}{\partial t^2} = \kappa g\left(\frac{\partial \zeta}{\partial x} + x\frac{\partial^2 \zeta}{\partial x^2}\right)$$

It can be shown that a solution can take the form:

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$

Note that $J_0(u)$ satisfies the equation: $\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + 1\right)J_0(u) = 0$

Therefore, for $u = \frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}$

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx}\right)J_0(u) = \frac{\omega^2}{\kappa g}\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du}\right)J_0(u) = -\frac{\omega^2}{\kappa g}J_0(u)$$



More details pertaining to the homework problem.

Therefore, for $u=\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x} \quad \Rightarrow \frac{1}{\sqrt{x}}=\frac{2\omega}{\sqrt{\kappa g}}\frac{1}{u}$

$$\left(x\frac{d^2}{dx^2}+\frac{d}{dx}\right)J_0(u)=\frac{\omega^2}{\kappa g}\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}\right)J_0(u)=-\frac{\omega^2}{\kappa g}J_0(u)$$

Detail: $\frac{dJ_0(u)}{dx}=\frac{dJ_0(u)}{du}\frac{\omega}{\sqrt{\kappa g}}\frac{1}{\sqrt{x}}$

$$\frac{d^2J_0(u)}{dx^2}=\frac{d^2J_0(u)}{du^2}\left(\frac{\omega}{\sqrt{\kappa g}}\frac{1}{\sqrt{x}}\right)^2-\frac{dJ_0(u)}{du}\frac{\omega}{2\sqrt{\kappa g}}\frac{1}{x\sqrt{x}}$$

Therefore: $\left(x\frac{d^2}{dx^2}+\frac{d}{dx}\right)J_0(u)=\left(\frac{\omega^2}{\kappa g}\frac{d^2J_0(u)}{du^2}+\frac{dJ_0(u)}{du}\frac{\omega}{2\sqrt{\kappa g}}\frac{1}{\sqrt{x}}\right)$

$$=\frac{\omega^2}{\kappa g}\left(\frac{d^2J_0(u)}{du^2}+\frac{dJ_0(u)}{du}\frac{1}{u}\right)$$

Example continued

$$\frac{\partial^2 \zeta}{\partial t^2} = \kappa g \left( \frac{\partial \zeta}{\partial x} + x \frac{\partial^2 \zeta}{\partial x^2} \right)$$

$$\Rightarrow \zeta(x,t) = C J_0\left( \frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$

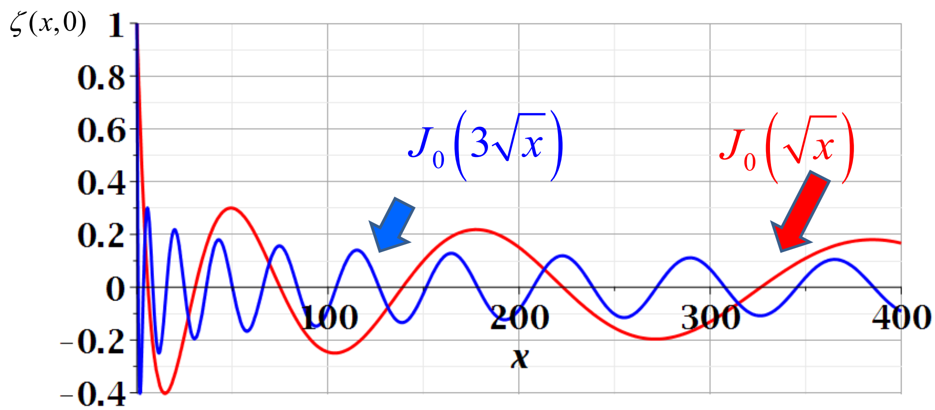
Check:

$$-\omega^2 C J_0\left( \frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t) = \kappa g \left( \frac{\partial}{\partial x} + x \frac{\partial^2}{\partial x^2} \right) C J_0\left( \frac{2\omega\sqrt{x}}{\sqrt{\kappa g}} \right) \cos(\omega t)$$



Continued.

$$\zeta(x,t) = CJ_0\left(\frac{2\omega}{\sqrt{\kappa g}}\sqrt{x}\right)\cos(\omega t)$$



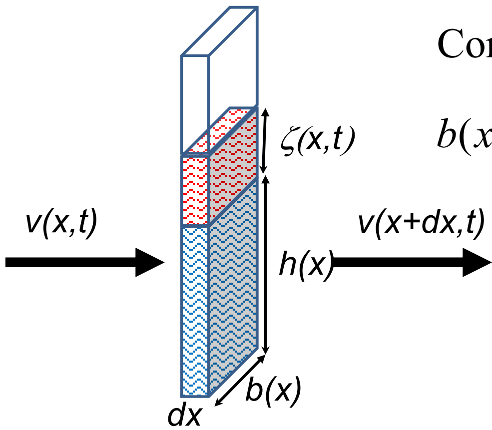



Continued.

