PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Plan for Lecture 34: Chapter 10 in F & W

Surface waves

- **Summary of linear surface wave solutions**
- **Non-linear contributions and soliton solutions**

In this lecture, we will continue analyzing surface waves in water including the special non-linear soliton solutions.

This material is covered in Chapter 10 of your textbook using similar notation.

33	Fri, 11/12/2021	Chap. 10	Surface waves in fluids	
34	Mon, 11/15/2021	Chap. 10	Surface waves in fluids; soliton solutions	
35	Wed, 11/17/2021	Chap. 11	Heat conduction	
	Fri, 11/19/2021		Presentations I	
	Mon, 11/22/2021		Presentations II	
	Wed, 11/24/2021		Thanksgiving	
	Fri, 11/26/2021		Thanksgiving	
36	Mon, 11/29/2021	Chap. 12	Viscous effects on hydrodynamics	
37	Wed, 12/01/2021	Chap. 1-12	Review	
38	Fri, 12/03/2021	Chap. 1-12	Review	

Schedule.

Sign up will be available after class -- revised as discussed --

Schedule for Monday, November 22, 2021

Time	Name	Topic
10:00-10:20	Owen	
10:20-10:40	Manikata	
10:40-11:00	Wells	
11:00-11:20	Can	
11:20-11:40	Ramesh	

PHYSICS Ph.D. DEFENCE

2:30 PM ZSR-404

TUESDAY

•

NOVEMBER 16, 2021

"Metal Halide Perovskite: Innovations in Applications and Processing"

Metal halide perovskites (MHPs) are an exciting class of materials that have been a topic of great interest in material science and semiconductor research. These materials possess intriguing properties, such as cost-effective processing, high charge carrier mobilities, band gap tunability, and ionic conduction. With such a wide array of properties, MHPs have shown that they can be used in a variety of electronic devices. There is much to learn about MHPs, both in terms of improving device performance and in developing cost-effective and nonhazardous processing MHP components. These studies focus on understanding MHPs in the context of device applications and on processing techniques for MHPs.

We have successfully reduced the contact resistance in 2D perovskite transistors by chemically treating the surface of both the electrodes and the dielectric. We further found that the application of a chemical barrier layer between the electrodes and the perovskite inhibits detrimental chemical reactions at the electrode/perovskite interface, resulting in the lowest reported value for a 2D bottom-gate, bottom-

