# PHY 711 Classical Mechanics and Mathematical Methods
# 10-10:50 AM MWF in Olin 103

## Lecture notes for Lecture 4

## Scattering analysis in the center of mass reference frame – Chap 1 F&W

1. Review of scattering ideas.
2. In center of mass frame, analytical evaluation of the differential scattering cross section in general and for Rutherford scattering.

# PHY 711 Classical Mechanics and Mathematical Methods

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| **MWF 10 AM-10:50 AM** | **OPL 103** | **http://www.wfu.edu/~natalie/f21phy711/** |

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| **Instructor:** Natalie Holzwarth | **Office:**300 OPL | **e-mail:**natalie@wfu.edu |

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## Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

| | Date | F&W Reading | Topic | Assignment | Due |
|---|---|---|---|---|---|
| **1** | Mon, 8/23/2021 | Chap. 1 | Introduction | #1 | 8/27/2021 |
| **2** | Wed, 8/25/2021 | Chap. 1 | Scattering theory | #2 | 8/30/2021 |
| **3** | Fri, 8/27/2021 | Chap. 1 | Scattering theory | | |
| **4** | Mon, 8/30/2021 | Chap. 1 | Scattering theory | #3 | 9/01/2021 |
| **5** | Wed, 9/01/2021 | Chap. 1 | Summary of scattering theory | | |
| **6** | Fri, 9/03/2021 | Chap. 2 | Non-inertial coordinate systems | | |

# PHY 711 -- Assignment #3

Aug. 30, 2021

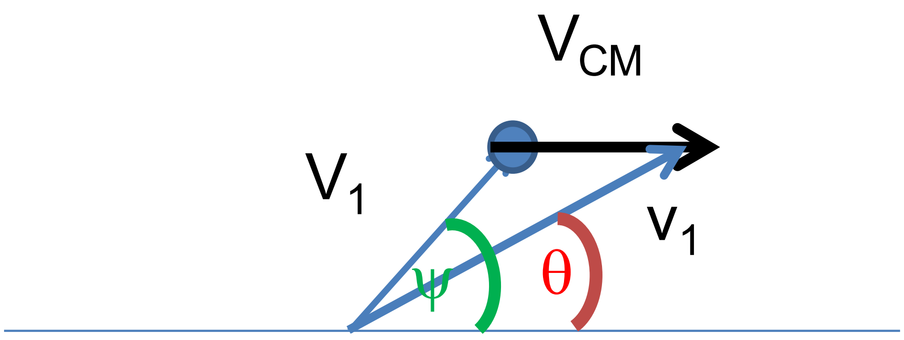
Finish reading Chapter 1 in **Fetter & Walecka**.

1. Work Problem #1.16 at the end of Chapter 1 in **Fetter and Walecka**. Note that you might want to use the equation in FW #1.15 or the equivalent equation derivated in class.

# Your questions –
## From Owen –

1. We have seen the parabolic trajectory of a light particle being deflected by a stationary heavy particle. If the second particle were lighter and allowed to move, how would it alter the trajectory of the first particle? Would it no longer be an ideal "conic section"?

Comment: Relationship between center of mass and laboratory frames of reference for the scattering particle 1. Here $V_1$ represents particle in center of mass frame and $v_1$ represents particle in lab frame. Both $V_1$ and $v_1$ change in time but $V_{CM}$ remains constant.



# Reviewing – scattering in the laboratory

Figure from F&W:

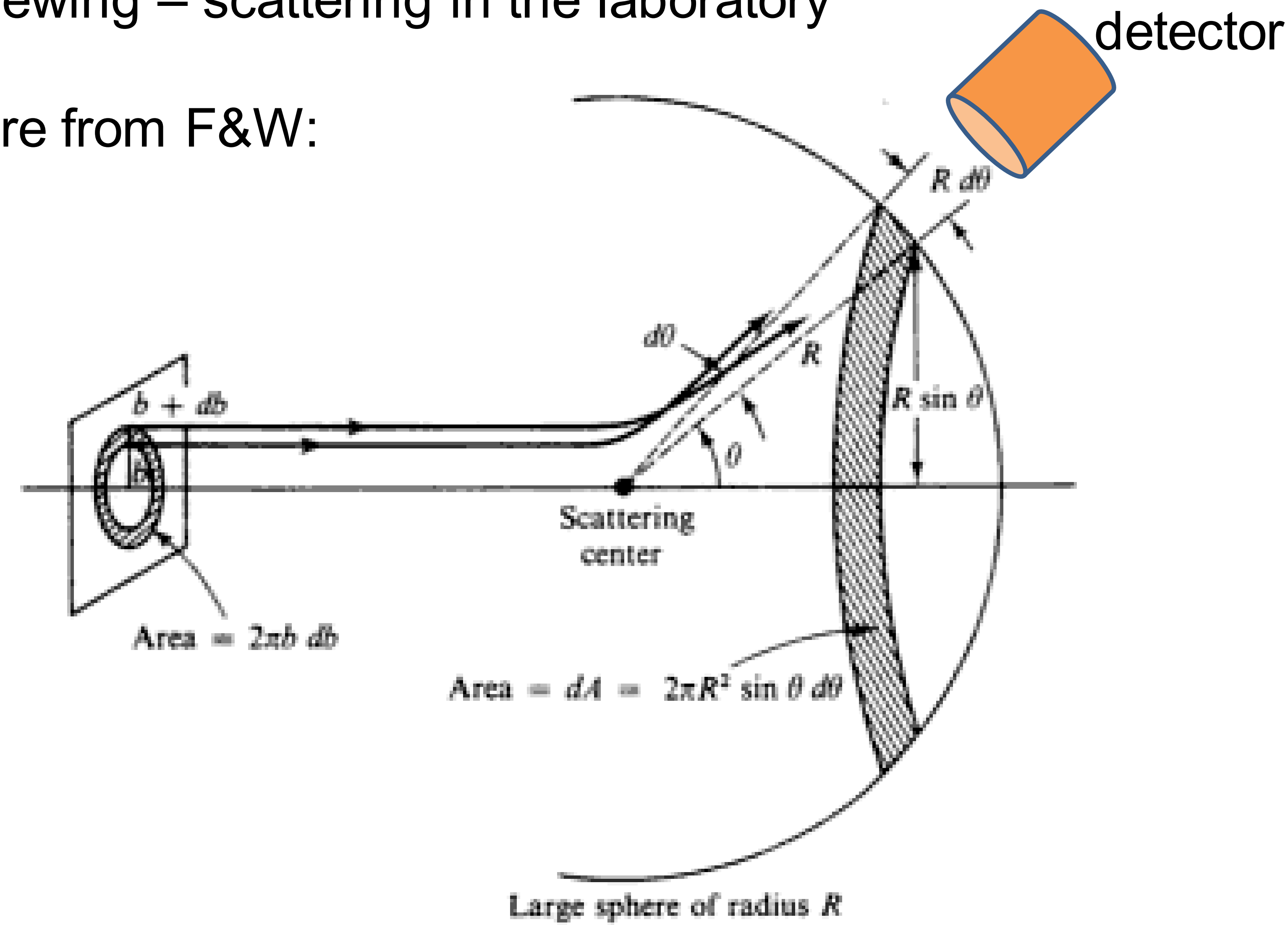
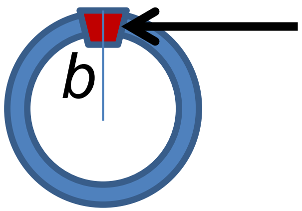



**Figure 5.5** The scattering problem and relation of cross section to impact parameter.

Differential cross section

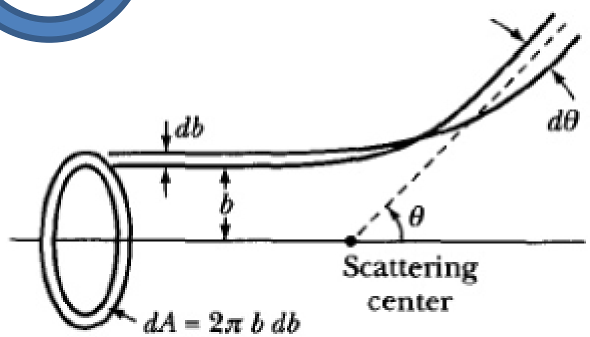
$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector at angle } \theta$$




$$d\sigma = d\varphi b db$$

$$d\Omega = d\varphi \sin\theta d\theta$$




$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

**Note that the same formula applies to the center of mass analysis.**

Total energy of system:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

constants

vary in time

For scattering analysis only need to know trajectory **before** and **after** the collision.

Some details --

Relationship between center of mass and laboratory frames of reference. At and time *t*, the following relationships apply --

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = (m_1 + m_2)\mathbf{R}_{CM}$$

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = (m_1 + m_2)\dot{\mathbf{R}}_{CM} = (m_1 + m_2)\mathbf{V}_{CM}$$

Note that $\dot{\mathbf{R}}_{CM} \equiv \frac{d\mathbf{R}_{CM}}{dt} \equiv \mathbf{V}_{CM}$

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu|\mathbf{v}_1 - \mathbf{v}_2|^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

where: $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$

## More details

Total energy of system:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r) \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$E = \quad E_{\text{Center of mass}} \quad + \quad E_{rel}$$

Recall that $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Focus on relative motion: $E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$

Since **r**(t) represents motion in a plane, we will analyze the system in that plane and use polar coordinates.