Imagine watching the waves at a beach – can you visualize the configuration for the surface wave pattern to approximation this situation?

a. Long flat beach
b. Beach in which average water level increases
c. Beach in which average water level decreases

A simplier example:




Continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\big(h(x)b(x)v(x,t)\big)$$

Special case, where *b* and *h* are constant --

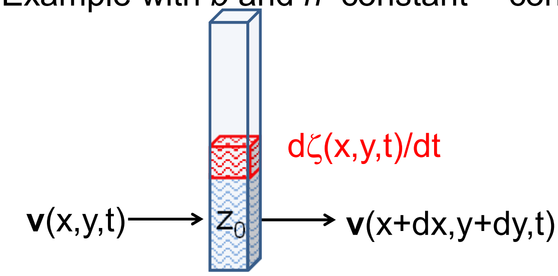
For constant $b$ and $h$:

$$\frac{\partial \zeta}{\partial t} = -h\frac{\partial}{\partial x}\big(v(x,t)\big)$$



A simpler example.

## Example with *b* and *h* constant -- continued




Continuity condition for flow of incompressible fluid:

$$\frac{\partial \zeta}{\partial t} + h\nabla \cdot \mathbf{v} = 0$$

From horizontal flow relations: $$\frac{\partial \mathbf{v}}{\partial t} = -g\nabla\zeta$$

Equation for surface function: $$\frac{\partial^2 \zeta}{\partial t^2} - gh\nabla^2\zeta = 0$$



Considering the surface height.

For uniform channel:

Surface wave equation:

$$\frac{\partial^2 \zeta}{\partial t^2} - c^2 \nabla^2 \zeta = 0 \qquad c^2 = gh$$

More complete analysis finds:

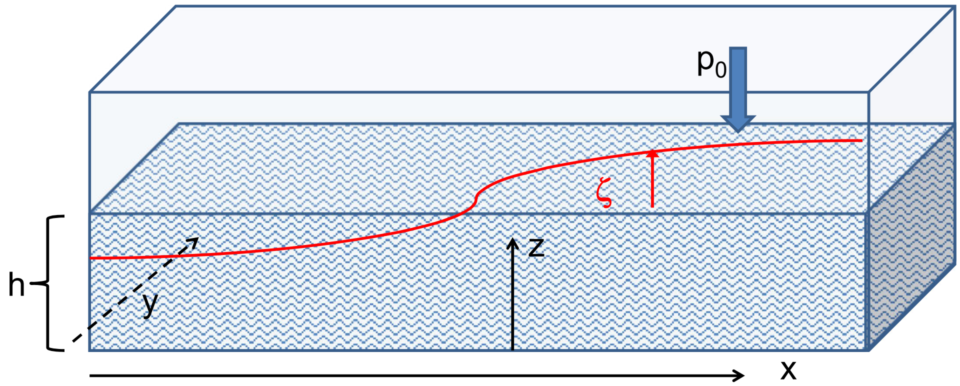
$$c^2 = \frac{g}{k}\tanh(kh) \qquad \text{where } k = \frac{2\pi}{\lambda}$$



For the simple case, we find the wave equation for the surface height. In the following slides, we will find a more complete solution depends on the wavelength the of surface wave.

More details: -- recall setup --

Consider a container of water with average height h and surface h+$\zeta$(x,y,t)

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Some details for the more general case.

Equations describing fluid itself (without boundaries)

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) + \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = -\nabla U - \frac{\nabla p}{\rho}$$

Assume that $\nabla\times\mathbf{v} = 0$ (irrotational flow) $\Rightarrow \mathbf{v} = -\nabla\Phi$

$$\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho}\right) = 0$$

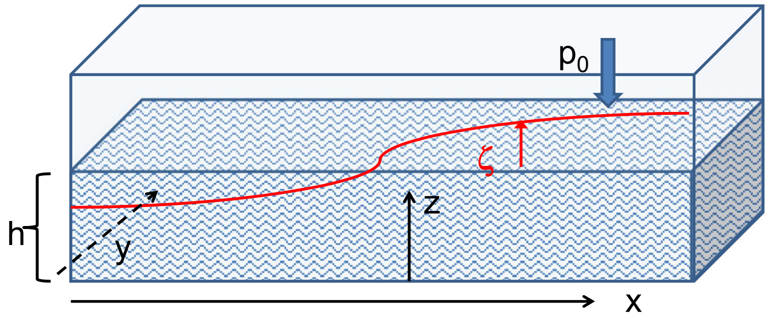
$$\Rightarrow -\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho} = \text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

$$\nabla\cdot\mathbf{v} = -\nabla^2\Phi = 0$$



Considering the case of irrotational flow.




Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} v^2 + g\left(z - h\right) = \text{constant}$$

(We have absorbed $p_0$ in "constant")

$$-\nabla^2 \Phi = 0$$

At surface: $z = h + \zeta$ with $\zeta = \zeta\left(x, y, t\right)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y}$$

where $v_{x,y} = v_{x,y}\left(x, y, h + \zeta, t\right)$



Considering the equations within the wave and at the surface.

