PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion notes for Lecture 5

Review of classical mechanical scattering theory – Chap 1 F&W

1. Some numerical considerations
2. Discussion questions
3. Review

PHYSICS COLLOQUIUM

THURSDAY • SEPTEMBER 2, 2021

Program

Student Research Presentations
Arina Yu (O. Jurchescu, Mentor)
Caleb Sawyer (M. Guthold, Mentor)
Leon Lu (G. Cook, Mentor)
Zach Scofield (P. Anderson, Mentor)

Welcome
Professor Natalie Holzwarth
Professor Dany Kim-Shapiro

Department Introductions and Announcements
- Introduction of Physics Faculty and Staff along with overview of the department
- Special programs -- SPS, Affiliated Centers and Collaborations
- Introduction of Physics Majors/Minors and Announcements (such as honors in physics)

WELCOME BACK TO IN-PERSON COLLOQUIUMS!

Reception – 3:30 PM
Olin Lounge*

* We encourage all to wander out to the front entrance of the building or up to the Observatory Deck on the 3rd floor to enjoy their refreshments.

Colloquium 4:00 pm
Olin 101*
*Video link also available

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 AM	**OPL 103**	**http://www.wfu.edu/~natalie/f21phy711/**

Instructor: Natalie Holzwarth	**Office:**300 OPL	**e-mail:**natalie@wfu.edu

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/23/2021	Chap. 1	Introduction	#1	8/27/2021
2	Wed, 8/25/2021	Chap. 1	Scattering theory	#2	8/30/2021
3	Fri, 8/27/2021	Chap. 1	Scattering theory		
4	Mon, 8/30/2021	Chap. 1	Scattering theory	#3	9/01/2021
5	Wed, 9/01/2021	Chap. 1	Summary of scattering theory	#4	9/03/2021
6	Fri, 9/03/2021	Chap. 2	Non-inertial coordinate systems		

PHY 711 – Assignment #4

09/01/2021

1. Equation 5.28 in Fetter and Walecka represents the differential equation evaluated in the center of mass parameters of an alpha particle ($z = 2$) having center mass energy E acting on a gold particle ($Z = 79$) assumed to be initially at rest. In the lab frame, the initial velocity of the alpha particle is 5 mega electron volts (MeV).

   (a) Fetter and Walecka use cgs Gaussian units for the Coulomb interaction. Using SI units for the Coulomb interaction, rewrite the equation for the differential cross section.

   (b) Using reliable sources for the fundamental constants such as those from the NIST website `https://physics.nist.gov/cuu/Constants/index.html` in order to evaluate the expression numerically.

   (c) Evaluate the differential equation at various center of mass scattering angles θ at least for $\theta = 45, 90$, and 135 degrees.

Comments on numerical evaluations

- Fetter and Walecka like many older textbooks use cgs Gaussian units (centimeters, grams, seconds plus miscellaneous factors of 4π) while "modern" texts use SI units (meters, kilograms, seconds plus other miscellaneous factors of $4\pi\varepsilon_0$)

Coulomb force between *ze* and *Ze* at separation **r**:

cgs Gaussian units

$$\mathbf{F}_{Coulomb} = \frac{zZe^2\mathbf{r}}{r^3}$$

SI units

$$\mathbf{F}_{Coulomb} = \frac{zZe^2\mathbf{r}}{4\pi\epsilon_0 r^3}$$

Comments on numerical evaluations -- continued

- MeV is a convenient unit of energy related to SI unit of Joules

  1 eV=$1.602176634\times10^{-19}$ J
  1 MeV= $1.602176634\times10^{-13}$ J

- More generally a reliable source for fundamental constants is available at the NIST website https://physics.nist.gov/cuu/Constants/index.html

The NIST Reference on
Constants, Units, and Uncertainty

Information at the foundation of modern science and technology from the Physical Measurement Laboratory of NIST

Constants Topics:

- **Values**
- Energy Equivalents
- Searchable Bibliography
- Background

- Constants Bibliography

- Constants, Units & Uncertainty home page

CODATA Internationally recommended 2018 values of the Fundamental Physical Constants

Version history and disclaimer

(e.g., **electron mass**, most misspellings okay)

Search by name [ ] Search

Display ◉ alphabetical list, ○ table (image), or ○ table (pdf)

by clicking a category below

Universal	Defined constants	Frequently used constants
Electromagnetic	Non-SI units	Extensive listings
Atomic and nuclear	Conversion factors for energy equivalents	All values (ascii)
Physico-chemical	X-ray values	

Find the correlation coefficient between any pair of constants

What constants will you need to know?

What units will your answer have?

Your questions –

From Wells –

1. What if we considered an attractive potential which caused the incident particle to orbit the target particle? Would the analysis be similar to the hyperbolic orbits in gravitational potentials section except start by choosing an eccentricity of < 1? There would also be issues in the analysis such as multivalued functions caused by the orbit, correct?

From Manikanta –

1. Why the total cross section is not a finite value. I feel performing the integral over all solid angles should give 1 since it tells there is a 100% chance of finding a particle?
2. When $\boldsymbol{\Theta}$ = 0, differential cross section goes to infinity. By this do we mean we can trace an infinite number of particles ?

From Can –

1. My question is why on slide the angle scattering equation has negative Pi, whereas on the book is positive Pi.

From Owen –

1. Are non-central potentials ever used in scattering theory?
2. Are there any scattering experiments that test non-central potentials?

Reviewing – scattering in the laboratory

Figure from F&W:

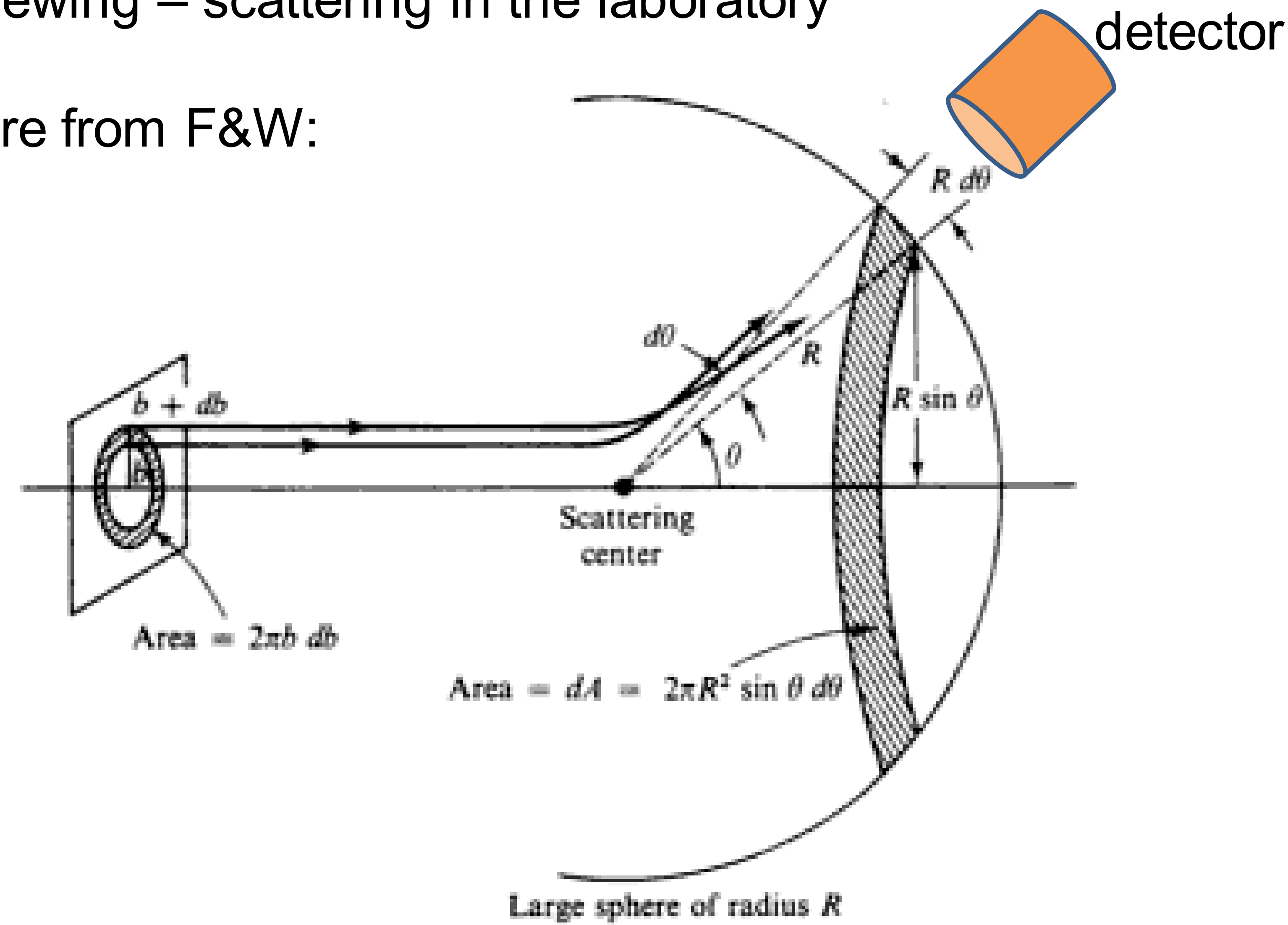


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Owen is asking about angularly dependent interaction potentials *V(**r**)*

Comments -- These certainly occur in nature and are important. However, the equations we have used here have to be modified. The experimental set up will be the same, but the differential cross section will depend both on θ and ϕ.

