PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin103

Lecture notes for Lecture 7
Chapter 3.17 of F&W

Introduction to the calculus of variations

1. **Mathematical construction**
2. **Practical use**
3. **Examples**

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 AM	**OPL 103**	**http://www.wfu.edu/~natalie/f21phy711/**

Instructor: Natalie Holzwarth	**Office:**300 OPL	**e-mail:**natalie@wfu.edu

---

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/23/2021	Chap. 1	Introduction	#1	8/27/2021
2	Wed, 8/25/2021	Chap. 1	Scattering theory	#2	8/30/2021
3	Fri, 8/27/2021	Chap. 1	Scattering theory		
4	Mon, 8/30/2021	Chap. 1	Scattering theory	#3	9/01/2021
5	Wed, 9/01/2021	Chap. 1	Summary of scattering theory	#4	9/03/2021
6	Fri, 9/03/2021	Chap. 2	Non-inertial coordinate systems	#5	9/06/2021
7	Mon, 9/06/2021	Chap. 3	Calculus of Variation	#6	9/10/2021

PHY 711 -- Assignment #6

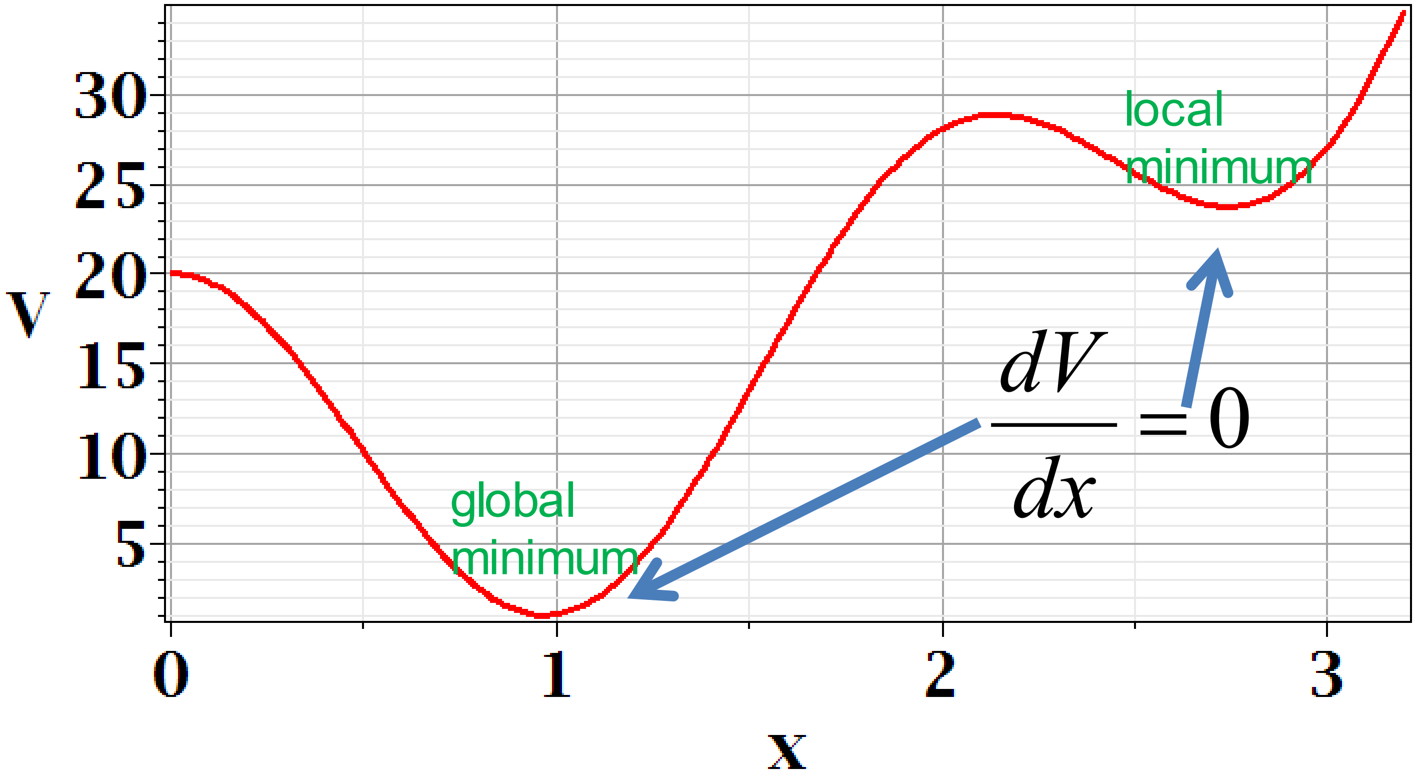
Sept. 6, 2021

Start reading Chapter 3, especially Section 17, in **Fetter & Walecka**.

1. Using calculus of variations, find the equation $y(x)$ of the shortest length "curve" which passes through the points ($x=0$, $y=0$) and ($x=2$, $y=8$).

In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton's laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as "the calculus of variation".

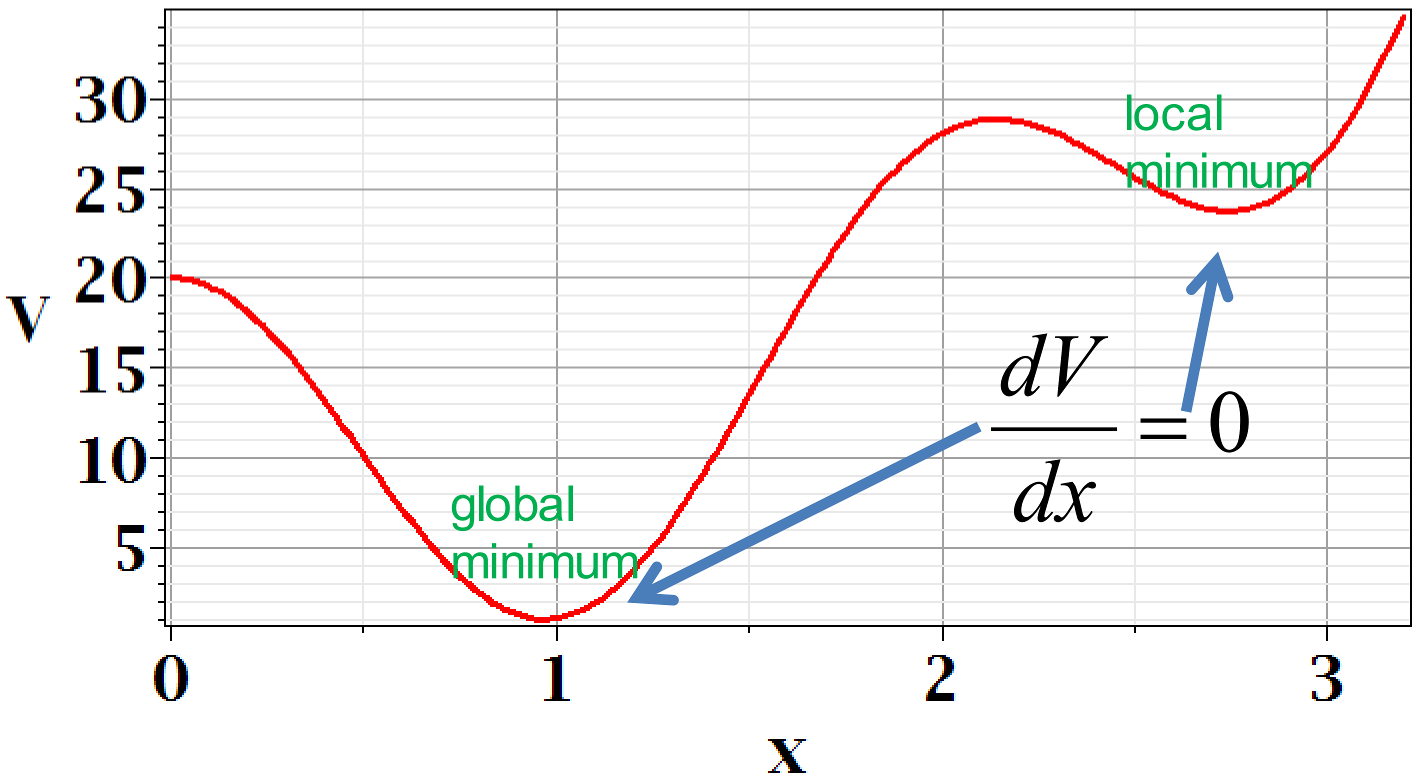
Minimization of a simple function



Minimization of a simple function

Given a function $V(x)$, find the value(s) of x for which $V(x)$ is minimized (or maximized).

