PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 36: Chap. 12 in F & W

Viscous fluids

1. Viscous stress tensor
2. Navier-Stokes equation
3. Example for incompressible fluid – Stokes "law"

11/19/2021 PHY 711 Fall 2021 -- Lecture 31 1

In this lecture, we will consider some effects of viscosity on the motion of fluids, following Chapter 12 of your textbook.

32	Wed, 11/10/2021	Chap. 9	Non linear effects in sound waves and shocks
33	Fri, 11/12/2021	Chap. 10	Surface waves in fluids
34	Mon, 11/15/2021	Chap. 10	Surface waves in fluids; soliton solutions
35	Wed, 11/17/2021	Chap. 11	Heat conduction
36	Fri, 11/19/2021	Chap. 12	Viscous effects on hydrodynamics
	Mon, 11/22/2021		Presentations
	Wed, 11/24/2021		Thanksgiving
	Fri, 11/26/2021		Thanksgiving
37	Mon, 11/29/2021	Chap. 13	Elasticity
38	Wed, 12/01/2021	Chap. 1-13	Review
39	Fri, 12/03/2021	Chap. 1-13	Review

Schedule.

Schedule for Monday, November 22, 2021

Time	Name	Topic
10:00-10:20	Owen	Molecular dynamics-theory comparison for spring and scattering systems
10:20-10:40	Manikata	
10:40-11:00	Wells	Eigenvector Continuation of some undecided system
11:00-11:20	Can	Euler Angle in EPR spectra
11:20-11:40	Ramesh	Nature of orbits in conservative central force fields

Schedule for Monday --

Equations for motion of non-viscous fluid

Newton-Euler equation of motion:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} = \rho \mathbf{f}_{applied} - \nabla p$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) = 0 \qquad \Rightarrow \; \mathbf{v} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) \right) = 0$$

Add the two equations:

$$\underbrace{\rho \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \rho}{\partial t} \mathbf{v}}_{\frac{\partial \left(\rho \mathbf{v} \right)}{\partial t}} + \underbrace{\rho \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} + \mathbf{v} \nabla \cdot \left(\rho \mathbf{v} \right)}_{\sum_{j=1}^{3} \frac{\partial \left(\rho v_j \mathbf{v} \right)}{\partial x_j}} = \rho \mathbf{f}_{applied} - \nabla p$$

Reviewing the fluid equations that we have discussed previously, combining Newton’s equations with the continuity equation to find a new convenient form.

Equations for motion of non-viscous fluid -- continued

Newton-Euler equation in terms of fluid momentum:

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j\mathbf{v})}{\partial x_j}=\rho\mathbf{f}_{applied}-\nabla p$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j\mathbf{v})}{\partial x_j}+\nabla p=\rho\mathbf{f}_{applied}$$

Fluid momentum: $\rho\mathbf{v}$

Stress tensor: $T_{ij}\equiv\rho v_i v_j+p\delta_{ij}$

i^{th} component of Newton-Euler equation:

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{j=1}^{3}\frac{\partial T_{ij}}{\partial x_j}=\rho f_i$$

Here we recognize terms that have the units of force/area and can be described as a stress tensor Tij.

Now consider the effects of viscosity

In terms of stress tensor:

$$T_{ij} = T_{ij}^{\text{ideal}} + T_{ij}^{\text{viscous}}$$

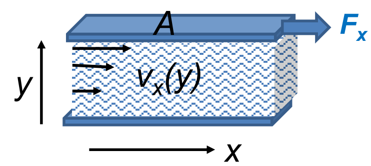
$$T_{ij}^{\text{ideal}} = \rho v_i v_j + p\delta_{ij} = T_{ji}^{\text{ideal}}$$

As an example of a viscous effect, consider --

Newton's "law" of viscosity

$$\frac{F_x}{A} = \eta \frac{\partial v_x}{\partial y}$$

material dependent parameter



The next step is to imagine that the additional effects of viscosity should/can be represented as a viscous stress tensor. The example of sheer force suggests that the viscous stress tensor involves derivatives of the velocity of the fluid.

