PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Plan for Lecture 38

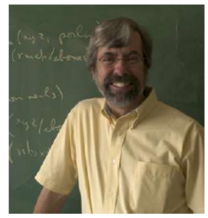
Review

1. **Mathematical methods**
2. **Classical mechanics concepts**

PHYSICS COLLOQUIUM

THURSDAY • DECEMBER 2, 2021

"Collapse of the Collapse: Physicists Return to Reality"



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We note the recent demise[1] of the collapse hypothesis, that an integral part of quantum mechanics is the "collapse" of the state of a system when measured. Using recent work of Anthony Rizzi, we show that the Ensemble Interpretation provides a simple and natural resolution to the problem of measurement[2,3] in quantum mechanics. Along the way, we give a fuller explanation of the Ensemble Interpretation[4] (in spite of the familiar-sounding name, this is almost entirely unknown), measurement theory in general (correcting many common misconceptions), the resolution of the Schrodinger Cat experiment, Wigner's Friend experiment, and the Extended Wigner's Friend experiment. At the end we encourage discussion of how to deal with the pedagogical situation, given that almost all current QM textbooks[5] are based on the (now defunct) collapse hypothesis.

[1] D. Frauchiger & R. Renner, "Quantum theory cannot consistently describe the use of itself", Nature Comm. [2018]

[2] A. Rizzi, "How the natural interpretation of QM avoids the recent no-go theorem" [Foundations of Physics, 2020]

4:00 pm - Olin 101*

*Link provided for those unable to attend in person.

Note: For additional information on the seminar or to obtain the video conference link, contact wfuphys@wfu.edu

	Mon, 11/22/2021		Presentations
	Wed, 11/24/2021		Thanksgiving
	Fri, 11/26/2021		Thanksgiving
37	Mon, 11/29/2021	Chap. 13	Elasticity
38	Wed, 12/01/2021	Chap. 1-13	Review
39	Fri, 12/03/2021	Chap. 1-13	Review

Comments on take-home final –

Similar to mid-term in form

Available: Mon. Dec. 6, 2021

Due before: Mon. Dec. 13, 2021 before 11 AM

(final grades are due 12/15/2021)

December 2021

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

Review of mathematical methods

Some useful identities for vectors and vector operators

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

Vector relations for spherical polar coordinates

$$\nabla\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]$$

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$$

$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$$

$$\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$$

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \cos\theta\cos\phi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \cos\theta\sin\phi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{\partial}{\partial\theta}$$

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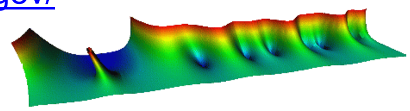


NIST Digital Library of Mathematical Functions

Project News

2018-09-15 *DLMF Update; Version 1.0.20*
2018-06-22 *DLMF Update; Version 1.0.19*
2018-06-22 *Philip J. Davis, A&S Author, dies at age 95*
2018-03-27 *DLMF Update; Version 1.0.18*
More news

Example – special functions

10 Bessel Functions
Bessel and Hankel Functions

10.1 Special Notation | 10.3 Graphics

§10.2 Definitions

Contents

§10.2(i) Bessel's Equation

10.2.1
$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2) w = 0.$$

This differential equation has a regular singularity at $z = 0$ with indices $\pm\nu$, and an irregular singularity at $z = \infty$ of rank 1; compare §§2.7(i) and 2.7(ii).

§10.2(ii) Standard Solutions

Bessel Function of the First Kind

10.2.2
$$J_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{(\tfrac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)}.$$

This solution of (10.2.1) is an analytic function of $z \in \mathbb{C}$, except for a branch point at $z = 0$ when ν is not an integer. The *principal branch* of $J_\nu(z)$ corresponds to the principal value of $(\tfrac{1}{2}z)^\nu$ (§4.2(iv)) and is analytic in the z-plane cut along the interval $(-\infty, 0]$.

Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\, e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re\big(f(z)\big) + i\Im\big(f(z)\big) \equiv u(x,y) + iv(x,y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i\frac{\partial u(z)}{\partial y}$$

Argue that $\frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y}$ and $\frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

➔A closed integral of an analytic function is zero.