Necessary condition : $\frac{dV}{dx} = 0$



Functional minimization

Consider a family of functions $y(x)$, with fixed end points

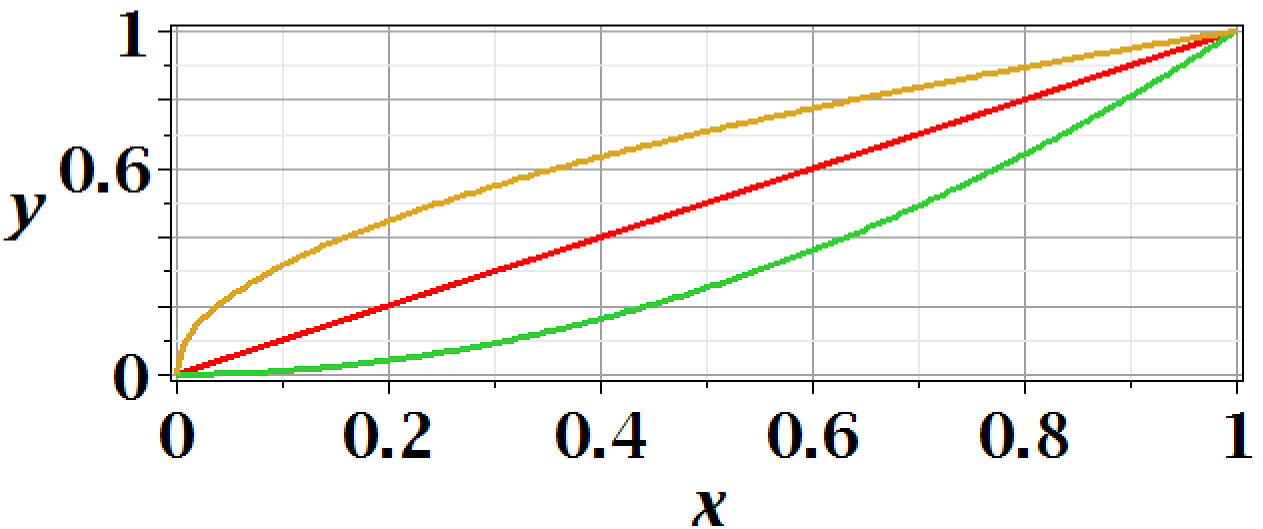
$y(x_i) = y_i$ and $y(x_f) = y_f$ and a function $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition: $\delta L = 0$

Example:

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$



Difference between minimization of a function V(x) and the minimization in the calculus of variation.

Minimization of a function

➔Know V(x) ➔Find x_0 such that $V(x_0)$ is a minimum.

Calculus of variation

For $x_i \le x \le x_f$ want to find a function $y(x)$ that minimizes an integral that depends on $y(x)$.

The analysis involves deriving and solving a differential equation for $y(x)$.