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$$

$$x(t) = r(t)\cos(\chi(t))$$

$$y(t) = r(t)\sin(\chi(t))$$

Note that $\left|\dot{\mathbf{r}}(t)\right|^2 = \dot{x}^2(t) + \dot{y}^2(t)$

$$= \dot{r}^2(t) + r^2(t)\dot{\chi}^2(t)$$

Clarification –

$$E_{total} = E_{\text{Center of mass}} + E_{rel}$$

Energy of the center mass motion

Energy within the center of mass reference frame

In the following slides $E_{rel}$ is written $E$

Also note that the relative angular momentum of the system is a constant

$$\ell = \mu r^2 \dot{\chi}$$

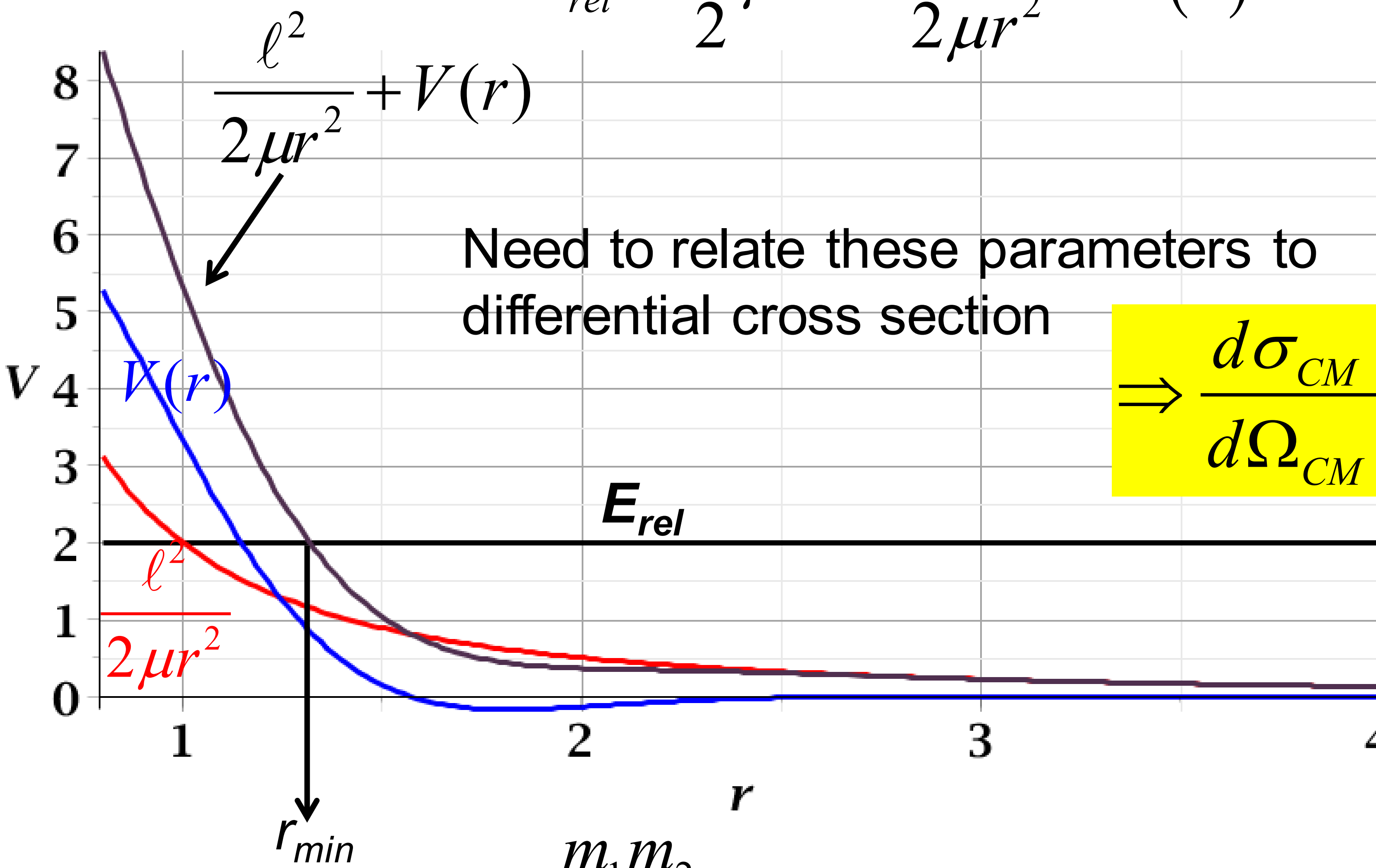
So that $\frac{1}{2}\mu \left|\dot{\mathbf{r}}(t)\right|^2 = \frac{1}{2}\mu\left(\dot{r}^2(t) + r^2(t)\dot{\chi}^2(t)\right)$