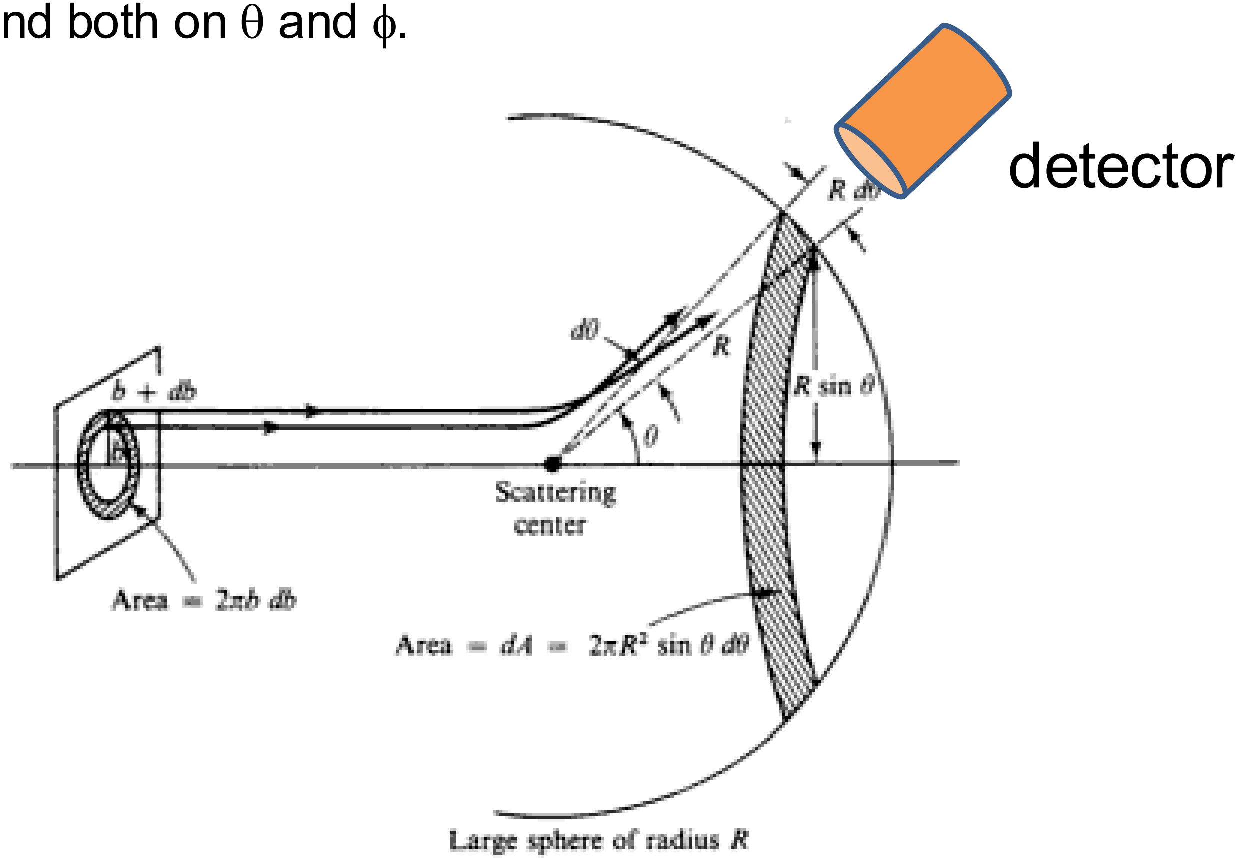


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

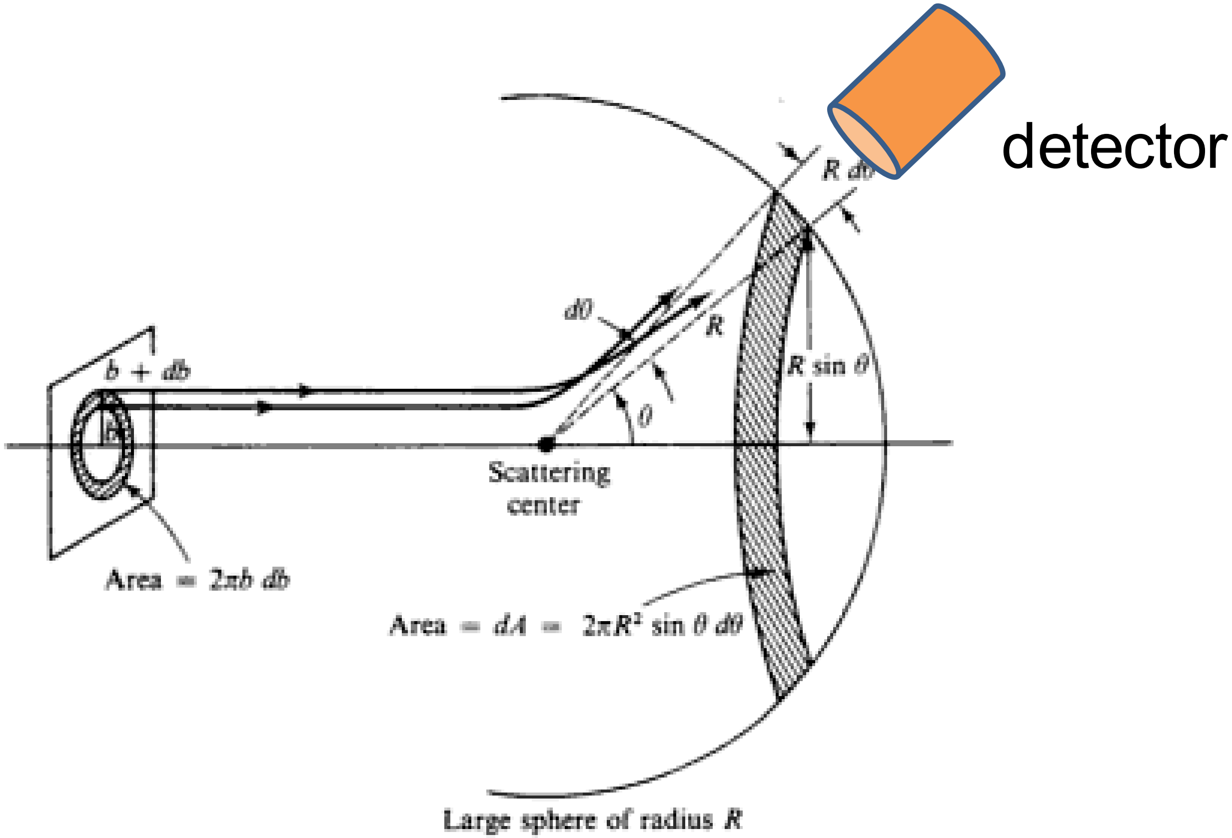
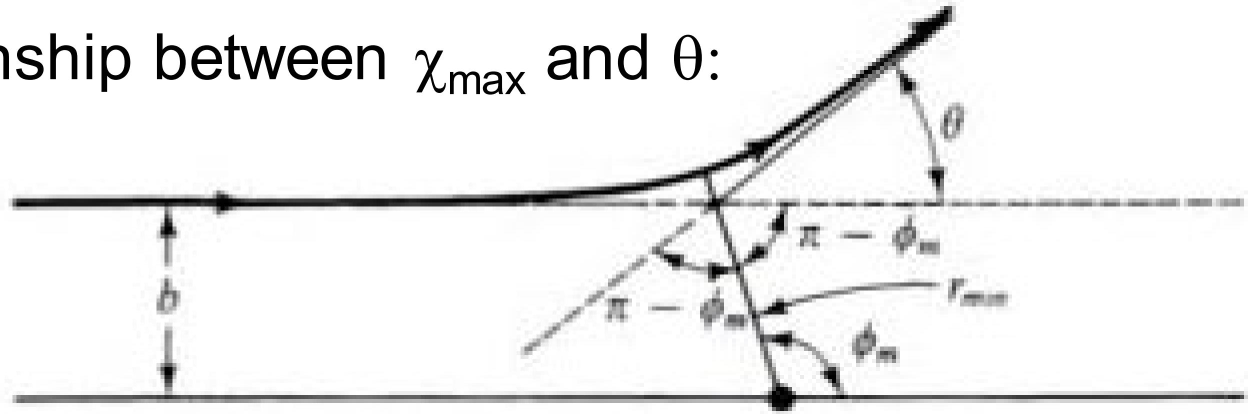


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Note that this diagram implies a repulsive interaction. Wells is asking how would it look if the interaction was attractive?

Close up of repulsive interaction

Relationship between χ_{max} and θ:



$$2(\pi - \chi_{\max}) + \theta = \pi$$

$$\Rightarrow \chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

Using the diagram from your text, θ represents the scattering angle in the center of mass frame and ϕ is used instead of χ.