Functional minimization

Consider a family of functions $y(x)$, with fixed end points

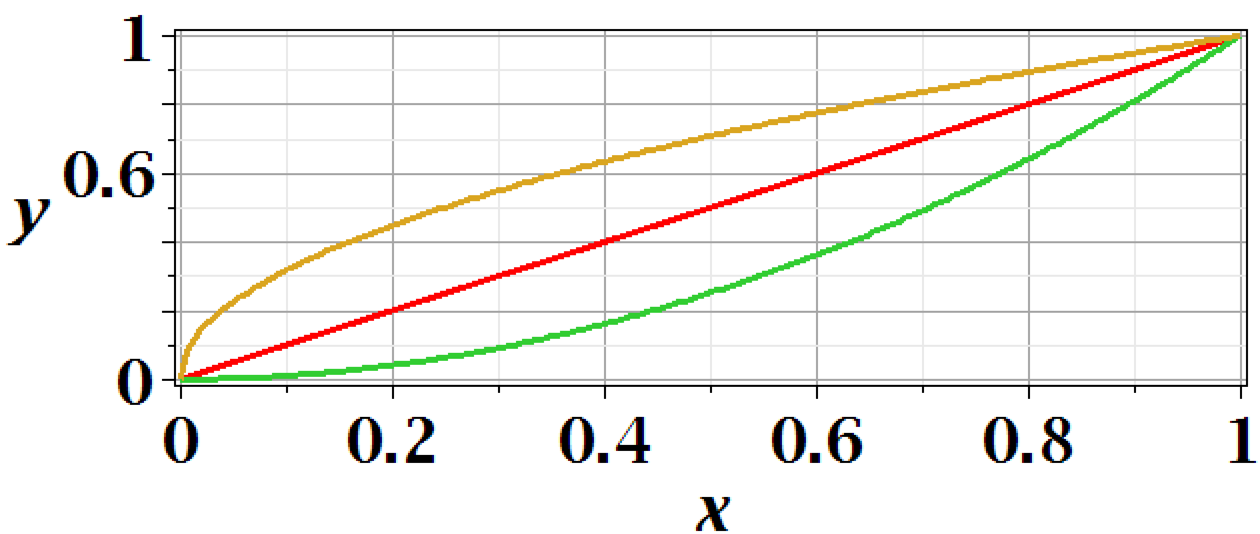
$y(x_i) = y_i$ and $y(x_f) = y_f$ and a function $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition: $\delta L = 0$

Example:

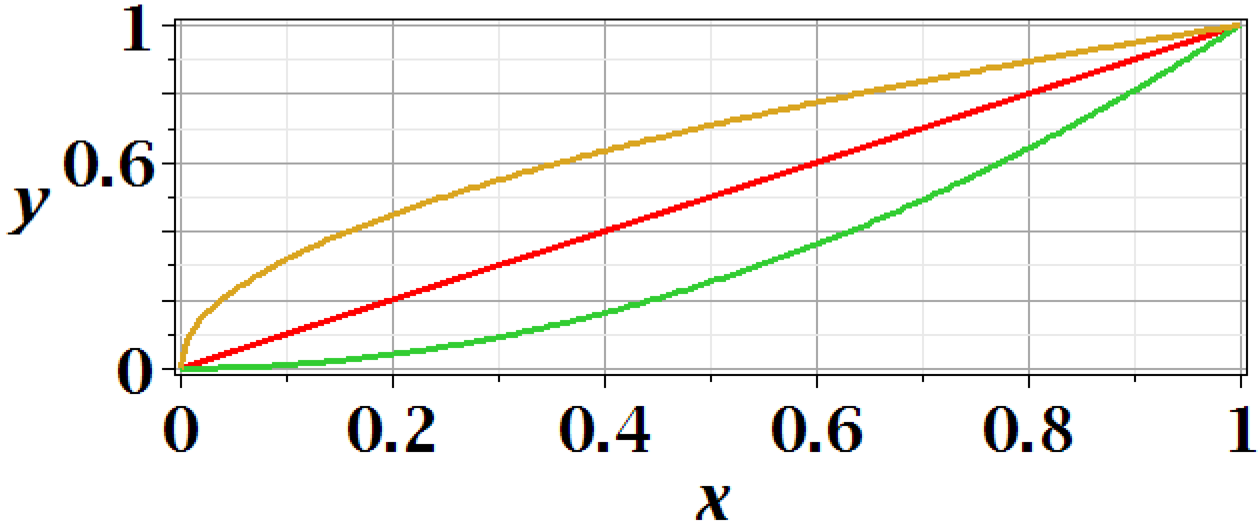
$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$



Example:

$$L = \int_{(0,0)}^{(1,1)} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Sample functions :

$y_1(x) = \sqrt{x}$ $\qquad L = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx = 1.4789$