Colin Tyznik

Mentor: Dr. Oana Jurchescu

Department of Physics

Wake Forest University

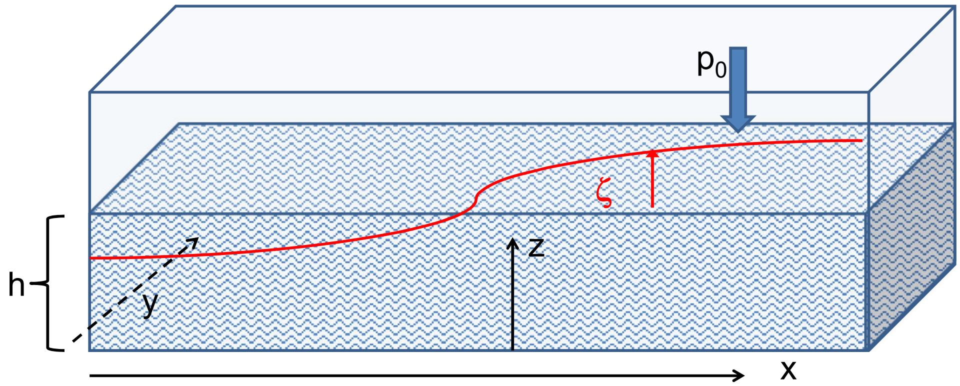
Public Presentation - 2:30 pm

ZSR Library Auditorium and Zoom

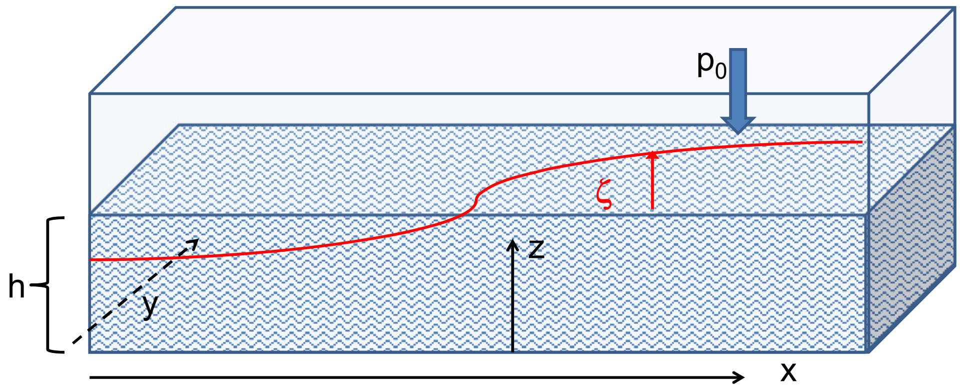
(followed by Private Defense)*

Consider a container of water with average height h and surface h+ζ(x,y,t)

Atmospheric pressure p_0 is in equilibrium at the surface



Reference system and notation.



Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -\nabla U - \frac{\nabla p}{\rho}$$

Continuity equation within the fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{v} = 0$$

For irrotational flow -- $\mathbf{v} = -\nabla \Phi$

Linearized equation: $\nabla\left(-\frac{\partial \Phi}{\partial t} + g(z-h) + \frac{p}{\rho} \right) = 0$

At surface: $z = h + \zeta \qquad -\frac{\partial \Phi}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

Summarizing the linear analysis.

Keep only linear terms and assume that horizontal variation is only along *x*:

For $0 \le z \le h+\zeta$: $\nabla^2\Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}\right)\Phi(x,z,t) = 0$

Consider and periodic waveform: $\Phi(x,z,t) = Z(z)\cos\big(k(x-ct)\big)$

$$\Rightarrow \left(\frac{d^2}{dz^2} - k^2\right)Z(z) = 0$$

Boundary condition at bottom of tank: $v_z(x,0,t) = 0$

$$\Rightarrow \frac{dZ}{dz}(0) = 0 \qquad Z(z) = A\cosh(kz)$$

At surface: $z = h+\zeta$ $\quad \frac{\partial \zeta}{\partial t} = v_z\left(x, h+\zeta, t\right) = -\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial z}$

Also: $\quad -\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

$$\Rightarrow -\frac{\partial^2 \Phi\left(x,h+\zeta,t\right)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi\left(x,h+\zeta,t\right)}{\partial t^2} - g\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial z} = 0$$

Continue analysis of linear equations.

Velocity potential: $\Phi(x,z,t) = A\cosh(kz)\cos\left(k(x-ct)\right)$

At surface: $\Phi(x,(h+\zeta),t) = A\cosh(k(h+\zeta))\cos\left(k(x-ct)\right)$

$$A\cosh(k(h+\zeta))\cos\left(k(x-ct)\right)\left(k^2c^2 - gk\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}\right) = 0$$

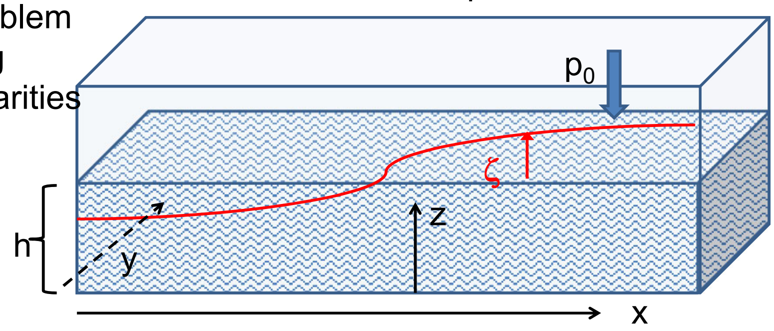
$$\Rightarrow c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))} \approx \frac{g}{k}\tanh(kh)$$

Note that this solution represents a pure plane wave. More likely, there would be a linear combination of wavevectors *k*. Additionally, your text considers the effects of surface tension. In this lecture, we will focus on the effects of the non-linear effects of Euler and continuity equations.

Consistent analysis of the wave speed.