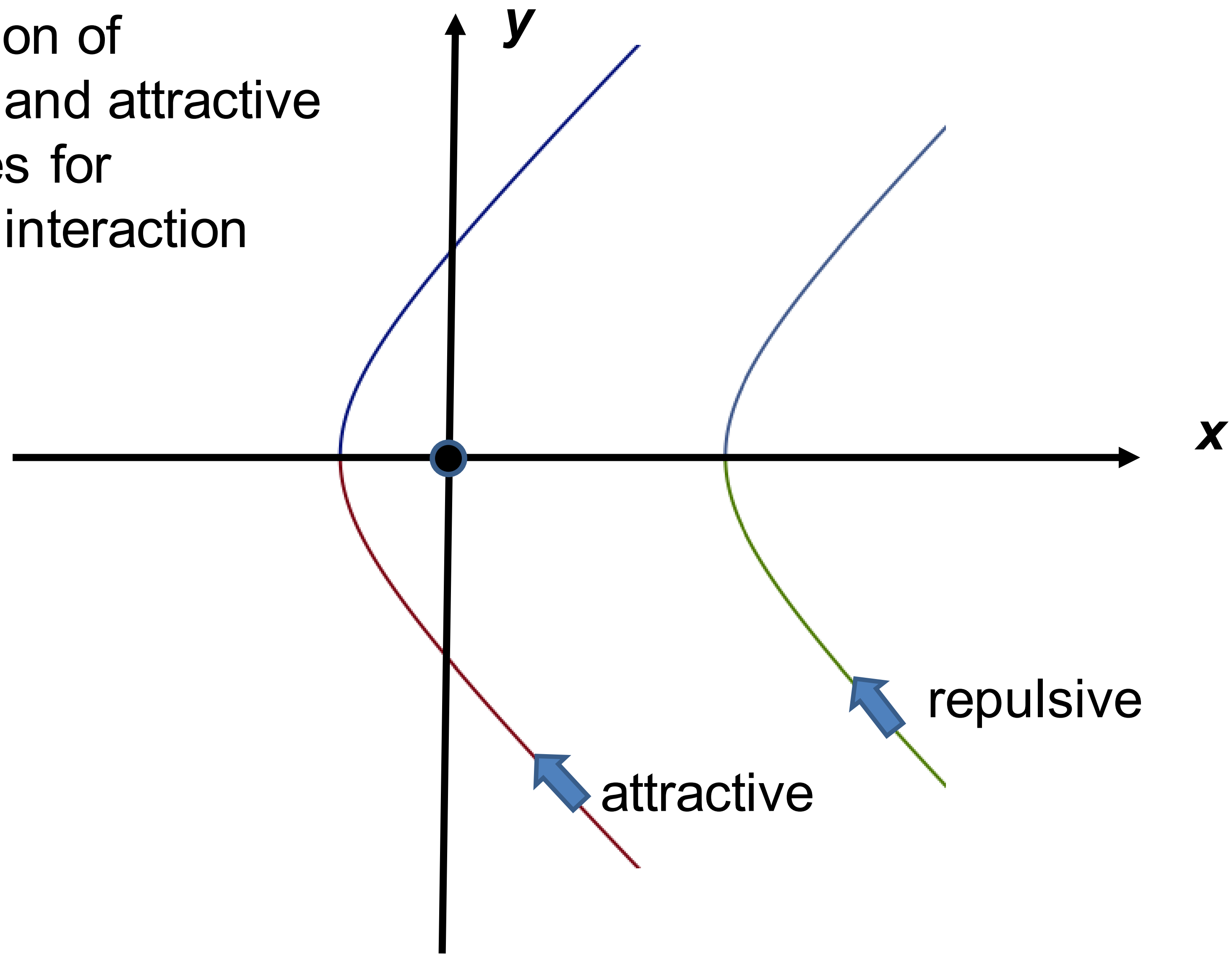
Can is asking about the sign of angles in these diagrams

Comment –

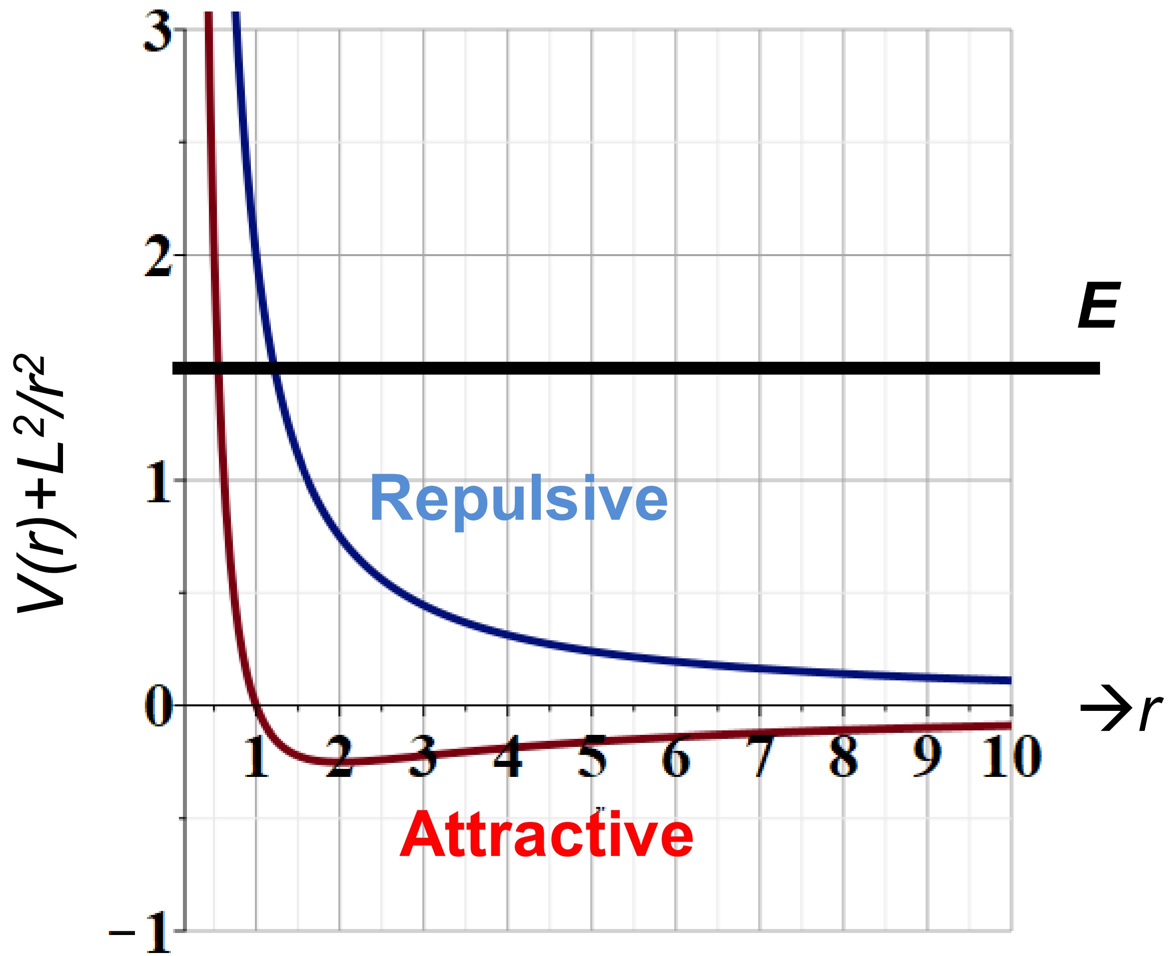
1. Never trust my signs.
2. Sometimes sign choices are equivalent.
3. For any system you are analyzing make sure the sign conventions are internally consistent.