$y_2(x) = x$ $\qquad L = \int_0^1 \sqrt{1+1}\, dx = \sqrt{2} = 1.4142$

$y_2(x) = x^2$ $\qquad L = \int_0^1 \sqrt{1 + 4x^2}\, dx = 1.4789$

Calculus of variation example for a pure integral functions

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$

where $$L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx.$$

Necessary condition : $\delta L = 0$

At any x, let $y(x) \rightarrow y(x) + \delta y(x)$

$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta \frac{dy(x)}{dx}$$

Formally :

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta\left(\frac{dy}{dx} \right) \right] \right] dx.$$

Comment about notation concerning functional dependence and partial derivatives

Suppose x, y, z represent independent variables that determine a function f :

We write $f(x, y, z)$. A partial derivative with respect to x implies that we hold y, z fixed and infinitessimally change x

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}\right)$$

After some derivations, we find

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y \, dx = 0 \quad \text{for all } x_i \le x \le x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Note that this is a "total" derivative

"Some" derivations --

Consider the term

$$\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]dx:$$

If $y(x)$ is a well-defined function, then $\delta\left(\frac{dy}{dx}\right)=\frac{d}{dx}\delta y$ *

$$\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]dx=\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\frac{d}{dx}\delta y\right]dx$$

$$=\int_{x_i}^{x_f}\left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]-\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

*Clarification -- what is the meaning of the following statement:

$$\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}\delta y$$

Up to now, the operator δ is not well defined and meant to be general.

Now let us suppose that it implies an infinitessimal difference to its function.

As an example, suppose that $y(x,\eta)$ where x and η are independent such as

$y(x,\eta) = x^\eta$ For $\eta > 0$, and $0 \le x \le 1$

assume $\eta > 0$

$$\frac{d}{d\eta}\frac{d}{dx}y(x,\eta) = \frac{d}{dx}\frac{d}{d\eta}y(x,\eta) = \left(1+\eta\ln(x)\right)x^{\eta-1}$$

Note that the construction of this system is that

$y(x_i,\eta)$ has the same value for all η and

$y(x_f,\eta)$ has the same value for all η.

Example $y(x,\eta) = x^\eta$ for $x_i = 0$ and $x_f = 1$

$y_i = y(0,\eta) = 0$ and $y_f = y(1,\eta) = 1$

Note that the δy notation is meant to imply a general infinitessimal variation of the function $y(x)$

“Some” derivations (continued)--

$$\int_{x_i}^{x_f}\left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]-\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]_{x_i}^{x_f}-\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$= \qquad 0 \qquad -\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

Euler-Lagrange equation:

$$\Rightarrow\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0 \quad \text{for all } x_i\le x\le x_f$$

Clarfication – Why does this term go to zero?

$$\int_{x_i}^{x_f}\left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]-\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]_{x_i}^{x_f}-\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\quad 0 \quad -\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

Answer --

By construction $\delta y(x_i)=\delta y(x_f)=0$

Recap --

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y \, dx = 0 \quad \text{for all } x_i \le x \le x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Here we conclude that the integrand has to vanish at every argument in order for the integral to be zero

a. Necessary?
b. Overkill?

Example: End points -- $y(0)=0;\, y(1)=1$

$$L=\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=K \qquad \frac{dy}{dx}=K'\equiv\frac{K}{\sqrt{1-K^2}}$$