Surface waves in an incompressible fluid

General problem including non-linearities



Within fluid: $\quad 0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}v^2 + g\left(z-h\right) = \text{constant} \qquad \Phi = \Phi(x,y,z,t)$$

$$-\nabla^2 \Phi = 0 \qquad \mathbf{v} = \mathbf{v}(x,y,z,t) = -\nabla \Phi(x,y,z,t)$$

At surface: $\quad z = h + \zeta \qquad$ with $\zeta = \zeta\left(x,y,t\right)$

$$v_z(h+\zeta) = \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} = -\left.\frac{\partial \Phi(x,y,z,t)}{\partial z}\right|_{z=h+\zeta} \quad \text{where } v_{x,y} = v_{x,y}\left(x,y,h+\zeta,t\right)$$

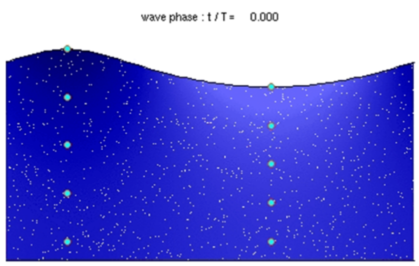
Returning to the full problem with non-linearities.

Some relationships at surface --

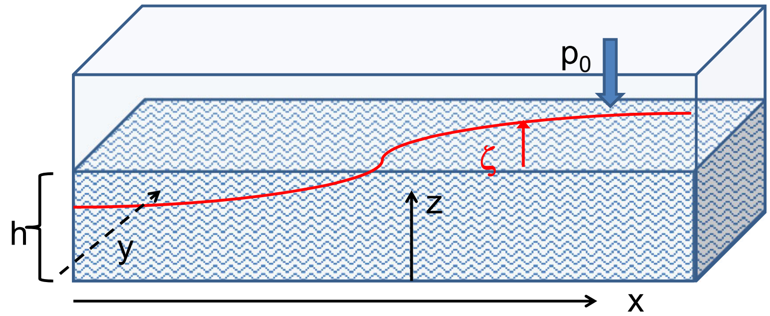
At surface: $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} = -\left.\frac{\partial \Phi(x,y,z,t)}{\partial z}\right|_{z=h+\zeta} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$$

Note that $v_z(x, y, h+\zeta, t) = \frac{d\zeta}{dt}$



From wikipedia



Further simplifications; assume trivial y - dependence

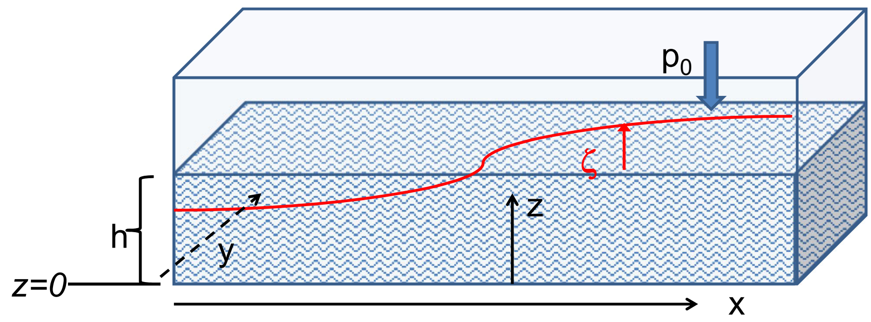
$\Phi = \Phi(x, z, t) \qquad \zeta = \zeta(x, t)$

Within fluid: $\quad 0 \le z \le h + \zeta$

At surface: $\quad v_z(x, z = h + \zeta, t) = -\frac{\partial \Phi}{\partial z} = \frac{d\zeta}{dt}$

Specializing to motion along the x direction and surface direction in the z direction.

Non-linear effects in surface waves:



Dominant non-linear effects $\Rightarrow$ soliton solutions

$$\zeta(x,t) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right) \qquad \eta_0 = \text{constant}$$

$$\text{where } c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)$$

Answer that we will find for the soliton solution.

Detailed analysis of non-linear surface waves

[Note that these derivations follow Alexander L. Fetter and John Dirk Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw Hill, 1980), Chapt. 10.]

We assume that we have an incompressible fluid: $\rho = \text{constant}$

Velocity potential: $\Phi(x,z,t)$; $\mathbf{v}(x,z,t) = -\nabla\Phi(x,z,t)$

The surface of the fluid is described by $z=h+\zeta(x,t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$.