However:

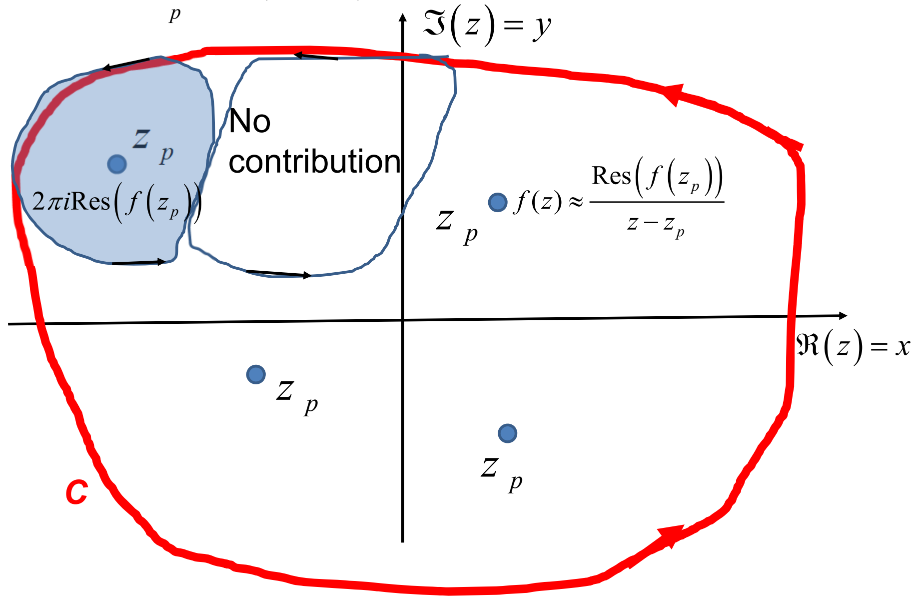
Behavior of $f(z) = \frac{1}{z^n}$ about the point $z = 0$:

For an integer n, consider

$$\oint \frac{1}{z^n}\, dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i \, d\phi}{\rho^n e^{in\phi}} = \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} i \, d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

Contour integration methods --

$$\oint_C f(z)dz = 2\pi i \sum_p \text{Res}\left(f(z_p)\right)$$



General formula for determining residue:

Suppose that in the neighborhood of z_p, $f(z) \approx \frac{g(z)}{\left(z-z_p\right)^m} \underset{z \to z_p}{\equiv} \frac{\text{Res}\left(f(z_p)\right)}{z-z_p}$

Since $g(z)$ is analytic near z_p, we can make a Taylor exansion about z_p :

$$g(z) \approx g(z_p) + \left(z-z_p\right)\frac{dg(z_p)}{dz} + \ldots + \frac{\left(z-z_p\right)^{m-1}}{(m-1)!}\frac{d^{m-1}g(z_p)}{dz^{m-1}} + \ldots.$$

$$\Rightarrow \text{Res}\left(f(z_p)\right) = \lim_{z \to z_p}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}\left(\left(z-z_p\right)^m f(z)\right)}{dz^{m-1}}\right\}$$

Fourier transforms --

Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega \, F(\omega) \, e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, f(t) \, e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \, f(t') \, e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt' \, f(t') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt' \, f(t') \, \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt\left(f(t)\right)^* f(t) = 2\pi\int_{-\infty}^{\infty} d\omega\ \left(F(\omega)\right)^* F(\omega)$$

Check: $$\int_{-\infty}^{\infty} dt\left(f(t)\right)^* f(t) = \int_{-\infty}^{\infty} dt\left(\left(\int_{-\infty}^{\infty} d\omega F(\omega)e^{i\omega t}\right)^* \int_{-\infty}^{\infty} d\omega' F(\omega')e^{i\omega' t}\right)$$

$$=\int_{-\infty}^{\infty} d\omega F^*(\omega)\int_{-\infty}^{\infty} d\omega' F(\omega')\int_{-\infty}^{\infty} dt\ e^{i(\omega'-\omega)t}$$