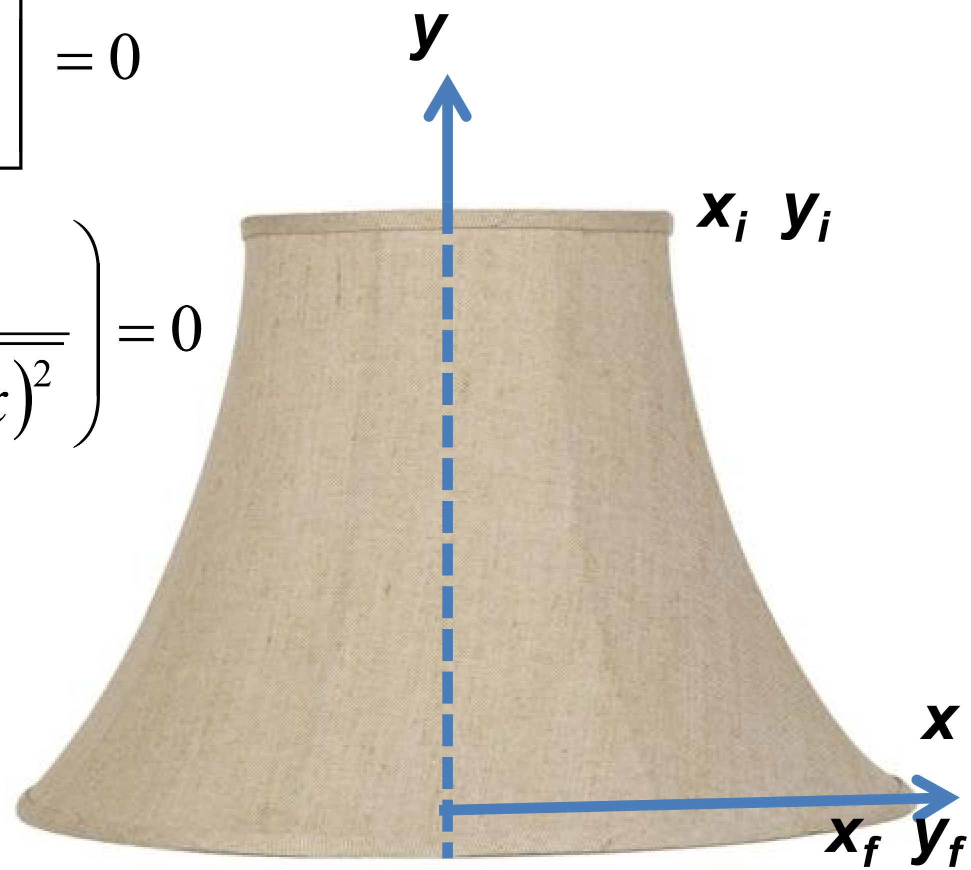
$$\Rightarrow y(x)=K'x+C \qquad y(x)=x$$

Example: Lamp shade shape $y(x)$

$$A = 2\pi \int_{x_i}^{x_f} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$



$$-\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

$$\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}=K_1$$

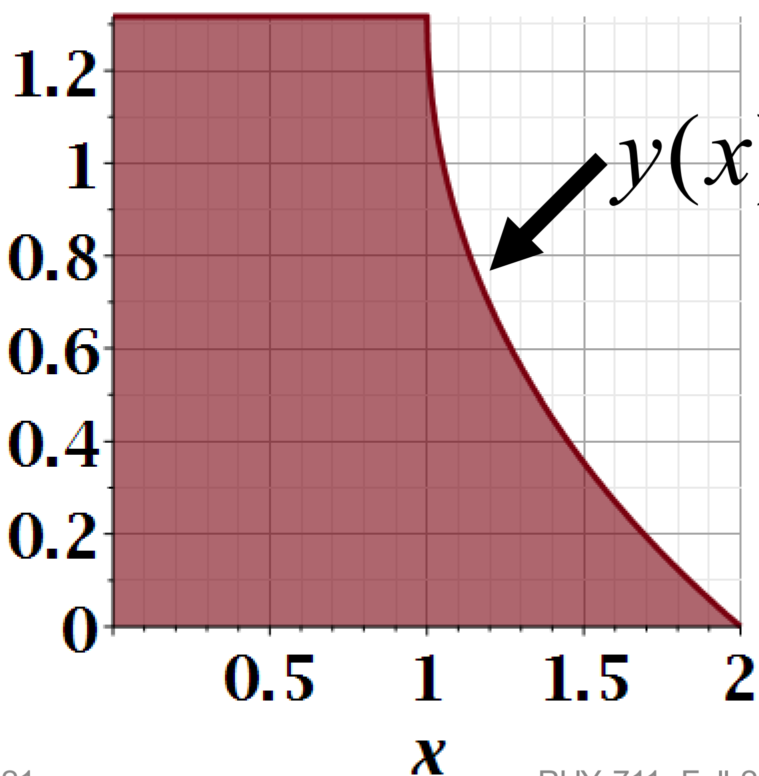
$$\frac{dy}{dx}=-\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2-1}}$$

$$\Rightarrow y(x)=K_2-K_1\ln\left(\frac{x}{K_1}+\sqrt{\frac{x^2}{K_1^2}-1}\right)$$