Question about repulsive and attractive interaction potentials

Comparison of repulsive and attractive trajectories for Coulomb interaction



Energetic view



Divergences in differential cross sections

Rutherford case when $V(r) = \frac{\kappa E}{r}$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$

In this example, the differential cross section diverges as $\theta \rightarrow 0$ and the total cross section also diverges.

What to do?

1. Become a mathematician to better control the equations
2. Think about possible intervening physics for the $\theta \rightarrow 0$ case.

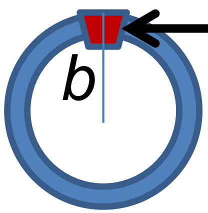
Review of important equations --

Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

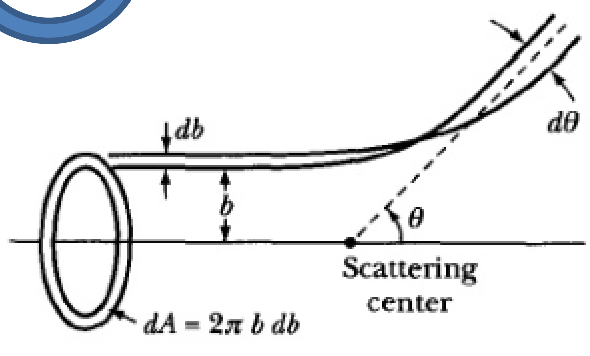
$$= \text{Area of incident beam that is scattered into detector}$$

$$\text{at angle } \theta$$



$$d\sigma = d\varphi b db$$

$$d\Omega = d\varphi \sin\theta d\theta$$



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Note that the same formula applies to the center of mass analysis.

Total energy of system:

$$E = \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

constants

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

vary in time

For scattering analysis only need to know trajectory **before** and **after** the collision.

Some details --

Relationship between center of mass and laboratory frames of reference. At and time *t*, the following relationships apply --

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = (m_1 + m_2)\mathbf{R}_{CM}$$

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = (m_1 + m_2)\dot{\mathbf{R}}_{CM} = (m_1 + m_2)\mathbf{V}_{CM}$$

Note that $\dot{\mathbf{R}}_{CM} \equiv \dfrac{d\mathbf{R}_{CM}}{dt} \equiv \mathbf{V}_{CM}$

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\left|\mathbf{v}_1 - \mathbf{v}_2\right|^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

where: $\mu \equiv \dfrac{m_1 m_2}{m_1 + m_2}$

More details

Total energy of system:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r) \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$E = \quad E_{\text{Center of mass}} \quad + \quad E_{rel}$$

Recall that $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Focus on relative motion: $E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$

Since **r**(t) represents motion in a plane, we will analyze the system in that plane and use polar coordinates.