Summary of assumptions for our analysis.

Defining equations for Φ*(x,z,t)* and ζ*(x,t)*

where $0 \leq z \leq h + \zeta(x,t)$

Continuity equation:

$$\nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Bernoulli equation (assuming irrotational flow) and gravitation potential energy

$$-\frac{\partial \Phi(x,z,t)}{\partial t} + \frac{1}{2}\left[\underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial x}\right)^2}_{v_x^2} + \underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial z}\right)^2}_{v_z^2}\right] + g(z-h) = 0.$$

Working through the equations within water.

Boundary conditions on functions –

Zero velocity at bottom of tank:

$$\frac{\partial \Phi(x,0,t)}{\partial z} = 0.$$

Consistent vertical velocity at water surface

$$v_z(x,z,t)\Big|_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v}\cdot\nabla\zeta + \frac{\partial \zeta}{\partial t}$$

$$= v_x\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t}$$

$$\Rightarrow -\frac{\partial \Phi(x,z,t)}{\partial z} + \frac{\partial \Phi(x,z,t)}{\partial x}\frac{\partial \zeta(x,t)}{\partial x} - \frac{\partial \zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta} = 0$$

11/15/2021 PHY 711 Fall 2021 -- Lecture 34 16

Boundary effects at the bottom of the channel and at the surface.

Analysis assuming water height *z* is small relative to variations in the direction of wave motion *(x)*
Taylor's expansion about *z* = 0:

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

Note that the zero vertical velocity at the bottom suggest that to a good approximation, that all odd derivatives $\frac{\partial^n \Phi}{\partial z^n}(x,0,t)$ vanish from the Taylor expansion. In addition, the Laplace equation allows us to convert all even derivatives with respect to *z* to derivatives with respect to *x*.

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

$$\Rightarrow \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Modified Taylor's expansion: $\Phi(x,z,t) \approx \Phi(x,0,t) - \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial x^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial x^4}(x,0,t)\cdots$

Here we start a number of steps to analyze the leading terms in the linearities. In this case we perform a Taylor's expansion about z=0 at the bottom of the channel.

Some details --

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

At bottom: $z = 0$ and $v_z(x,0,t) = 0 \quad \Rightarrow \frac{\partial \Phi}{\partial z}(x,0,t) = 0$

Further, you textbook argues that using Fourier transforms,

$$\Phi(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \cosh(kz) e^{ikx} \tilde{f}(k,t) \approx \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left(1 + \frac{(kz)^2}{2!} + \frac{(kz)^4}{4!} + \ldots\right) e^{ikx} \tilde{f}(k,t)$$

$$\Phi(x,z,t) \approx \Phi(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

Check linearized equations and their solutions:

Bernoulli equations --

Bernoulli equation evaluated at $z = h + \zeta(x,t)$

$$-\frac{\partial\Phi(x,h,t)}{\partial t} + g\zeta(x,t) = 0$$

Consistent vertical velocity at $z = h + \zeta(x,t)$

$$-\frac{\partial\Phi(x,z,t)}{\partial z} - \frac{\partial\zeta(x,t)}{\partial t}\bigg|_{z=h+\zeta} = 0$$

Using Taylor's expansion results to lowest order

$$-\frac{\partial\Phi(x,h,t)}{\partial z} \approx h\frac{\partial^2\Phi(x,0,t)}{\partial x^2} = -\frac{\partial\zeta(x,t)}{\partial t} \qquad -\frac{\partial\Phi(x,h,t)}{\partial t} \approx -\frac{\partial\Phi(x,0,t)}{\partial t} = -g\zeta(x,t)$$

Decoupled equations: $\dfrac{\partial^2\Phi(x,0,t)}{\partial t^2} = gh\dfrac{\partial^2\Phi(x,0,t)}{\partial x^2}.$

➔linear wave equation with $c^2=gh$

11/15/2021 PHY 711 Fall 2021 -- Lecture 34 19

Checking lowest order (linear) term.