$$=\int_{-\infty}^{\infty} d\omega F^*(\omega)\int_{-\infty}^{\infty} d\omega' F(\omega')\ 2\pi\delta(\omega'-\omega)$$

$$=2\pi\int_{-\infty}^{\infty} d\omega F^*(\omega)F(\omega)$$

Doubly discrete Fourier Transforms

Doubly periodic functions

$$\omega \to \frac{2\pi\nu}{T} \qquad t \to \frac{\mu T}{2N+1} \qquad (N,\ \nu,\ \text{and}\ \mu\ \text{integers})$$

$$\tilde{f}_\mu = \frac{1}{2N+1}\sum_{\nu=-N}^{N} \tilde{F}_\nu e^{-i2\pi\nu\mu/(2N+1)}$$

$$\tilde{F}_\nu = \sum_{\mu=-N}^{N} \tilde{f}_\mu e^{i2\pi\nu\mu/(2N+1)}$$

➔Fast Fourier Transforms (FFT)

Notions of eigenvalues and eigenvectors

In the context of linear algebra --

Eigenvalue properties of matrices $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Hermitian matrix: $\mathbf{H}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha \qquad H_{ij} = H^*_{ji}$

Theorem for Hermitian matrices:

λ_α have real values and $\mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$

Unitary matrix: $\mathbf{U}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha \qquad \mathbf{U}^H\mathbf{U} = \mathbf{I}$

$|\lambda_\alpha| = 1$ and $\mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$

In the context of Sturm-Liouville differential equations --

Notions of eigenvalues and eigenvectors -- continued

Sturm Liouville differential equations, in terms of given functions $\tau(x)$, $v(x)$, and $\sigma(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Orthogonality of eigenfunctions: $\int_a^b \sigma(x) f_n(x) f_m(x) dx = \delta_{nm} N_n$,

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2 dx.$

Calculus of variation – a method to find a function (*y(x))* which optimizes a particular integral relationship.

For $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)_{x,y}\right]=0$$

Euler-Lagrange equation

Lagrangian in the presence of electromagnetic forces

Lagrangian: (using cartesian coordinates)

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) \equiv T - U$$

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\Phi\left(\mathbf{r}, t\right) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}\left(\mathbf{r}, t\right)$$

where $\mathbf{E}\left(\mathbf{r}, t\right) = -\nabla\Phi\left(\mathbf{r}, t\right) - \frac{1}{c}\frac{\partial \mathbf{A}\left(\mathbf{r}, t\right)}{\partial t}$ $\qquad \mathbf{B}\left(\mathbf{r}, t\right) = \nabla \times \mathbf{A}\left(\mathbf{r}, t\right)$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi\left(\mathbf{r}, t\right) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}\left(\mathbf{r}, t\right)$$

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$
2. Compute generalized momenta : $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function : $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Note that the equations of motion should yield equivalent trajectories for the Lagrangian and Hamiltonian formulations.

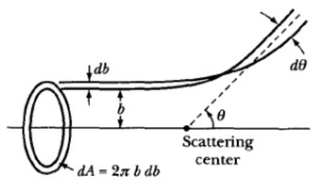
Mechanics topics

- Scattering theory
- Lagrangian mechanics
- Hamiltonian mechanics
- Liouville theorem
- Rigid body motion
- Normal modes of oscillation about equilibrium
- Wave motion
- Fluid mechanics (ideal or including viscosity; linear and nonlinear)
- Heat conduction
- Elasticity

Note: The following review slides are necessarily brief. Please refer to the original “Extra” lecture slides for details. Please email: natalie@wfu.edu with any corrections/suggestions

Scattering theory

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



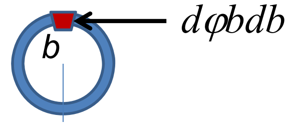


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in ϕ

Lagrangian mechanics

Given the Lagrangian function: $L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) \equiv T - U,$

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

Are those which minimize the action: $S = \int L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)dt$

Euler-Lagrange equations:

$$\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0 \quad \Rightarrow \text{for each } \sigma: \quad \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