Effects of viscosity

Argue that viscosity is due to shear forces in a fluid of the form:

$$\frac{F_{drag}}{A} = \eta \frac{\partial v_x}{\partial y}$$

Formulate viscosity stress tensor with traceless and diagonal terms:

$$T_{kl}^{\text{viscous}} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

viscosity

bulk viscosity

Total stress tensor: $T_{kl} = T_{kl}^{\text{ideal}} + T_{kl}^{\text{viscous}}$

$$T_{kl}^{\text{ideal}} = \rho v_k v_l + p\delta_{kl}$$

$$T_{kl}^{\text{viscous}} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

Imagining the most general form of the viscous tensor, we consider all derivatives of all components of fluid velocity, separating out the terms with zero trace, with the remaining terms proportional to the divergence of the velocity and representing the “bulk” viscosity.

Effects of viscosity -- continued

Incorporating generalized stress tensor into Newton-Euler equations

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x_j}=\rho f_i$$

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{j=1}^{3}\frac{\partial\left(\rho v_i v_j\right)}{\partial x_j}=\rho f_i-\frac{\partial p}{\partial x_i}+\eta\sum_{j=1}^{3}\frac{\partial^2 v_i}{\partial x_j^2}+\left(\zeta+\frac{1}{3}\eta\right)\sum_{j=1}^{3}\frac{\partial^2 v_j}{\partial x_i\partial x_j}$$

Continuity equation

$$\frac{\partial\rho}{\partial t}+\sum_{j=1}^{3}\frac{\partial\left(\rho v_j\right)}{\partial x_j}=0$$

Vector form (Navier-Stokes equation)

$$\frac{\partial\mathbf{v}}{\partial t}+\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla\left(\nabla\cdot\mathbf{v}\right)$$

Continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\mathbf{v}\right)=0$$

Now we can write the fluid equations with the full stress tensor. The continuity equation still applies. The so-called Navier-Stokes equation summarizes the expected behavior of fluids in terms of the material dependent viscosity parameters eta and zera.

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Typical viscosities at 20^o C and 1 atm:

Fluid	η/ρ (m^2/s)	η (Pa s)
Water	1.00×10^{-6}	1×10^{-3}
Air	14.9×10^{-6}	0.018×10^{-3}
Ethyl alcohol	1.52×10^{-6}	1.2×10^{-3}
Glycerine	1183×10^{-6}	1490×10^{-3}

Here is a list of some typical values of the viscosity parameter eta.

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R*

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Note that $\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$

Incompressible fluid $\Rightarrow$ $\nabla\cdot\mathbf{v} = 0$

Steady flow $\Rightarrow$ $\frac{\partial \mathbf{v}}{\partial t} = 0$

Irrotational flow $\Rightarrow$ $\nabla\times\mathbf{v} = 0$

No applied force $\Rightarrow$ $\mathbf{f}=0$

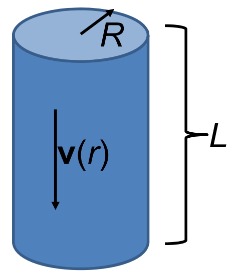
Neglect non-linear terms $\Rightarrow \nabla\left(v^2\right) = 0$

Example of a measurement of viscosity for irrotational flow.

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius R -- continued

Navier-Stokes equation becomes:

$$0 = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v}$$



Assume that $\mathbf{v}(\mathbf{r},t) = v_z(r)\hat{\mathbf{z}}$

$$\frac{\partial p}{\partial z} = \eta \nabla^2 v_z(r) \quad \text{(independent of } z\text{)}$$

Suppose that $\frac{\partial p}{\partial z} = -\frac{\Delta p}{L}$ (uniform pressure gradient)

$$\Rightarrow \nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

Continued analysis of simple viscous flowl

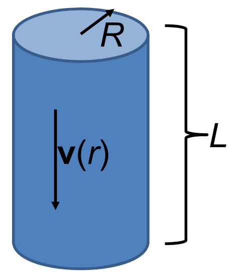
Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r}\frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + \mathrm{C}_2$$

$$\Rightarrow C_1 = 0 \qquad v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + \mathrm{C}_2$$