$$= \frac{1}{2}\mu\dot{r}^2(t) + \frac{\ell^2}{2\mu r^2}$$

➔ $E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$




Need to relate these parameters to differential cross section

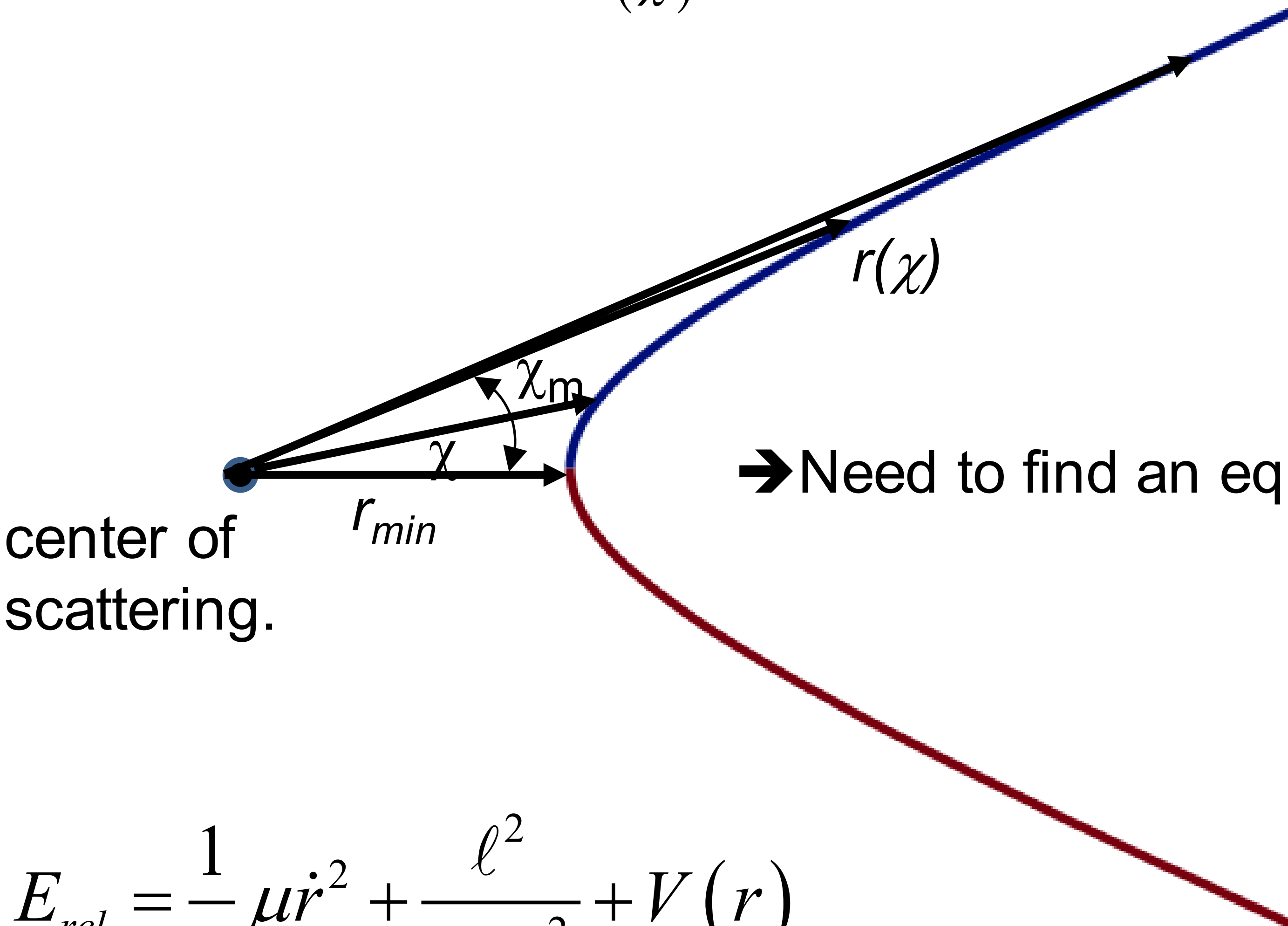
$$\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