Analysis of non-linear equations --

Bernoulli equation evaluated at surface:

$$-\frac{\partial\Phi(x,z,t)}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi(x,z,t)}{\partial x}\right)^2+\left(\frac{\partial\Phi(x,z,t)}{\partial z}\right)^2\right]\Bigg|_{z=h+\zeta}+g\zeta(x,t)=0.$$

Consistency of surface velocity

$$-\frac{\partial\Phi(x,z,t)}{\partial z}+\frac{\partial\Phi(x,z,t)}{\partial x}\frac{\partial\zeta(x,t)}{\partial x}-\frac{\partial\zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta}=0$$

Representation of velocity potential from Taylor′s expansion:

$$\Phi(x,z,t)\approx\Phi(x,0,t)-\frac{z^2}{2}\frac{\partial^2\Phi}{\partial x^2}(x,0,t)+\frac{z^4}{4!}\frac{\partial^4\Phi}{\partial x^4}(x,0,t)\cdots$$

Back to non-linear equations using Taylor’s expansion.

Analysis of non-linear equations -- keeping the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms. Let $\phi(x,t) \equiv \Phi(x,0,t)$

Approximate form of Bernoulli equation evaluated at surface: $z = h + \zeta$

$$-\frac{\partial \phi}{\partial t} + \frac{(h+\zeta)^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left((h+\zeta)\frac{\partial^2 \phi}{\partial x^2}\right)^2\right] + g\zeta = 0$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g\zeta = 0.$$

Approximate form of surface velocity expression :

$$\Rightarrow \frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

These equations represent non-linear coupling of $\phi(x,t)$ and $\zeta(x,t)$.

Systematic keeping/limiting terms in non-linearity and in high order derivatives. The highlighted equations are the coupled equations that we will analyze.

Coupled equations: $-\frac{\partial \phi}{\partial t}+\frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2}+\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2+g\zeta=0.$

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right)-\frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4}-\frac{\partial \zeta}{\partial t}=0.$$

Traveling wave solutions with new notation:

$u \equiv x-ct \qquad \phi(x,t)\equiv \chi(u) \qquad$ and $\qquad \zeta(x,t)\equiv \eta(u)$

Note that the wave "speed" c will be consistently determined

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$

Decoupling the equations.

Integrating and re-arranging coupled equations

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\chi'=-\frac{g}{c}\eta+\frac{h^2}{2}\chi'''-\frac{1}{2c}(\chi')^2\approx-\frac{g}{c}\eta-\frac{h^2g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$

$$\Rightarrow (h+\eta)\frac{d\chi(u)}{du}-\frac{h^3}{6}\frac{d^3\chi(u)}{du^3}+c\eta(u)=0$$

Now we can express $\frac{d\chi(u)}{du}=\chi'$ in terms of η:

$$\chi'\approx-\frac{g}{c}\eta-\frac{h^2g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

Analysis continued.

Integrating and re-arranging coupled equations – continued --
Expressing modified surface velocity equation in terms of $\eta(u)$:

$$(h+\eta)\left(-\frac{g}{c}\eta-\frac{h^2g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2\right)+\frac{h^3g}{6c}\eta''+c\eta=0$$

$$\Rightarrow\left(1-\frac{gh}{c^2}\right)\eta-\frac{gh^3}{3c^2}\eta''-\frac{g}{c^2}\left(1+\frac{gh}{2c^2}\right)\eta^2=0$$

$$\Rightarrow\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Note: $c^2=gh+\ldots$

More derivations.

Solution of the famous Korteweg-de Vries equation

Modified surface amplitude equation in terms of η

$$\Rightarrow \left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Soliton solution

$$\zeta(x,t)=\eta(x-ct)=\eta_0 \ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\ \frac{x-ct}{2h}\right)$$

$$c=\sqrt{\frac{gh}{1-\eta_0/h}}\approx\sqrt{gh}\left(1+\frac{\eta_0}{2h}\right) \quad \text{where } \eta_0 \text{ is a constant}$$

Finally arriving at the famous equation and the famous soliton solution.