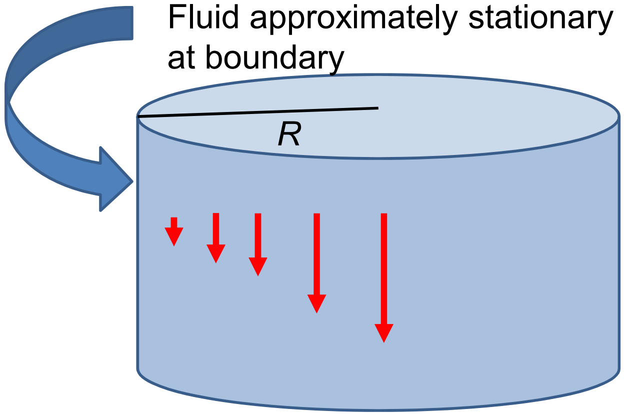
$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

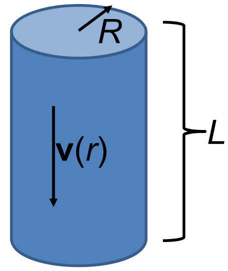
Solving for the velocity profile.

Comment on boundary condition

$$v_z(R) = 0$$

Fluid approximately stationary at boundary



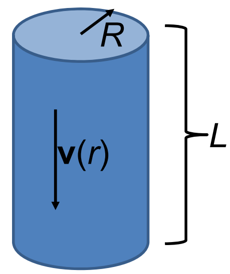


Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

Mass flow rate through the pipe:

$$\frac{dM}{dt} = 2\pi\rho\int_0^R r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}$$

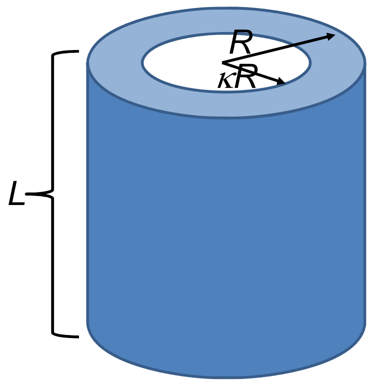


Poiseuille formula;

➔Method for measuring η

This analysis is useful for measuring eta.

Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius *R* and inner radius *κR*



$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r}\frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + \mathrm{C}_2$$

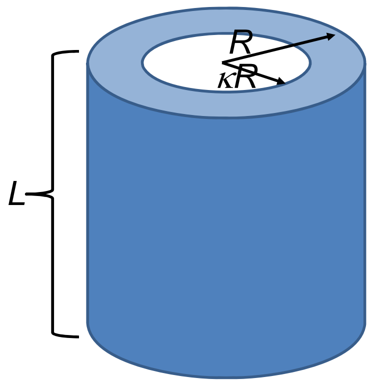
$$v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_1 \ln(R) + \mathrm{C}_2$$

$$v_z(\kappa R) = 0 = -\frac{\Delta p \kappa^2 R^2}{4\eta L} + C_1 \ln(\kappa R) + \mathrm{C}_2$$

11/19/2021 PHY 711 Fall 2021 -- Lecture 31 15

Another related system with a cylindrical shell.

Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius *R* and inner radius *κR* -- continued



Solving for C_1 and C_2 :

$$v_z(r) = \frac{\Delta p R^2}{4\eta L}\left(1-\left(\frac{r}{R}\right)^2 - \frac{1-\kappa^2}{\ln\kappa}\ln\left(\frac{r}{R}\right)\right)$$

Mass flow rate through the pipe:

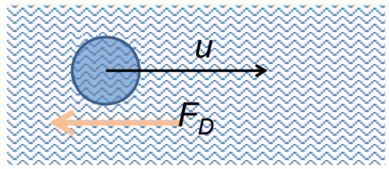
$$\frac{dM}{dt} = 2\pi\rho\int_{\kappa R}^{R} r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}\left(1-\kappa^4 + \frac{\left(1-\kappa^2\right)^2}{\ln\kappa}\right)$$

The final result again can be used to measure the viscosity.