$\ell$=angular momentum

Trajectory of relative vector in center of mass frame

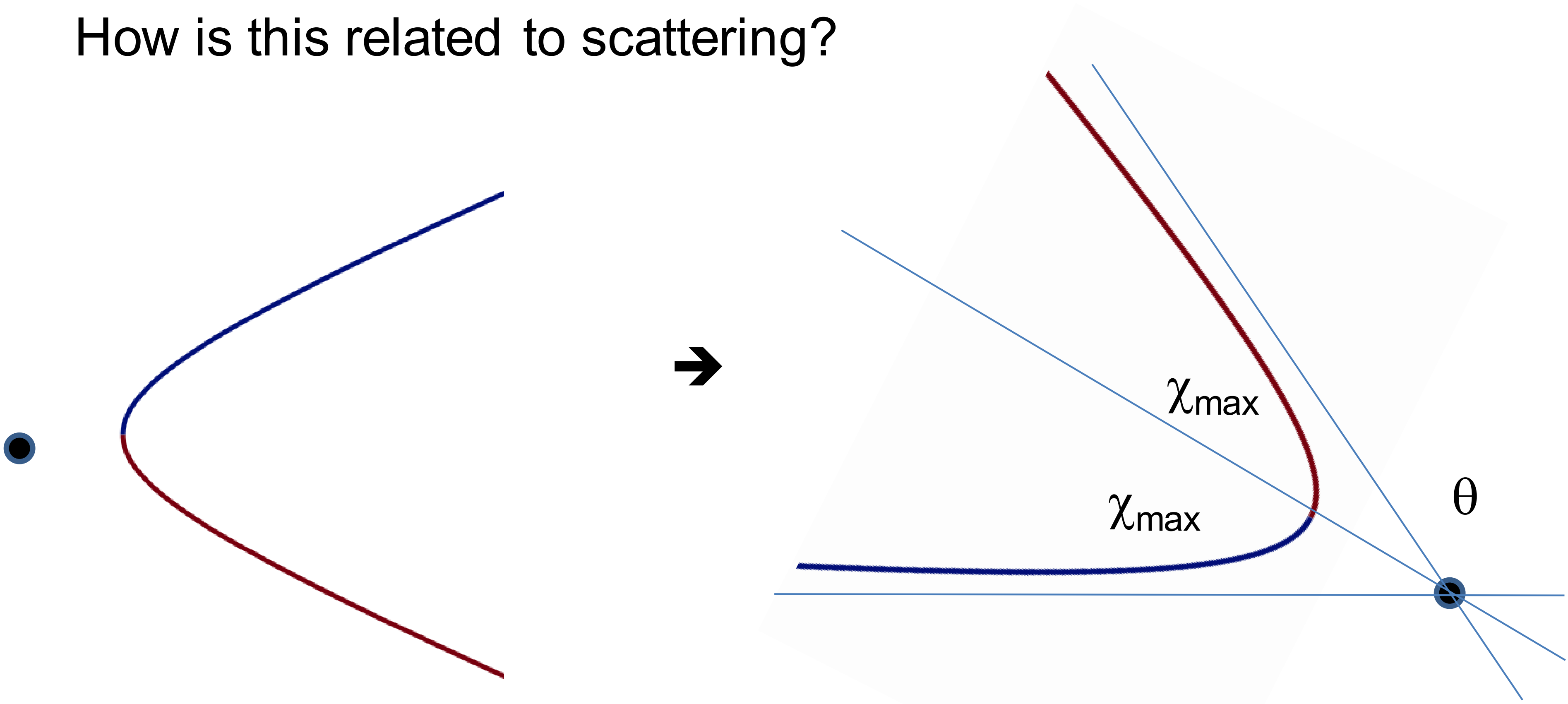
$r(\chi)$




➔Need to find an equation for $r(\chi)$

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

How is this related to scattering?




Note that here $\theta$ is used for the scattering angle

Note that we have use $\psi$ to denote the scattering angle in the center of mass frame, but your textbook uses $\theta$ (which we had used to denote the scattering angle in the lab frame). In this lecture our analysis is entirely in the center of mass frame and some of the equations use $\theta$ to denote the scattering angle.

Questions:

1. How can we find $r(\chi)$?
2. If we find $r(\chi)$, how can we relate $\chi$ to $\psi$?
   (Here $\psi$ is CM scattering angle.)
3. How can we find $b(\psi)$?

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{b}{\sin\psi}\left|\frac{db}{d\psi}\right|$$

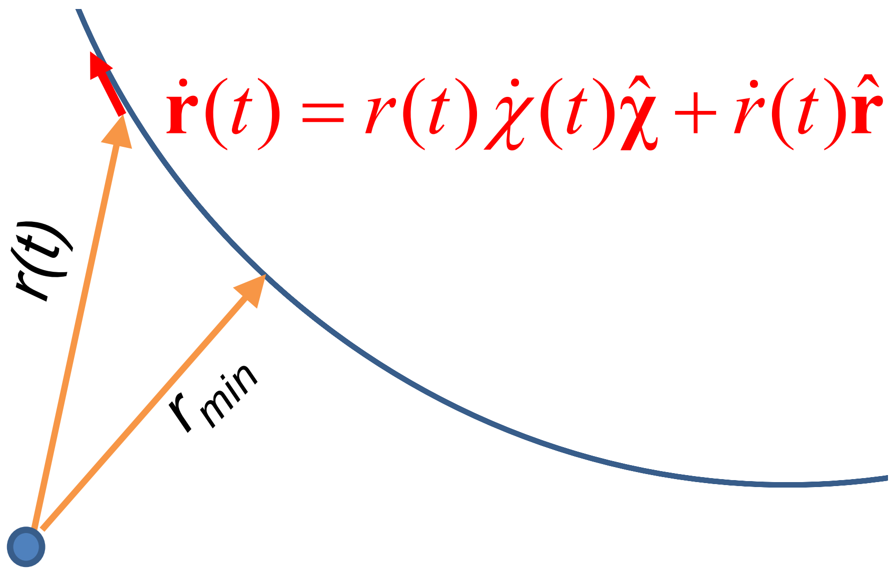
Evaluation of constants far from scattering center --

$$\ell = \mathbf{r} \times \left( \mu \dot{\mathbf{r}} \right) = r \mu r \frac{d\chi}{dt} = \mu r^2 \frac{d\chi}{dt}$$

$$\text{also:} \quad \ell = b \mu \dot{r}(t = -\infty)$$

$$E_{rel} = \frac{1}{2} \mu \left( \dot{r}(t = -\infty) \right)^2$$

$$\Rightarrow \quad \ell = b \sqrt{2 \mu E_{rel}}$$



Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\chi)$$

$$\frac{dr}{dt} = \frac{dr}{d\chi}\frac{d\chi}{dt} = \frac{dr}{d\chi}\frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2 \left(\frac{d\chi}{dt}\right)$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{d\chi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\chi) \Leftrightarrow \chi(r)$:

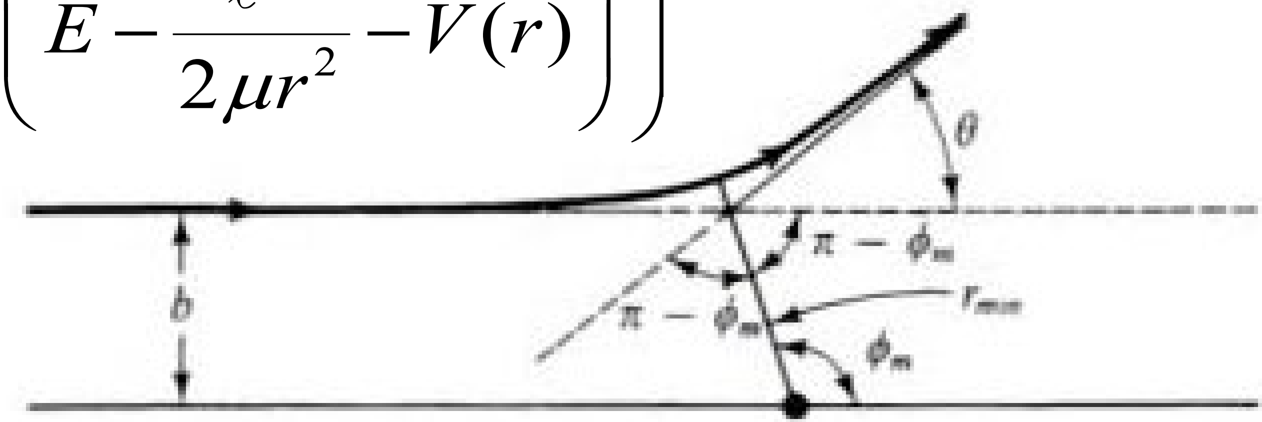
From: $$E = \frac{1}{2}\mu\left(\frac{dr}{d\chi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$\left(\frac{dr}{d\chi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\chi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\chi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$v_\infty \rightarrow$




Special values at large separation ($r \to \infty$):

$$\ell = \mu \left|\mathbf{r} \times \mathbf{v}\right|_{r\to\infty} = \mu v_\infty b$$

$$E = \frac{1}{2}\mu v_\infty^2$$

$$\Rightarrow \ell = \sqrt{2\mu E}\, b$$

Notation switch –
Notes: $\chi$
Text: $\phi$

# When the dust clears:

$$d\chi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\chi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

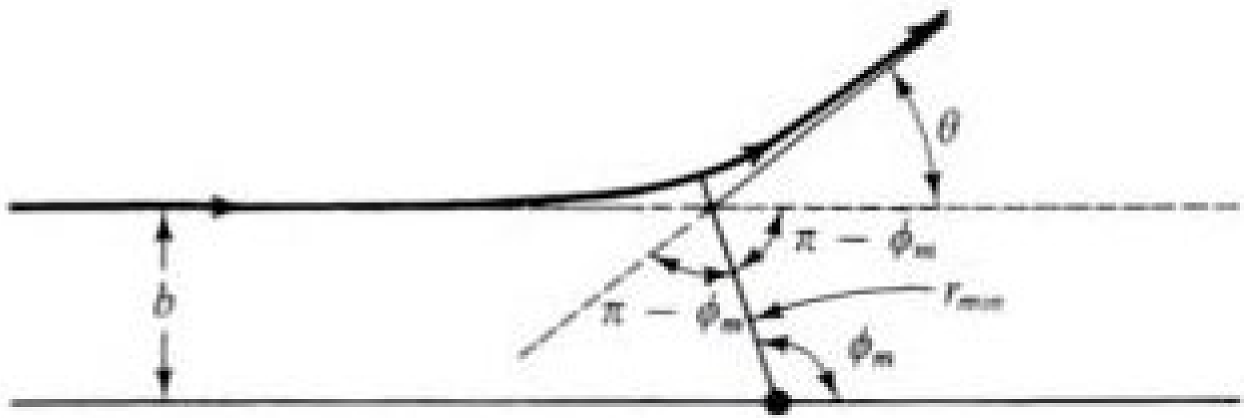
$$\Rightarrow \chi_{\max}(b,E) = \chi(r \to \infty) - \chi(r = r_{\min})$$

$$\int_0^{\chi_{max}} d\chi = \int_{r_{min}}^{\infty} dr \left( \frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{min}^2} - \frac{V(r_{min})}{E} = 0$$

Relationship between $\chi_{max}$ and $\theta$:




$$2(\pi - \chi_{\max}) + \theta = \pi$$

$$\Rightarrow \chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

Using the diagram from your text, $\theta$ represents the scattering angle in the center of mass frame and $\phi$ is used instead of $\chi$.

