PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion of Lecture 8 – Chap. 3 F & W

Calculus of variation

1. Various examples – Area of lamp shade
2. Brachistochrone problem
3. Calculus of variation with constraints

PHYSICS COLLOQUIUM

THURSDAY • SEPTEMBER 9, 2021

Richard Williams
"Scientist, Mentor, Pioneer"

Richard Williams, Emeritus Reynolds Professor of Physics who passed away on July 5 at the age of 75 after a brief struggle with AML leukemia, was a product of Wake Forest University and truly embodied the school's three pillars of academic scholarship of teaching, research and service. In addition to being an internationally recognized expert on the physics of defects, scintillation and excitons, he helped establish the Physics Department's graduate program and mentored its first PhD recipients and played a leading role in establishing several research institutes and organized conferences on campus that brought international recognition to WFU.

We would like to dedicate this colloquium to honor his legacy and his contributions to science, to our department and to our lives.

Memorial Honoring Dr. Richard Williams

Hosts: Dr. Burak Ucer
Dr. Weronika Wolszczak

4:00 pm - Olin 101

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 AM	OPL 103	http://www.wfu.edu/~natalie/f21phy711/

Instructor: Natalie Holzwarth	Office:300 OPL	e-mail:natalie@wfu.edu

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/23/2021	Chap. 1	Introduction	#1	8/27/2021
2	Wed, 8/25/2021	Chap. 1	Scattering theory	#2	8/30/2021
3	Fri, 8/27/2021	Chap. 1	Scattering theory		
4	Mon, 8/30/2021	Chap. 1	Scattering theory	#3	9/01/2021
5	Wed, 9/01/2021	Chap. 1	Summary of scattering theory	#4	9/03/2021
6	Fri, 9/03/2021	Chap. 2	Non-inertial coordinate systems	#5	9/06/2021
7	Mon, 9/06/2021	Chap. 3	Calculus of Variation	#6	9/10/2021
8	Wed, 9/08/2021	Chap. 3	Calculus of Variation		
9	Fri, 9/10/2021	Chap. 3 & 6	Lagrangian Mechanics	#7	9/13/2021

Your questions –
From Can –

1. What type of problems that we can and can‘t use the method of calculus of variation to solve？

Comment – Typically we need to optimize an integral involving an unknown function and its derivative. This type of problem occurs in many contexts including mechanics.

From Wells –

1. Is the Lagrange multiplier always a scalar?

Comment -- Lagrange multipliers in our examples are constant scalars, but they may be more complicated.

Summary of the method of calculus of variation --

Consider a family of functions $y(x)$, with the end points $y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral function

$$I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \int_{x_i}^{x_f} f\left(y(x), \frac{dy}{dx}; x\right) dx.$$

Find the function $y(x)$ which extremizes $I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

$\delta I = 0 \quad \Rightarrow$ Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Example: Find minimum curve between points -- $y(0)=0; y(1)=1$

$$L=\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=K \qquad \frac{dy}{dx}=K'\equiv\frac{K}{\sqrt{1-K^2}}$$

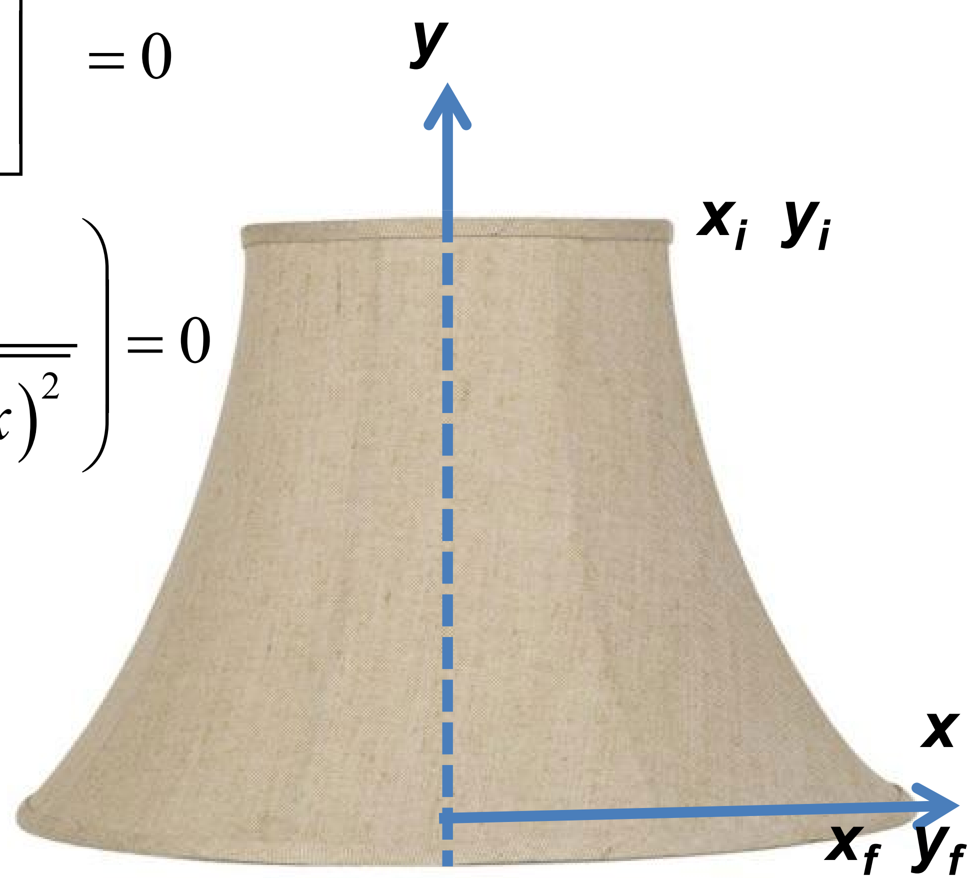
$$\Rightarrow y(x)=K'x+C \qquad y(x)=x$$