More discussion of viscous effects in incompressible fluids

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η:

$$F_D = -\eta\left(6\pi R u\right)$$



Plan:
1. Consider the general effects of viscosity on fluid equations
2. Consider the solution to the linearized equations for the case of steady-state flow of a sphere of radius R
3. Infer the drag force needed to maintain the steady-state flow

11/19/2021 PHY 711 Fall 2021 -- Lecture 31 17

Changing to an analysis of viscous flow as a drag force.

Have you ever encountered Stokes law in previous contexts?

a. Milliken oil drop experiment
b. A sphere falling due to gravity in a viscous fluid, reaching a terminal velocity
c. Other?

Newton-Euler equation for incompressible fluid, modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \underbrace{\frac{\eta}{\rho}}_{\upsilon} \nabla^2 \mathbf{v}$$

υ Kinematic viscosity

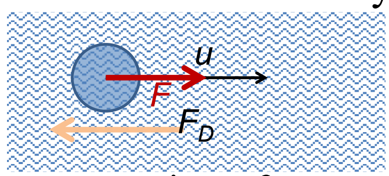
Typical kinematic viscosities at 20° C and 1 atm:

Fluid	ν (m^2/s)
Water	1.00×10^{-6}
Air	14.9×10^{-6}
Ethyl alcohol	1.52×10^{-6}
Glycerine	1183×10^{-6}

In this case, we will consider an incompressible fluid in which case eta/rho is the important parameter.

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η:

$$F_D = -\eta(6\pi R u)$$



Effects of drag force on motion of particle of mass m with constant force F:

$$F - 6\pi R\eta u = m\frac{du}{dt} \qquad \text{with } u(0) = 0$$

$$\Rightarrow u(t) = \frac{F}{6\pi R\eta}\left(1 - e^{-\frac{6\pi R\eta}{m}t}\right)$$

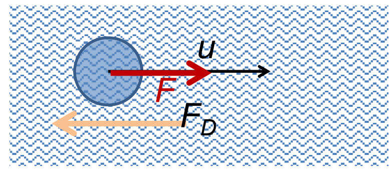
Before deriving Stokes law of viscous drag, it is interesting to recall its effects.

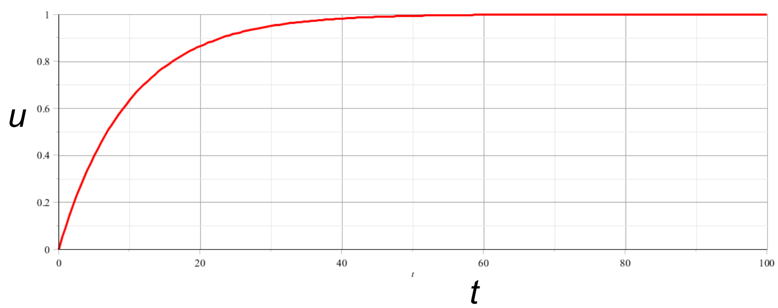
Effects of drag force on motion of

particle of mass m with constant force F :

$$F - 6\pi R\eta u = m\frac{du}{dt} \qquad \text{with } u(0) = 0$$

$$\Rightarrow u(t) = \frac{F}{6\pi R\eta}\left(1 - e^{-\frac{6\pi R\eta}{m}t}\right)$$



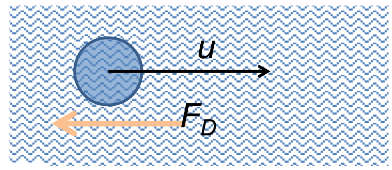


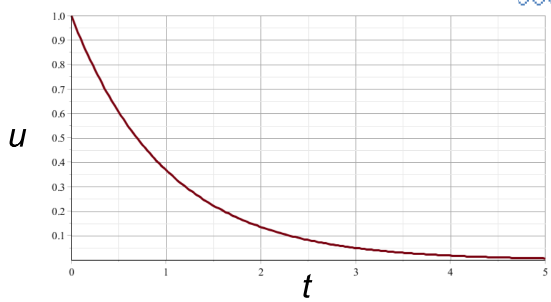
Objects moving in the presence of the Stokes viscous drag, tend to read a steady “terminal” velocity.