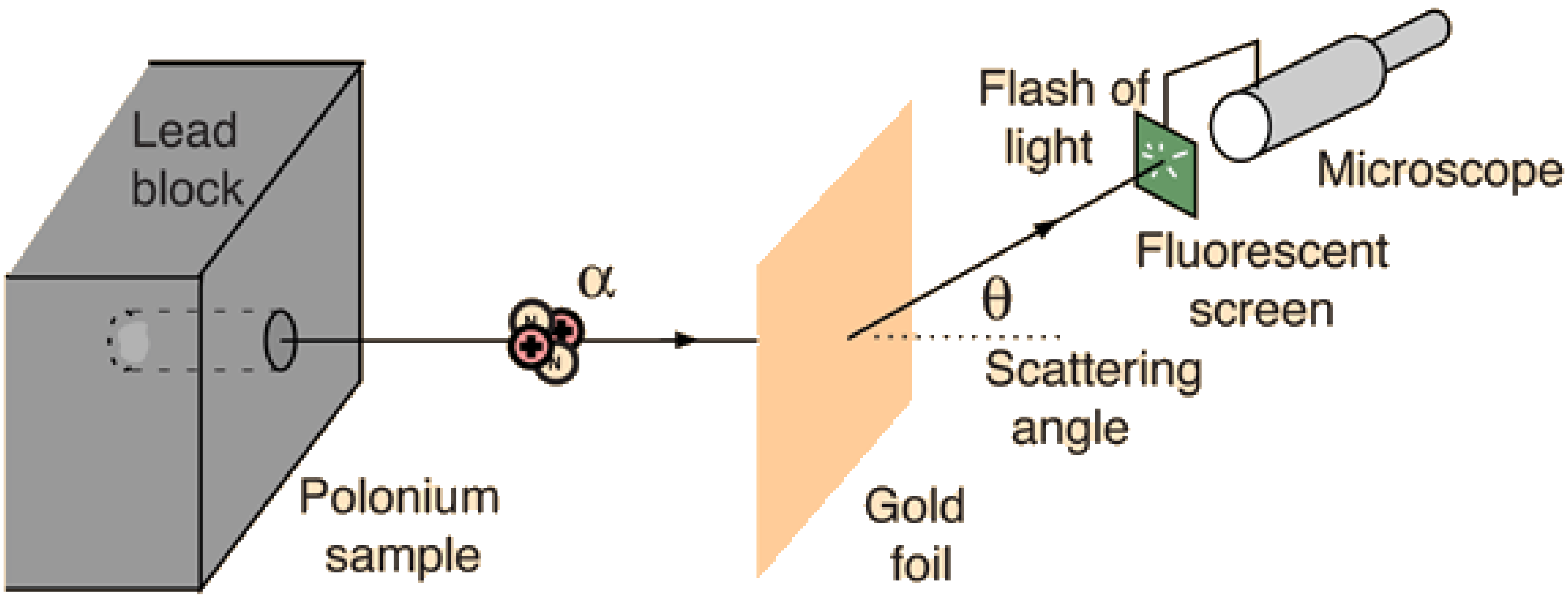
$$\chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left( \frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left( \frac{1 / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left( \frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

# Example: Diagram of Rutherford scattering experiment

http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left( \frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b}\left( -\frac{\kappa}{2b} + \sqrt{\left(\frac{\kappa}{2b}\right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left( \frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left( \frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

## Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2+1}}\right)$$

$$\frac{2b}{\kappa} = \left|\frac{\cos(\theta/2)}{\sin(\theta/2)}\right|$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$

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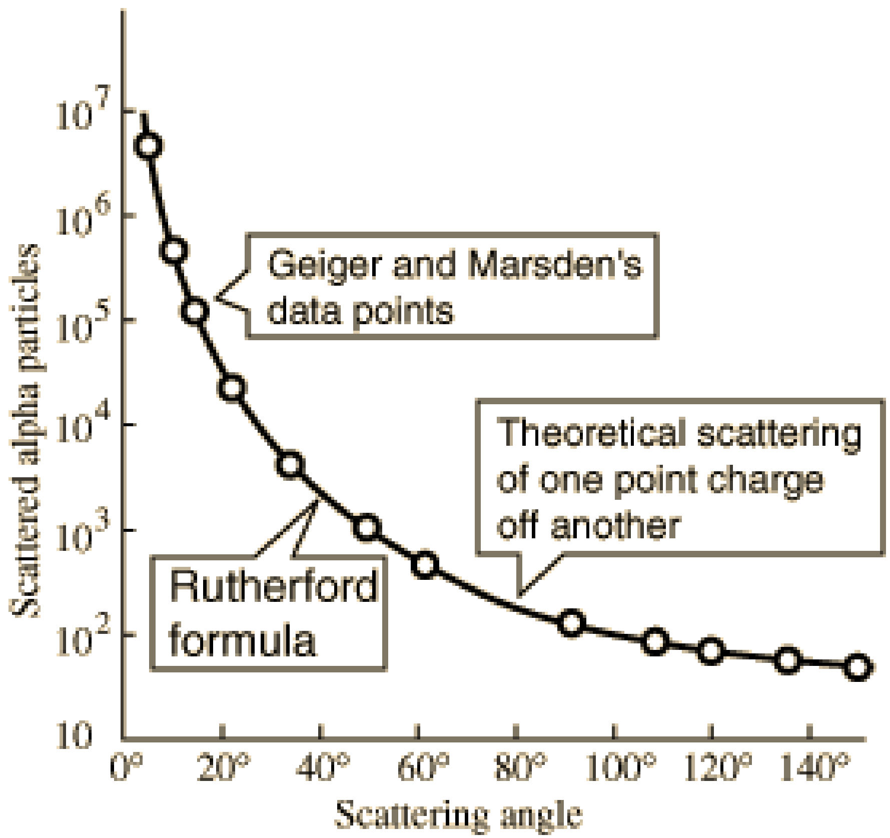



What happens as $\theta \rightarrow 0$?

From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

Original experiment performed with $\alpha$ particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{8\pi\epsilon_0 \mu v_\infty^2} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{16\pi\epsilon_0 E_{rel}}$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$

Question –

What do you think happens for $\theta \rightarrow 0$?

a. Big trouble; need to make sure experiment is designed to avoid that case.

b. No problem

  i. Physics is altered in that case and nothing explodes.

  ii. Rare event and rarely causes trouble.