Another example: Lamp shade shape $y(x)$

$$A = 2\pi \int_{x_i}^{x_f} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$



$$-\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+\left(dy/dx\right)^2}}\right)=0$$

$$\frac{x\,dy/dx}{\sqrt{1+\left(dy/dx\right)^2}}=K_1$$

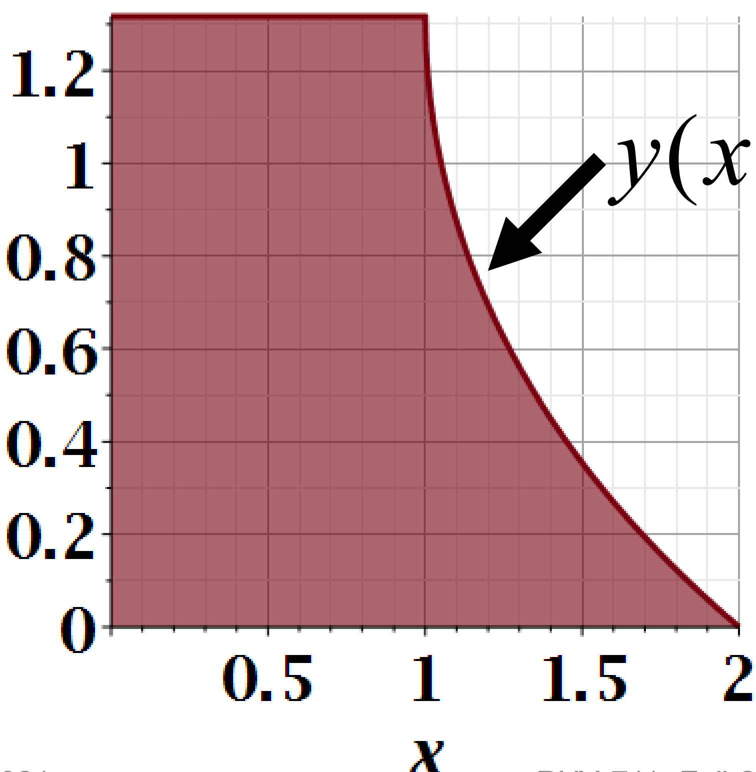
$$\frac{dy}{dx}=-\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2-1}}$$

$$\Rightarrow y(x)=K_2-K_1\ln\left(\frac{x}{K_1}+\sqrt{\frac{x^2}{K_1^2}-1}\right)$$