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$$

$$x(t) = r(t)\cos(\chi(t))$$

$$y(t) = r(t)\sin(\chi(t))$$

Note that $\left|\dot{\mathbf{r}}(t)\right|^2 = \dot{x}^2(t) + \dot{y}^2(t)$

$$= \dot{r}^2(t) + r^2(t)\dot{\chi}^2(t)$$

Clarification –

$$E_{total} = E_{\text{Center of mass}} + E_{rel}$$

Energy of the center mass motion

Energy within the center of mass reference frame

In the following slides E_{rel} is written E

Also note that the relative angular momentum of the system is a constant

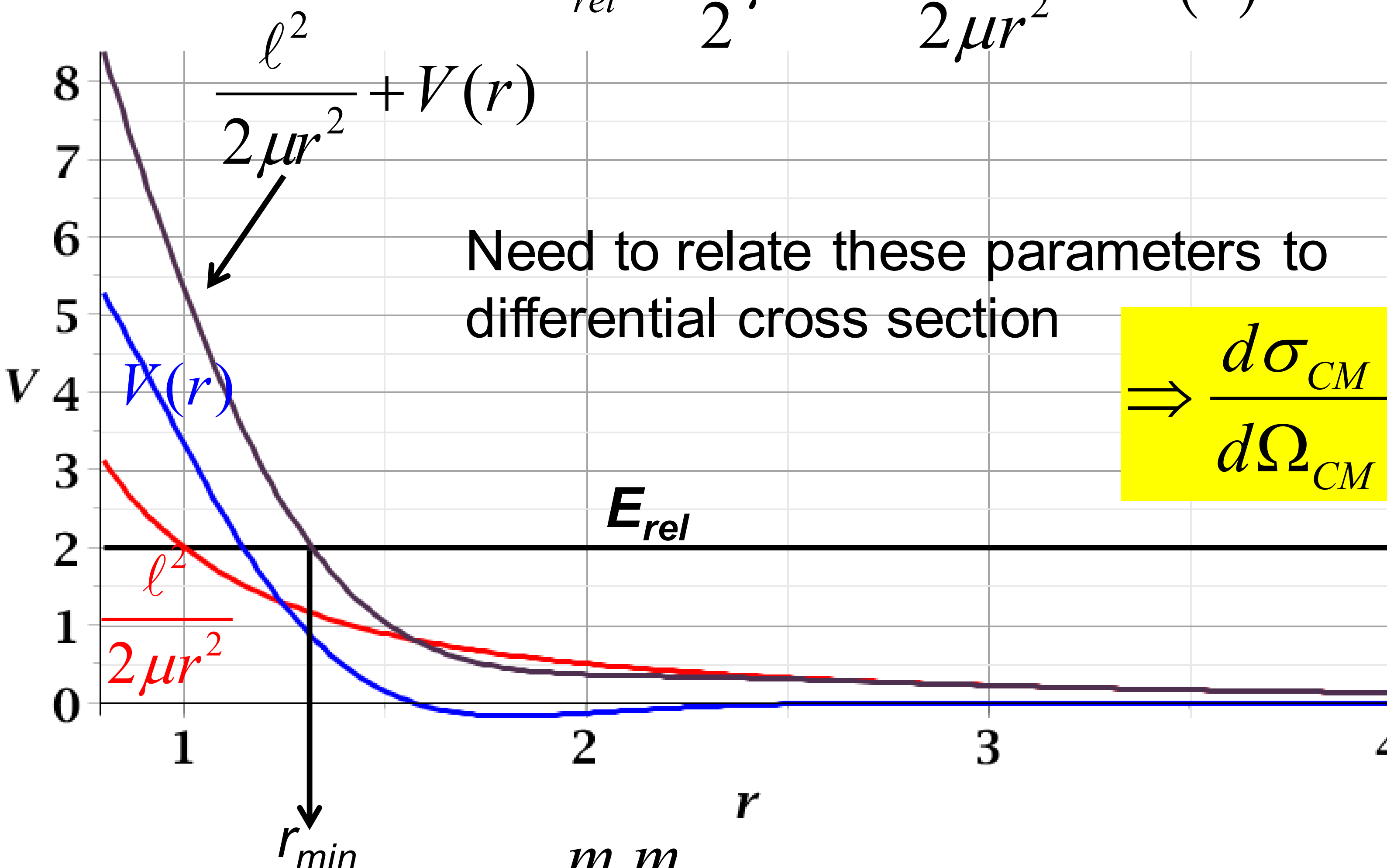
$\ell = \mu r^2 \dot{\chi}$

So that $$\frac{1}{2}\mu\, |\dot{\mathbf{r}}(t)|^2 = \frac{1}{2}\mu\left(\dot{r}^2(t) + r^2(t)\dot{\chi}^2(t)\right)$$
$$= \frac{1}{2}\mu\dot{r}^2(t) + \frac{\ell^2}{2\mu r^2}$$

$$\Rightarrow E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section

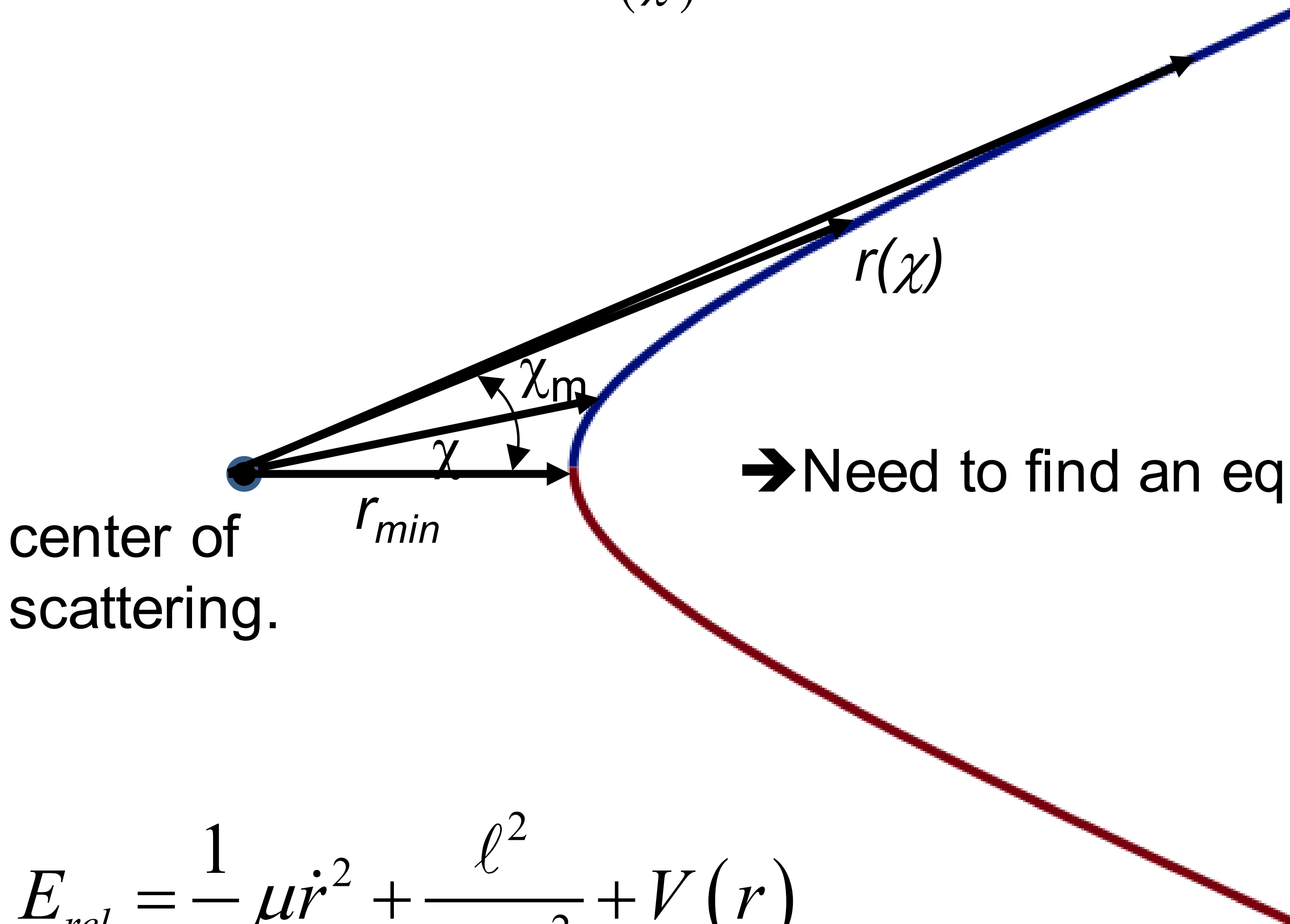
$$\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum

Trajectory of relative vector in center of mass frame

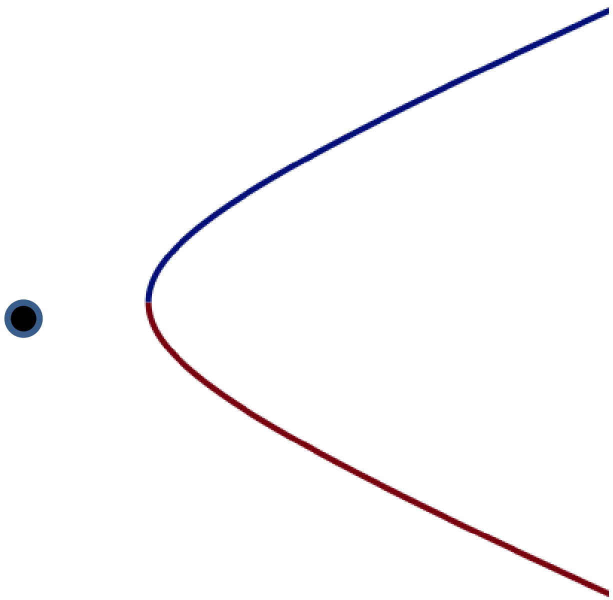
$r(\chi)$



➔Need to find an equation for $r(\chi)$

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

How is this related to scattering?