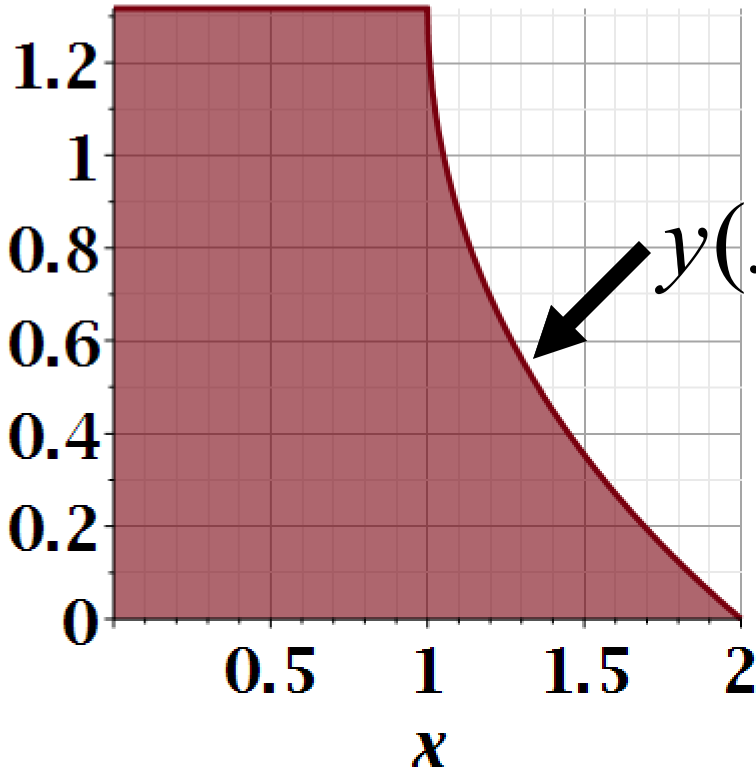
General form of solution --

$$y(x) = K_2 - K_1 \ln\left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1} \right)$$

Suppose $K_1 = 1$ and $K_2 = 2 + \sqrt{3}$



$$y(x) = \ln\left(\frac{2 + \sqrt{3}}{x + \sqrt{x^2 - 1}} \right)$$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x+\sqrt{x^2-1}}\right)$$

$$A = 2\pi\int_1^2 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 15.02014144$$

(according to Maple)

Another example:

(Courtesy of F. B. Hildebrand, Methods of Applied Mathematics)

Consider all curves $y(x)$ with $y(0)=0$ and $y(1)=1$

that minimize the integral:

$$I = \int_0^1 \left(\left(\frac{dy}{dx} \right)^2 - ay^2 \right) dx \quad \text{for constant } a > 0$$

Euler - Lagrange equation:

$$\frac{d^2y}{dx^2} + ay = 0$$

$$\Rightarrow y(x) = \frac{\sin\left(\sqrt{a}\,x\right)}{\sin\left(\sqrt{a}\right)}$$

Review : for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

⇦ Euler-Lagrange equation

Note that for $f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

⇦ Alternate Euler-Lagrange equation

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html

A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$?

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}\,dx \quad \text{because} \quad \tfrac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

Note that for the original form of Euler-Lagrange equation:

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = 0$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0,$$

differential equation is more complicated:

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}} - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a$$

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

Let $y=-2a\sin^2\frac{\theta}{2}=a(\cos\theta-1)$

$$\frac{dy}{dx}=-\sqrt{\frac{2a}{-y}-1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}}=dx$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=dx$$

$$x=\int_0^{\theta}a(1-\cos\theta')d\theta'=a(\theta-\sin\theta)$$

Parametric equations for Brachistochrone:

$$x=a(\theta-\sin\theta)$$
$$y=a(\cos\theta-1)$$

Parametric plot --

plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])