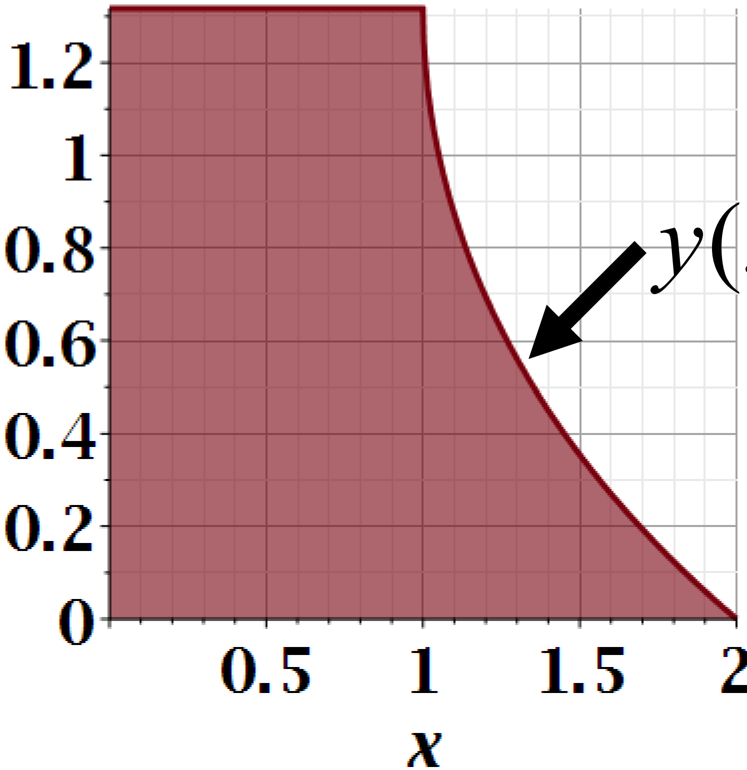
General form of solution --

$$y(x) = K_2 - K_1 \ln\left(\frac{x}{K_1} + \sqrt{\frac{x^2}{K_1^2} - 1}\right)$$

Suppose $K_1 = 1$ and $K_2 = \ln\left(2+\sqrt{3}\right)$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x+\sqrt{x^2-1}}\right)$$



$$y(x) = \ln\left(\frac{2+\sqrt{3}}{x+\sqrt{x^2-1}}\right)$$

$$A = 2\pi\int_1^2 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 15.02014144$$

(according to Maple)

Review : for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Euler-Lagrange equation

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Alternate Euler-Lagrange equation

A few more steps --

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right),$

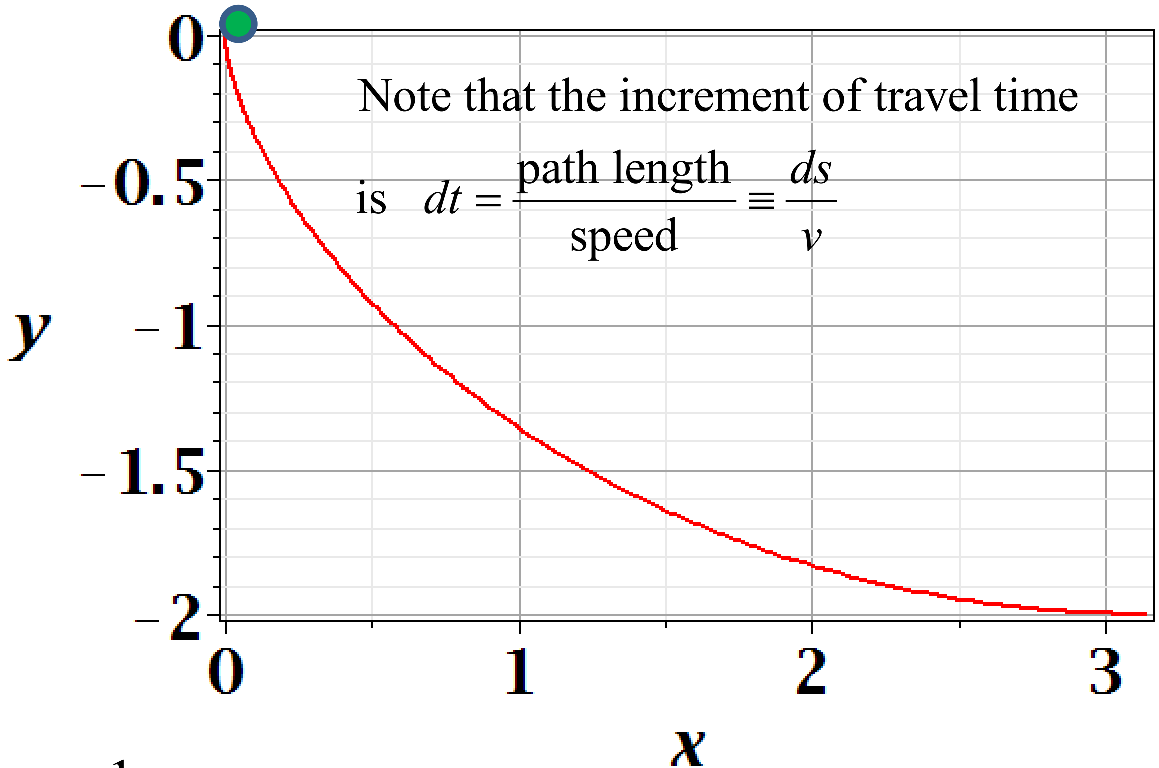
$$\frac{df}{dx}=\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx}+\left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx}+\left(\frac{\partial f}{\partial x}\right)$$

$$=\left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx}+\left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx}+\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html