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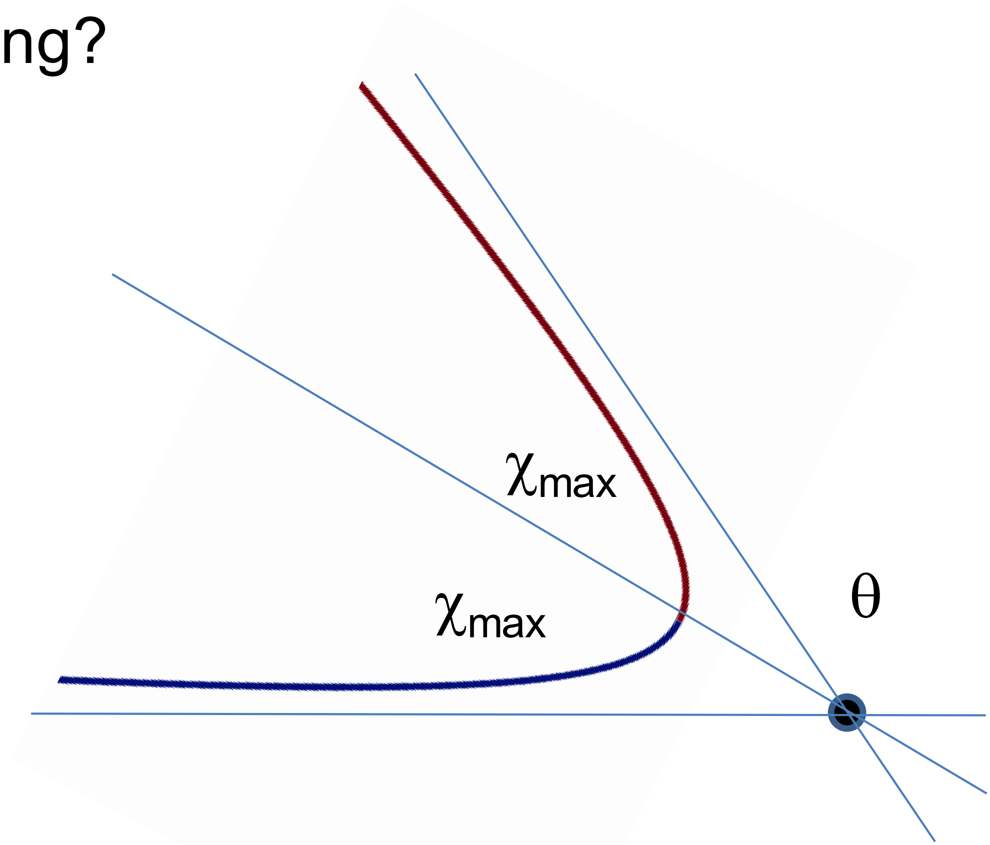


Note that here θ is used for the scattering angle

Note that we have use ψ to denote the scattering angle in the center of mass frame, but your textbook uses θ (which we had used to denote the scattering angle in the lab frame). In this lecture our analysis is entirely in the center of mass frame and some of the equations use θ to denote the scattering angle.

Questions:

1. How can we find $r(\chi)$?
2. If we find $r(\chi)$, how can we relate χ to ψ?
   (Here ψ is CM scattering angle.)
3. How can we find $b(\psi)$?

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{b}{\sin\psi}\left|\frac{db}{d\psi}\right|$$

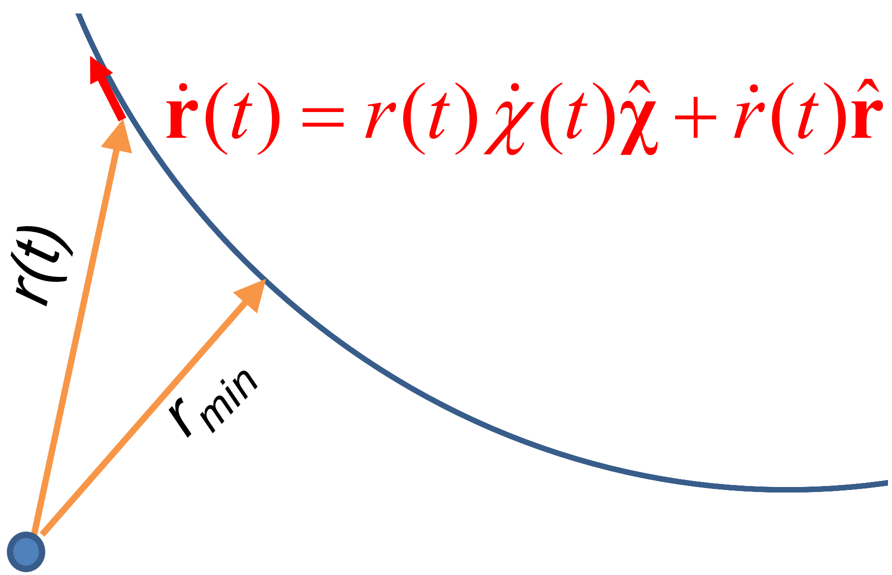
Evaluation of constants far from scattering center --

$$\ell = \mathbf{r} \times \left(\mu \dot{\mathbf{r}} \right) = r \mu r \frac{d\chi}{dt} = \mu r^2 \frac{d\chi}{dt}$$

$$\text{also: } \ell = b \mu \dot{r}(t = -\infty)$$

$$E_{rel} = \frac{1}{2} \mu \left(\dot{r}(t = -\infty) \right)^2$$

$$\Rightarrow \ell = b \sqrt{2 \mu E_{rel}}$$



Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\chi)$$

$$\frac{dr}{dt} = \frac{dr}{d\chi}\frac{d\chi}{dt} = \frac{dr}{d\chi}\frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2\left(\frac{d\chi}{dt}\right)$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{d\chi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\chi) \Leftrightarrow \chi(r)$:

From: $$E = \frac{1}{2}\mu\left(\frac{dr}{d\chi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$\left(\frac{dr}{d\chi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\chi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

When the dust clears:

$$d\chi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\chi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

$$\Rightarrow \chi_{\max}(b, E) = \chi\left(r \to \infty\right) - \chi\left(r = r_{\min}\right)$$

$$\int_0^{\chi_{\max}} d\chi = \int_{r_{\min}}^{\infty} dr \left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