Effects of drag force on motion of particle of mass m with an initial velocity with $u(0) = U_0$ and no external force

$$-6\pi R\eta u = m\frac{du}{dt}$$

$$\Rightarrow u(t) = U_0 e^{-\frac{6\pi R\eta}{m}t}$$





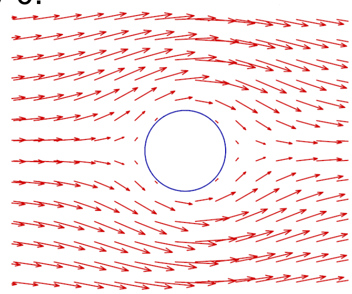
Or the velocity decays to zero.

Recall: PHY 711 -- Assignment #19 Nov. 01, 2021

Determine the form of the velocity potential for an incompressible fluid representing uniform velocity in the **z** direction at large distances from a spherical obstruction of radius *a*. Find the form of the velocity potential and the velocity field for all $r > a$. Assume that for $r = a$, the velocity in the radial direction is 0 but the velocity in the azimuthal direction is not necessarily 0.

$$\nabla^2\Phi = 0$$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$



In the present viscous case, we will assume that $\mathbf{v}(a)=0$.

In previous discussions without viscosity, the velocity near the sphere is not necessarily zero. How will this be affected in the presence of viscosity?

Newton-Euler equation for incompressible fluid,
modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \frac{\eta}{\rho}\nabla^2 \mathbf{v}$$

Continuity equation: $\nabla \cdot \mathbf{v} = 0$

Assume steady state: $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} = 0$

Assume non-linear effects small

Initially set $\mathbf{f}_{applied} = 0$;

$$\Rightarrow \nabla p = \eta \nabla^2 \mathbf{v}$$

Here we keep the dominant terms, finding a relationship between the pressure and the velocity.

$$\nabla p = \eta \nabla^2 \mathbf{v}$$

Take curl of both sides of equation:

$$\nabla \times (\nabla p) = 0 = \eta \nabla^2 (\nabla \times \mathbf{v})$$

Assume (with a little insight from Landau):

$$\mathbf{v} = \nabla \times (\nabla \times f(r)\mathbf{u}) + \mathbf{u}$$

where $f(r) \xrightarrow[r\to\infty]{} 0$

Note that:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

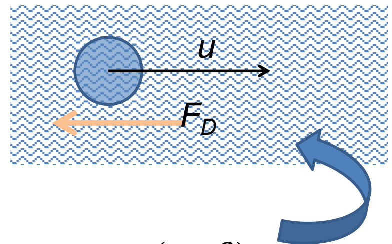
This analysis follows the treatment of Landau and Lifshitz.

Your question – why assume

Assume (with a little insight from Landau):

$$\mathbf{v} = \nabla \times \left(\nabla \times f(r)\mathbf{u} \right) + \mathbf{u}$$

where $f(r) \xrightarrow[r\to\infty]{} 0$



$$\mathbf{v}(r,\theta) \approx \mathbf{u}$$

Digression

Some comment on assumption: $\mathbf{v} = \nabla \times \left(\nabla \times f(r)\mathbf{u} \right) + \mathbf{u}$

$$\nabla \times \left(\nabla \times \mathbf{A} \right) = \nabla \left(\nabla \cdot \mathbf{A} \right) - \nabla^2 \mathbf{A}$$

Here $\mathbf{A} = f(r)\mathbf{u}$

$$\nabla \times \mathbf{v} = \nabla \times \left(\nabla \times \left(\nabla \times \mathbf{A} \right) \right) = -\nabla \times \left(\nabla^2 \mathbf{A} \right)$$

Also note: $\nabla p = \eta \nabla^2 \mathbf{v}$

$$\Rightarrow \nabla \times \nabla p = \nabla \times \eta \nabla^2 \mathbf{v} \qquad \text{or} \qquad \nabla^2 \left(\nabla \times \mathbf{v} \right) = 0$$

$$\nabla^2 \left(\nabla \times \nabla^2 \mathbf{A} \right) = \nabla^4 \left(\nabla \times \mathbf{A} \right) = 0$$