For the case that there both mechanical and electromagnetic contributions in terms of electric and magnetic fields:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

$$L = T - U_{\text{mech}} - q\Phi(\mathbf{r},t) + \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function: $L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$
2. Compute generalized momenta: $p_\sigma \equiv \dfrac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression: $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function: $H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$
5. Analyze canonical equations of motion:

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Question – When can you bypass the 5 step derivation process and directly write the Hamiltonian of the system as

$$H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)=\sum_\sigma \frac{p_\sigma^2}{2m_\sigma}+V(\{q_\sigma\})$$

1. Only when Natalie Holzwarth is not looking
2. When you have a simple system that has no explicit velocity and/or time dependence
3. Usually

Important tool for analyzing Lagrangian and/or Hamiltonian systems -- finding constants of the motion

In Lagrangian formulation --

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right) \qquad \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} = 0 \qquad \Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} =$ (constant)

Additionally: $$\frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) = \frac{\partial L}{\partial t}$$

For $\frac{\partial L}{\partial t} = 0 \qquad \Rightarrow L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma = -E$ (constant)

Constants of the motion in the Hamiltonian formulation

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \qquad \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \sum_\sigma \left(-\dot{p}_\sigma \dot{q}_\sigma + \dot{q}_\sigma \dot{p}_\sigma \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{ constant } H \text{ if } \frac{\partial H}{\partial t} = 0$$

Question – Why use this fancy formalism when simple conservation of energy or momentum intuitively apply?

a. You should use your intuition whenever possible.
b. You should never trust your intuition.
c. The equations should be consistent with correct intuitive solutions and also reveal additional solutions (perhaps beyond intuition)

Liouville's Theorem (1838)

The density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion.

Denote the density of particles in phase space: $D = D\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

$$\frac{dD}{dt} = \sum_\sigma \left(\frac{\partial D}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial D}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial D}{\partial t}$$

According to Liouville's theorem: $\dfrac{dD}{dt} = 0$

Rigid body motion

Moment of inertia tensor :

$$\overleftrightarrow{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p\right) \qquad \text{(dyad notation)}$$

In a reference frame attached to the object, there are 3 moments of inertia and 3 distinct principal axes

Representation of rotational kinetic energy:

$$T\left(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}\right) = \frac{1}{2} I_1 \widetilde{\omega}_1^2 + \frac{1}{2} I_2 \widetilde{\omega}_2^2 + \frac{1}{2} I_3 \widetilde{\omega}_3^2$$

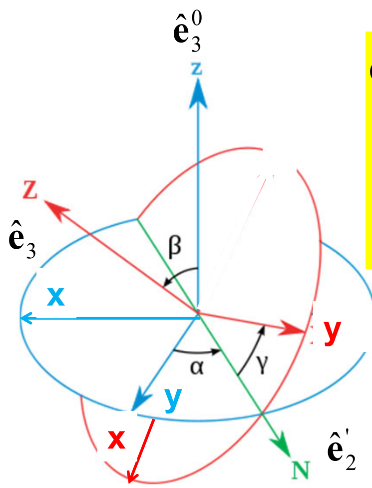
$$= \frac{1}{2} I_1 \left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right) + \dot{\beta}\sin\gamma\right]^2$$

$$+ \frac{1}{2} I_2 \left[\dot{\alpha}\left(\sin\beta\sin\gamma\right) + \dot{\beta}\cos\gamma\right]^2$$

$$+ \frac{1}{2} I_3 \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2$$

Euler's transformation between body fixed and inertial reference frames

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\, \hat{\mathbf{e}}_3^0 + \dot{\beta}\, \hat{\mathbf{e}}_2^{'} + \dot{\gamma}\, \hat{\mathbf{e}}_3$$