Full equations:

Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}v^2 + g(z-h) = \text{constant}$$

(We have absorbed $p_0$ in "constant")

$$-\nabla^2 \Phi = 0$$

At surface: $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$$

Linearized equations:

For $0 \le z \le h + \zeta$: $-\frac{\partial \Phi}{\partial t} + g(z-h) = 0 \qquad -\nabla^2 \Phi = 0$

At surface: $z = h + \zeta \qquad \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} = v_z(x, y, h+\zeta, t)$

$$-\frac{\partial \Phi(x, y, h+\zeta, t)}{\partial t} + g\zeta = 0$$



Taking the linear limit.

For simplicity, keep only linear terms and assume that horizontal variation is only along *x*:

$$\text{For} \quad 0 \le z \le h + \zeta : \qquad \nabla^2 \Phi = \left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right) \Phi(x,z,t) = 0$$

Consider and periodic waveform: $\Phi(x,z,t) = Z(z)\cos\big(k(x-ct)\big)$

$$\Rightarrow \left( \frac{d^2}{dz^2} - k^2 \right) Z(z) = 0$$

Boundary condition at bottom of tank: $v_z(x,0,t) = 0$

$$\Rightarrow \frac{dZ}{dz}\big(0\big) = 0 \qquad\qquad Z(z) = A\cosh(kz)$$



Solution for the linear equations.

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$\text{At surface:} \quad z = h + \zeta \qquad \frac{\partial \zeta}{\partial t} = v_z(x, h+\zeta, t) = -\frac{\partial \Phi(x, h+\zeta, t)}{\partial z}$$

$$-\frac{\partial \Phi(x, h+\zeta, t)}{\partial t} + g\zeta = 0$$

$$-\frac{\partial^2 \Phi(x, h+\zeta, t)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi(x, h+\zeta, t)}{\partial t^2} - g\frac{\partial \Phi(x, h+\zeta, t)}{\partial z} = 0$$

$$\text{For} \quad \Phi(x, (h+\zeta), t) = A\cosh(k(h+\zeta))\cos(k(x-ct))$$

$$A\cosh(k(h+\zeta))\cos(k(x-ct))\left(k^2c^2 - gk\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}\right) = 0$$

$$\Rightarrow \mathrm{c}^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}$$

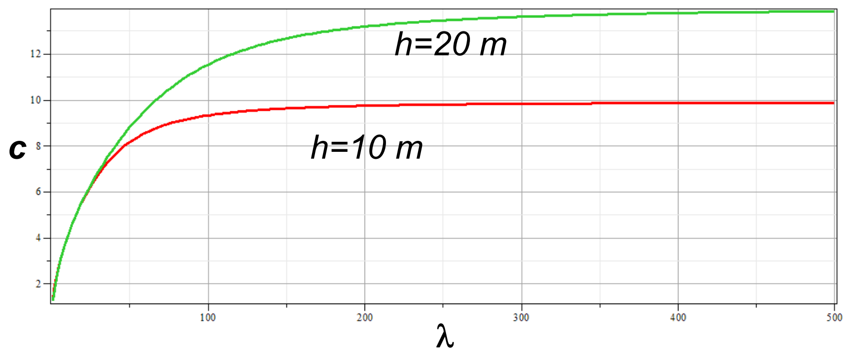


An expression for c.

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))} = \frac{g}{k}\tanh(k(h+\zeta))$$

Assuming $\zeta << h$: $\quad c^2 = \frac{g}{k}\tanh(kh) \qquad \lambda = \frac{2\pi}{k}$






Evaluating c as a function of wavelength.

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 \approx \frac{g}{k}\tanh(kh) \qquad \text{For } \lambda >> h,\ c^2 \approx gh$$

$$\Phi(x,z,t) = A\cosh(kz)\cos(k(x-ct))$$

$$\zeta(x,t) = \frac{1}{g}\frac{\partial\Phi(x,h+\zeta,t)}{\partial t} \approx \frac{kc}{g}A\cosh(kh)\sin(k(x-ct))$$

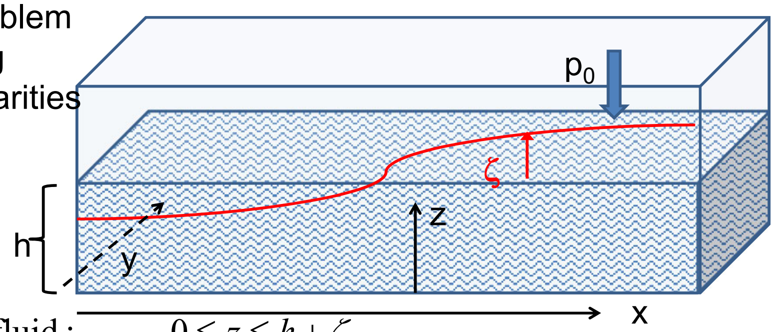
Note that for $\lambda >> h, \quad c^2 \approx gh$

(solutions are consistent with previous analysis)



Form of the surface wave form.

General problem including non-linearities




Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed}$$

$$-\nabla^2\Phi = 0 \qquad\qquad p_0 \text{ in our constant.)}$$

At surface: $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x\frac{\partial \zeta}{\partial x} + v_y\frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$$

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Introducing the equations beyond the linear approximation that we will cover next time.