Deducing the form of the velocity

$$\mathbf{v} = \nabla \times \left(\nabla \times f(r)\mathbf{u} \right) + \mathbf{u}$$

$$\mathbf{u} = u\hat{\mathbf{z}}$$

$$\nabla \times \left(\nabla \times f(r)\hat{\mathbf{z}} \right) = \nabla \left(\nabla \cdot f(r)\hat{\mathbf{z}} \right) - \nabla^2 f(r)\hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \nabla^2 \left(\nabla \times \mathbf{v} \right) = 0$$

$$\nabla^4 \left(\nabla \times f(r)\hat{\mathbf{z}} \right) = 0 \quad \Rightarrow \nabla^4 \left(\nabla f(r) \times \hat{\mathbf{z}} \right) = 0 \quad \Rightarrow \nabla^4 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$v_r = u\cos\theta \left(1 - \frac{2}{r}\frac{df}{dr} \right) = u\cos\theta \left(1 - 4C_1 - \frac{2C_2}{r} + \frac{2C_4}{r^3} \right)$$

$$v_\theta = -u\sin\theta \left(1 - \frac{d^2 f}{dr^2} - \frac{1}{r}\frac{df}{dr} \right) = -u\sin\theta \left(1 - 4C_1 - \frac{C_2}{r} - \frac{C_4}{r^3} \right)$$

Here we find the most general form of the equation that satisfies the differential equation.

Some details:

$$\nabla^4 f(r) = 0 \qquad \Rightarrow \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \right)^2 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$\mathbf{v} = u\left(\nabla \times \left(\nabla \times f\left(r\right)\hat{\mathbf{z}}\right) + \hat{\mathbf{z}}\right)$$

$$= u\left(\nabla\left(\nabla \cdot \left(f\left(r\right)\hat{\mathbf{z}}\right)\right) - \nabla^2 f(r)\hat{\mathbf{z}} + \hat{\mathbf{z}}\right)$$

Note that: $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$

$$\mathbf{v} = u\left(\nabla\left(\frac{df}{dr}\cos\theta\right) - \left(\nabla^2\left(f(r)\right) - 1\right)\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right)\right)$$

Some details.

$$v_r = u\cos\theta\left(1-\frac{2}{r}\frac{df}{dr}\right) = u\cos\theta\left(1-4C_1-\frac{2C_2}{r}+\frac{2C_4}{r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1-\frac{d^2 f}{dr^2}-\frac{1}{r}\frac{df}{dr}\right) = -u\sin\theta\left(1-4C_1-\frac{C_2}{r}-\frac{C_4}{r^3}\right)$$

To satisfy $\mathbf{v}(r\to\infty)=\mathbf{u}$: $\quad \Rightarrow C_1 = 0$

To satisfy $\mathbf{v}(R)=0$ solve for C_2, C_4

$$v_r = u\cos\theta\left(1-\frac{3R}{2r}+\frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1-\frac{3R}{4r}-\frac{R^3}{4r^3}\right)$$

Assume that the velocity achieves steady flue far from the sphere and is zero on the sphere boundary.

$$v_r = u\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)$$

Determining pressure:

$$\nabla p = \eta\nabla^2\mathbf{v} = -\eta\nabla\left(u\cos\theta\left(\frac{3R}{2r^2}\right)\right)$$

$$\Rightarrow p(r) = p_0 - \eta u\cos\theta\left(\frac{3R}{2r^2}\right)$$

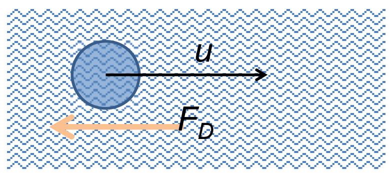
Finding all the constants and solving for the pressure .

$$p(r) = p_0 - \eta u \cos\theta \left(\frac{3R}{2r^2} \right)$$

Corresponds to:

$$F_D \cos\theta = \left(p(R) - p_0 \right) 4\pi R^2$$

$$\Rightarrow \mathrm{F}_D = -\eta u \left(6\pi R \right)$$



Deducing the drag force from the solution to the differential equation.