# Recap of equations for scattering cross section in the center of mass frame of reference

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\theta = -\pi + 2b\int_{r_{\min}}^{\infty} dr\left(\frac{1/r^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}}\right)$$

where $r_{\min}$ is found from

$$1-\frac{b^2}{r_{\min}^2}-\frac{V(r_{\min})}{E} = 0$$

Digression– In general, it is possible to determine the trajectory *r(χ)* --

$$\chi = \int_{r_{\min}}^{r} ds \left( \frac{b/s^2}{\sqrt{1 - \frac{b^2}{s^2} - \frac{V(s)}{E}}} \right)$$

$$\chi = b \int_{1/r}^{1/r_{\min}} du \left( \frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

For the Rutherford case --

$$V(r) = \frac{k}{r}$$

Find $r_{\min}$ :

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

$$r_{\min} = \frac{k / E}{2} + \frac{\sqrt{4b^2 + (k / E)^2}}{2}$$

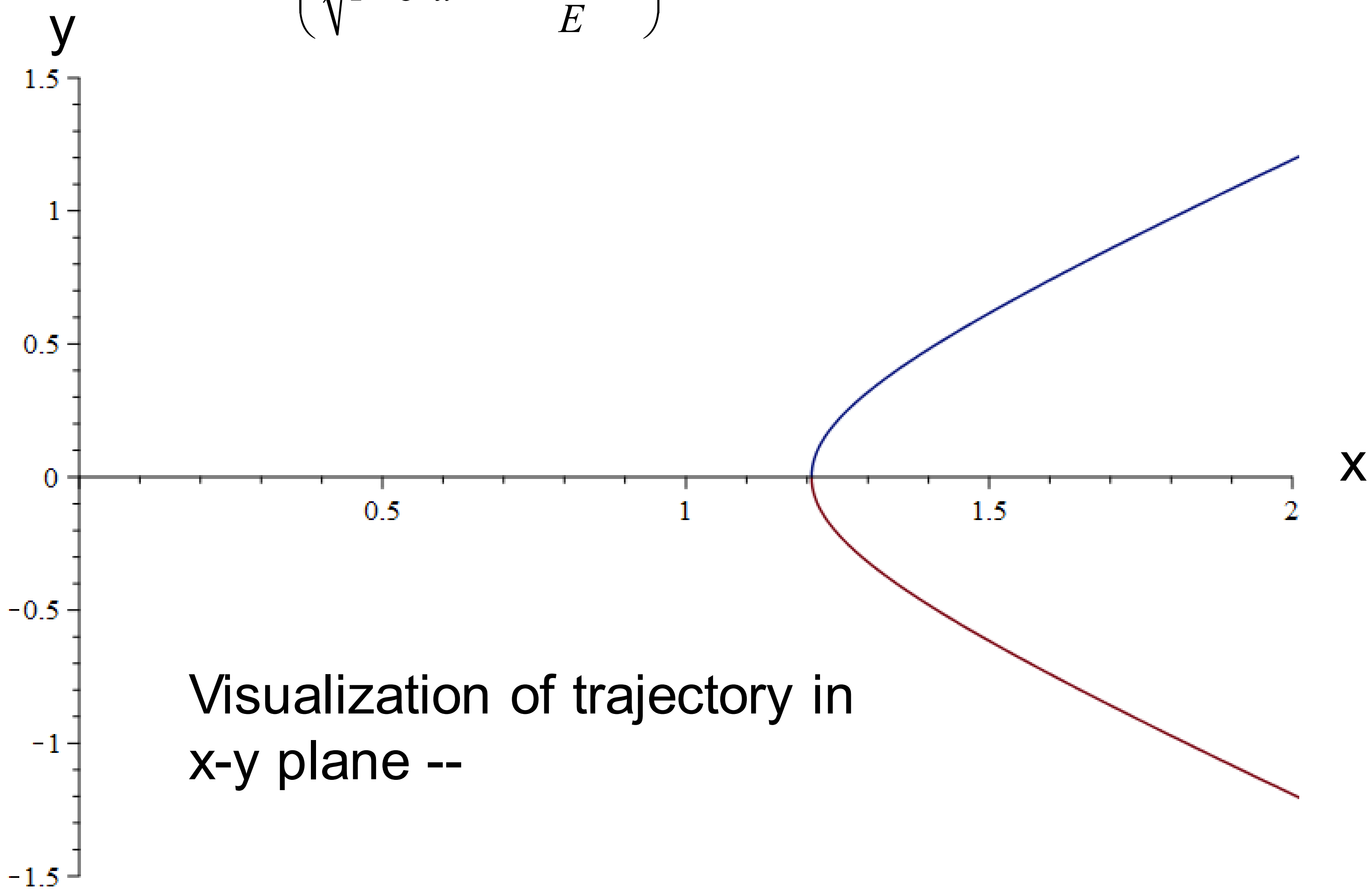
For the Rutherford case --

$$\chi = b \int_{1/r}^{1/r_{\min}} du \left( \frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right) = -\frac{\pi}{2} + \sin^{-1}\left( \frac{2b^2 + (k/E)r}{\sqrt{4b^2 + (k/E)^2}\, r} \right)$$

$$\Rightarrow r(\chi) = \frac{2b^2}{\sqrt{4b^2 + (k/E)^2}\cos(\chi) - k/E}$$

# For the Rutherford case --

$$\chi = b\int_{1/r}^{1/r_{\min}} du \left( \frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right) \quad \text{where } \frac{V(1/u)}{E} \equiv (k/E)u$$




Visualization of trajectory in x-y plane --