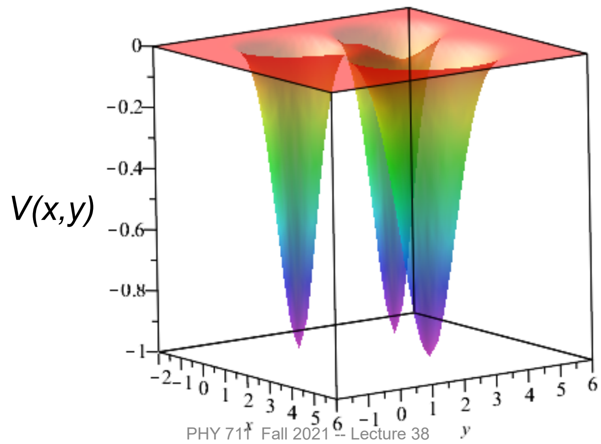


$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= \left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right) + \dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 \\ &+ \left[\dot{\alpha}\left(\sin\beta\sin\gamma\right) + \dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 \\ &+ \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\hat{\mathbf{e}}_3\end{aligned}$$

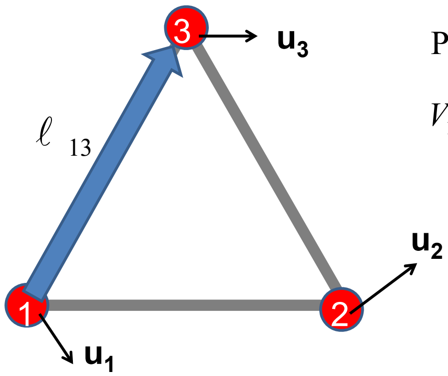
Normal modes of vibration -- potential in 2 and more dimensions

$$V(x,y) \approx V(x_{eq},y_{eq}) + \tfrac{1}{2}\left(x-x_{eq}\right)^2 \left.\frac{\partial^2 V}{\partial x^2}\right|_{x_{eq},y_{eq}}$$

$$+\tfrac{1}{2}\left(y-y_{eq}\right)^2 \left.\frac{\partial^2 V}{\partial y^2}\right|_{x_{eq},y_{eq}} + \left(x-x_{eq}\right)\left(y-y_{eq}\right)\left.\frac{\partial^2 V}{\partial x \partial y}\right|_{x_{eq},y_{eq}}$$



Example – normal modes of a system with the symmetry of an equilateral triangle -- continued



$$\boldsymbol{\ell}_{13} = |\boldsymbol{\ell}_{13}|\left(\frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{y}}\right)$$

Potential contribution for spring 13:

$$V_{13} = \frac{1}{2}k\left(|\boldsymbol{\ell}_{13} + \mathbf{u}_3 - \mathbf{u}_1| - |\boldsymbol{\ell}_{13}|\right)^2$$

$$\approx \frac{1}{2}k\left(\frac{\boldsymbol{\ell}_{13}\cdot(\mathbf{u}_3 - \mathbf{u}_1)}{|\boldsymbol{\ell}_{13}|}\right)^2$$

$$\approx \frac{1}{2}k\left(\frac{1}{2}(u_{x3} - u_{x1}) + \frac{\sqrt{3}}{2}(u_{y3} - u_{y1})\right)^2$$

Example – normal modes of a system with the symmetry of an equilateral triangle -- continued

Potential contributions: $V = V_{12} + V_{13} + V_{23}$

$$\approx \frac{1}{2}k\left(\frac{\ell_{12}\cdot(\mathbf{u}_2-\mathbf{u}_1)}{|\ell_{12}|}\right)^2 + \frac{1}{2}k\left(\frac{\ell_{13}\cdot(\mathbf{u}_3-\mathbf{u}_1)}{|\ell_{13}|}\right)^2$$

$$+\frac{1}{2}k\left(\frac{\ell_{23}\cdot(\mathbf{u}_3-\mathbf{u}_2)}{|\ell_{23}|}\right)^2$$

$$\approx \frac{1}{2}k\left(u_{x2}-u_{x1}\right)^2$$

$$+\frac{1}{2}k\left(\frac{1}{2}\left(u_{x3}-u_{x1}\right)+\frac{\sqrt{3}}{2}\left(u_{y3}-u_{y1}\right)\right)^2$$

$$+\frac{1}{2}k\left(\frac{1}{2}\left(u_{x2}-u_{x3}\right)-\frac{\sqrt{3}}{2}\left(u_{y2}-u_{y3}\right)\right)^2$$