Steps to solution

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Let $1-\frac{hg}{c^2}\equiv\frac{\eta_0}{h}$ $\quad\Rightarrow\frac{\eta_0}{h}\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$

Multiply equation by $\eta'(u)$ $\quad\Rightarrow\frac{d}{du}\left(\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)\right)=0$

Integrate wrt u and assume solution vanishes for $u\rightarrow\infty$

$$\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)=0$$

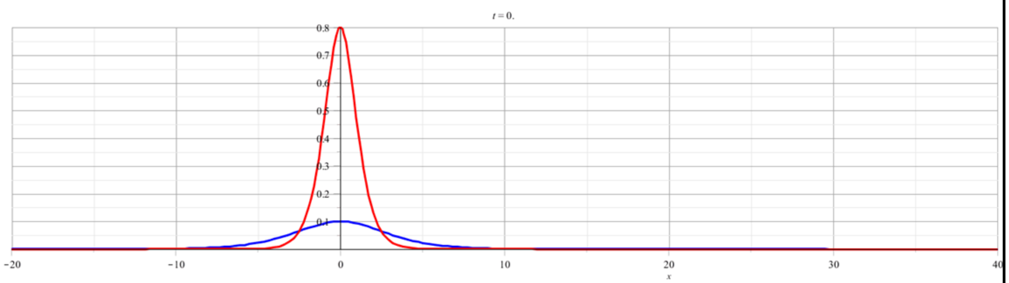
$$\eta'^2(u)=\frac{3}{h^3}\eta^2(u)\left(\eta_0-\eta(u)\right)$$

$$\frac{d\eta}{\eta\left(\eta_0-\eta\right)^{1/2}}=\sqrt{\frac{3}{h^3}}du\qquad\Rightarrow\eta(u)=\frac{\eta_0}{\cosh^2\left(\sqrt{\frac{3\eta_0}{4h^3}}u\right)}$$

More details.

$$\zeta(x,t)=\eta(x-ct)=\eta_0\ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\ \frac{x-ct}{2h}\right)$$

Two soliton solutions with different amplitudes --



Visualization

Relationship to "standard" form of Korteweg-de Vries equation

New variables:

$$\beta = 2\eta_0, \qquad \bar{x} = \sqrt{\frac{3}{2h}}\frac{x}{h}, \qquad \text{and} \qquad \bar{t} = \sqrt{\frac{3}{2h}}\frac{ct}{2\eta_0 h}.$$

Standard Korteweg-de Vries equation

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta\frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

Soliton solution:

$$\eta(\bar{x},\bar{t}) = \frac{\beta}{2}\operatorname{sech}^2\left[\frac{\sqrt{\beta}}{2}(\bar{x} - \beta\bar{t})\right].$$

Some notational manipulations.

More details

Modified surface amplitude equation in terms of η :

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Some identities: $\frac{\eta_0}{h}=1-\frac{gh}{c^2}$; $\frac{\partial\eta}{\partial t}=-c\frac{d\eta}{du}$; $\frac{\partial\eta}{\partial x}=\frac{d\eta}{du}$.

Derivative of surface amplitude equation:

$$\frac{\eta_0}{h}\eta'-\frac{h^2}{3}\eta'''-\frac{3}{h}\eta\eta'=0.$$

Expression in terms of x and t:

$$-\frac{\eta_0}{ch}\frac{\partial\eta}{\partial t}-\frac{h^2}{3}\frac{\partial^3\eta}{\partial x^3}-\frac{3}{h}\eta\frac{\partial\eta}{\partial x}=0.$$

Expression in terms of $\overline{x}$ and $\overline{t}$:

$$\frac{\partial\eta}{\partial\overline{t}}+6\eta\frac{\partial\eta}{\partial\overline{x}}+\frac{\partial^3\eta}{\partial\overline{x}^3}=0.$$

More details.

Summary

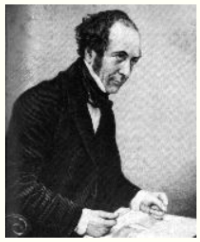
Soliton solution

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\ \frac{x-ct}{2h}\right)$$

$$c = \sqrt{\frac{gh}{1-\eta_0 / h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right) \quad \text{where } \eta_0 \text{ is a constant}$$

Summary.

John Scott Russell and the solitary wave



Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. As he described it in his "Report on Waves": (Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).

https://www.macs.hw.ac.uk/~chris/scott_russell.html

``I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".
(Cet passage en francais)

This event took place on the Union Canal at Hermiston, very close to the Riccarton campus of Heriot-Watt University, Edinburgh.

First observer of the soliton phenomenon.

Photo of canal soliton http://www.ma.hw.ac.uk/solitons/

Historic realization of the soliton wave in a channel.