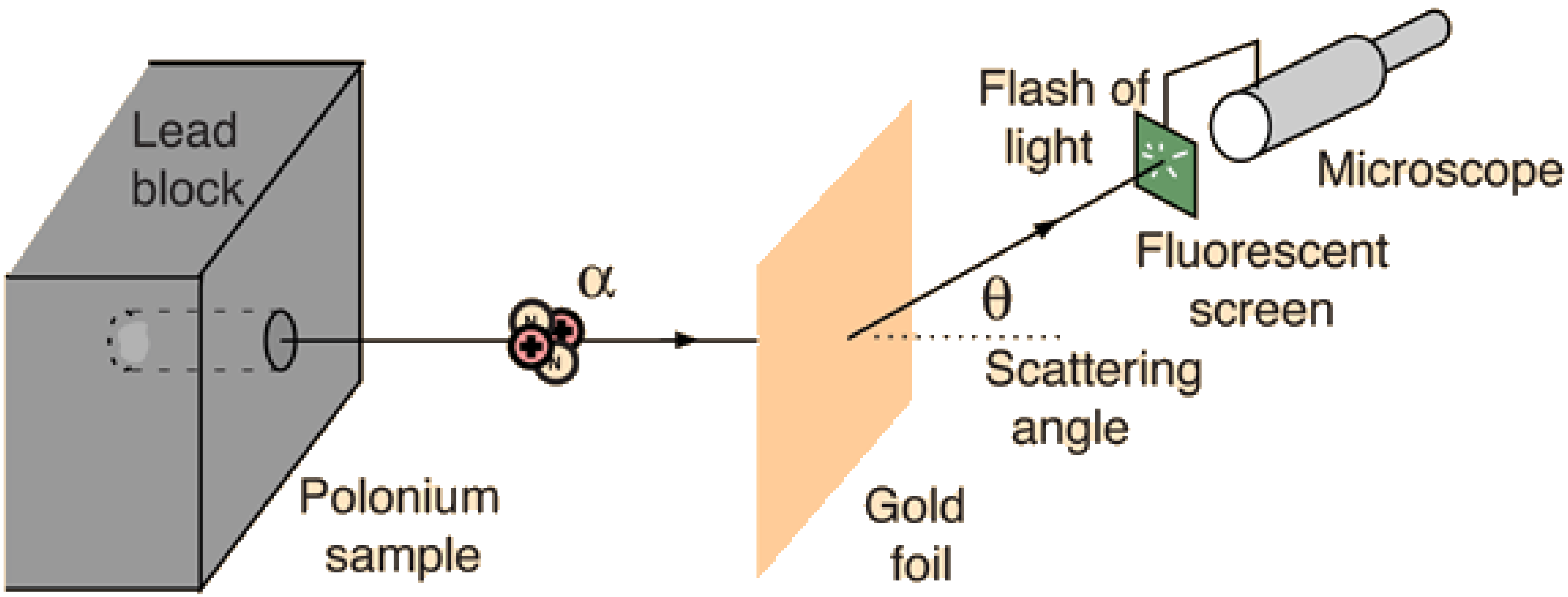
General equations for central potential V(r)

$$\chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b}\left(-\frac{\kappa}{2b} + \sqrt{\left(\frac{\kappa}{2b}\right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

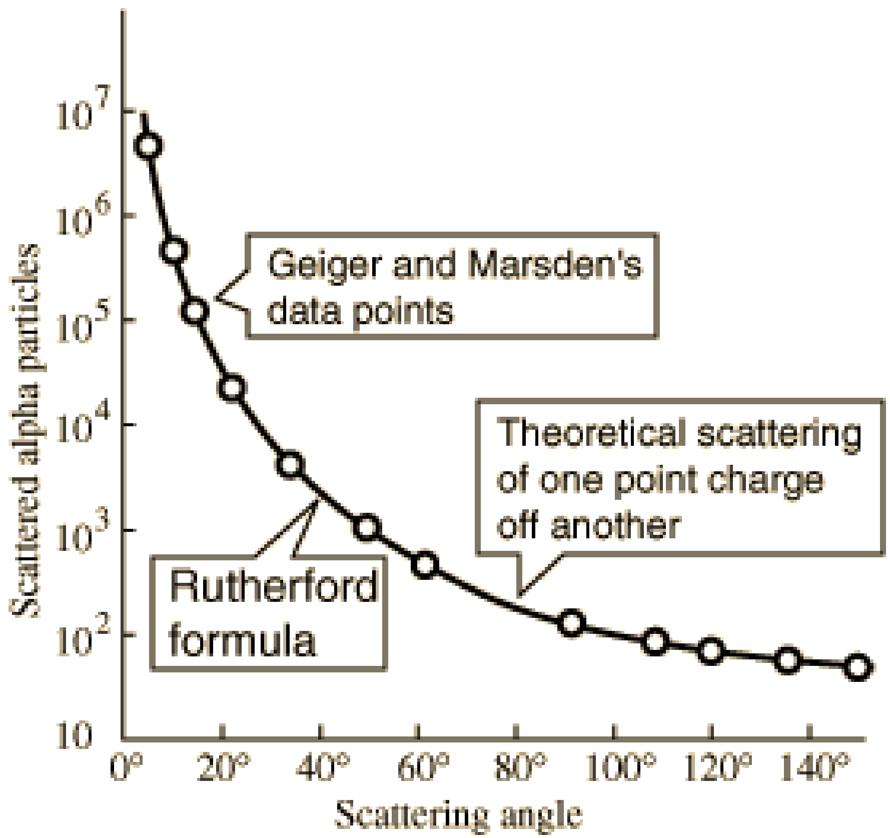
Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2+1}}\right)$$

$$\frac{2b}{\kappa} = \left|\frac{\cos(\theta/2)}{\sin(\theta/2)}\right|$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$



What happens as $\theta \rightarrow 0$?

From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

Original experiment performed with α particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{8\pi\epsilon_0 \mu v_\infty^2} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{16\pi\epsilon_0 E_{rel}}$$

(in SI units)

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$

Transformation between lab and center of mass results:
Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+\left(m_1/m_2\right)^2\right)^{3/2}}{\left(m_1/m_2\right)\cos\psi+1}$$

where: $$\tan\theta=\frac{\sin\psi}{\cos\psi+m_1/m_2}$$

For elastic scattering