Example – normal modes of a system with the symmetry of an equilateral triangle -- continued

$$\frac{k}{m}\begin{bmatrix} \frac{5}{4} & -1 & -\frac{1}{4} & \frac{1}{4}\sqrt{3} & 0 & -\frac{1}{4}\sqrt{3} \\ -1 & \frac{5}{4} & -\frac{1}{4} & 0 & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 \\ \frac{1}{4}\sqrt{3} & 0 & -\frac{1}{4}\sqrt{3} & \frac{3}{4} & 0 & -\frac{3}{4} \\ 0 & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 & \frac{3}{4} & -\frac{3}{4} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 & -\frac{3}{4} & -\frac{3}{4} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{y1} \\ u_{y2} \\ u_{y3} \end{bmatrix} = \omega^2 \begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{y1} \\ u_{y2} \\ u_{y3} \end{bmatrix}$$

Discrete particle interactions → continuous media →
The wave equation

Initial value solutions $\mu(x,t)$ to the wave equation;
attributed to D'Alembert:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \varphi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\left(\varphi(x-ct) + \varphi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Mechanical motion of fluids

Newton's equations for fluids

Use Euler formulation; following "particles" of fluid

Variables: Density $\rho(x,y,z,t)$

Pressure $p(x,y,z,t)$

Velocity $\mathbf{v}(x,y,z,t)$

Navier-Stokes equation

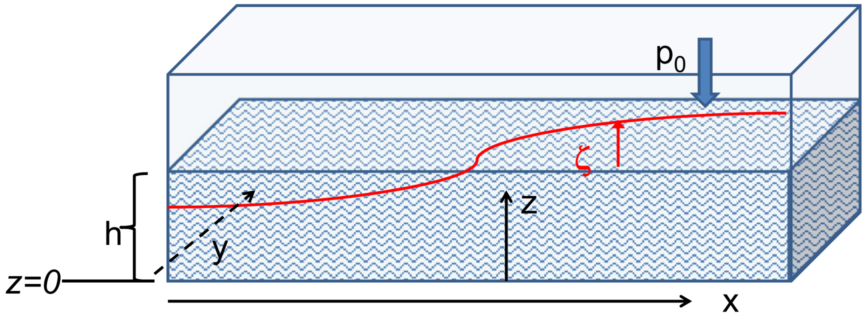
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \underbrace{\frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})}_{\text{Viscosity contributions}}$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Fluid mechanics of incompressible fluid plus surface

Non-linear effects in surface waves:



Dominant non-linear effects $\Rightarrow$ soliton solutions

$$\zeta(x,t) = \eta_0 \,\mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right) \qquad \eta_0 = \text{constant}$$

where $c = \sqrt{\dfrac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\dfrac{\eta_0}{2h}\right)$

Elements of elasticity theory -- Elastic stress tensor

$$-\sum_{j=1}^{3} T_{ij} dA_j \Rightarrow i^{th} \text{ component of force acting on surface } \hat{\mathbf{n}} dA \equiv d\mathbf{A}$$

Generalization of Hooke's law, $\mathrm{F}_x = -kx$:

Lame' coefficients : $$T_{ij} = -\lambda \delta_{ij} \nabla \cdot \mathbf{u} - \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = -\lambda \delta_{ij} \mathrm{Tr}(\epsilon) - 2\mu \epsilon_{ij}$$

material-dependent empirical parameters

Bulk modulus: $K = \lambda + \frac{2}{3}\mu$

Young's modulus: $E = \frac{9K\mu}{3K + \mu}$

Poisson ratio: $\sigma = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}$

Shear modulus: μ

Dynamical equations of elastic continuum

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \mu\nabla^2\mathbf{u} + \left(K + \frac{1}{3}\mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right) + \rho\mathbf{f}$$

In the absence of external forces, this reduces to two decoupled wave equations representing longitudinal and transverse modes:

$\mathbf{u}=\mathbf{u}_l + \mathbf{u}_t$

where $\nabla\times\mathbf{u}_l = 0$ and $\nabla\cdot\mathbf{u}_t = 0$

$$c_l = \left(\frac{K + \frac{4}{3}\mu}{\rho}\right)^{1/2} \quad \text{and} \quad c_t = \left(\frac{\mu}{\rho}\right)^{1/2}$$