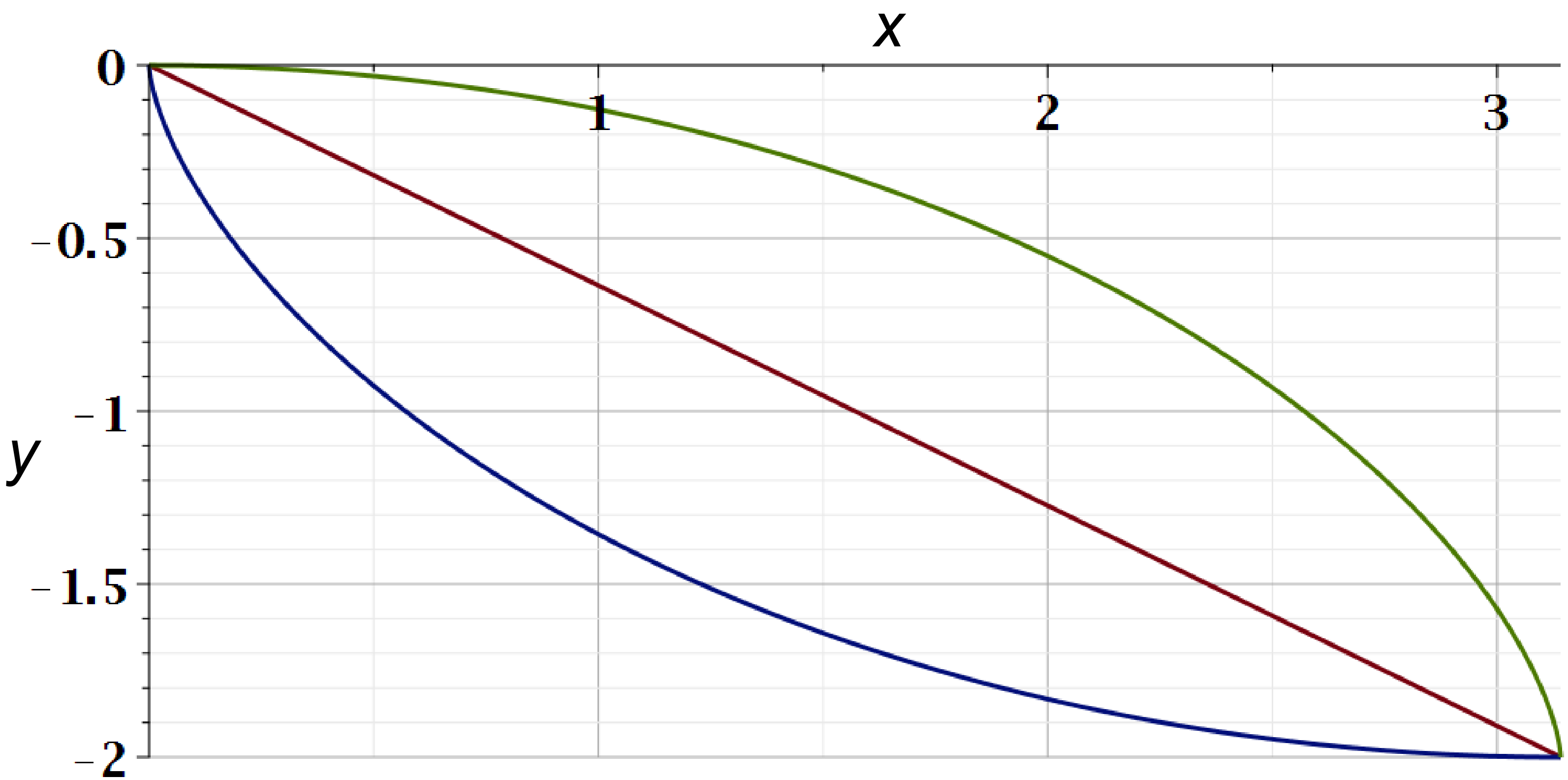
Note that the increment of travel time is $dt = \dfrac{\text{path length}}{\text{speed}} \equiv \dfrac{ds}{v}$

A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$?

$$E = \frac{1}{2}mv^2 + mgy$$

With the choice of initial conditions, $E = 0$

Vote for your favorite path



Which gives the shortest time?

a. Green
b. Red
c. Blue

$$T=\int_{x_i y_i}^{x_f y_f}\frac{ds}{v}=\int_{x_i}^{x_f}\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}dx \quad \text{because} \quad \tfrac{1}{2}mv^2=-mgy$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}}-\frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a$$

Question – why this choice?
Answer – because the answer will be more beautiful. (Be sure that was not my cleverness.)

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a$$

$$\frac{dy}{dx} = -\sqrt{\frac{2a}{-y}-1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}} = dx$$

Let $\quad y = -2a\sin^2\frac{\theta}{2} = a(\cos\theta - 1)$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}} = \frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}} = dx$$

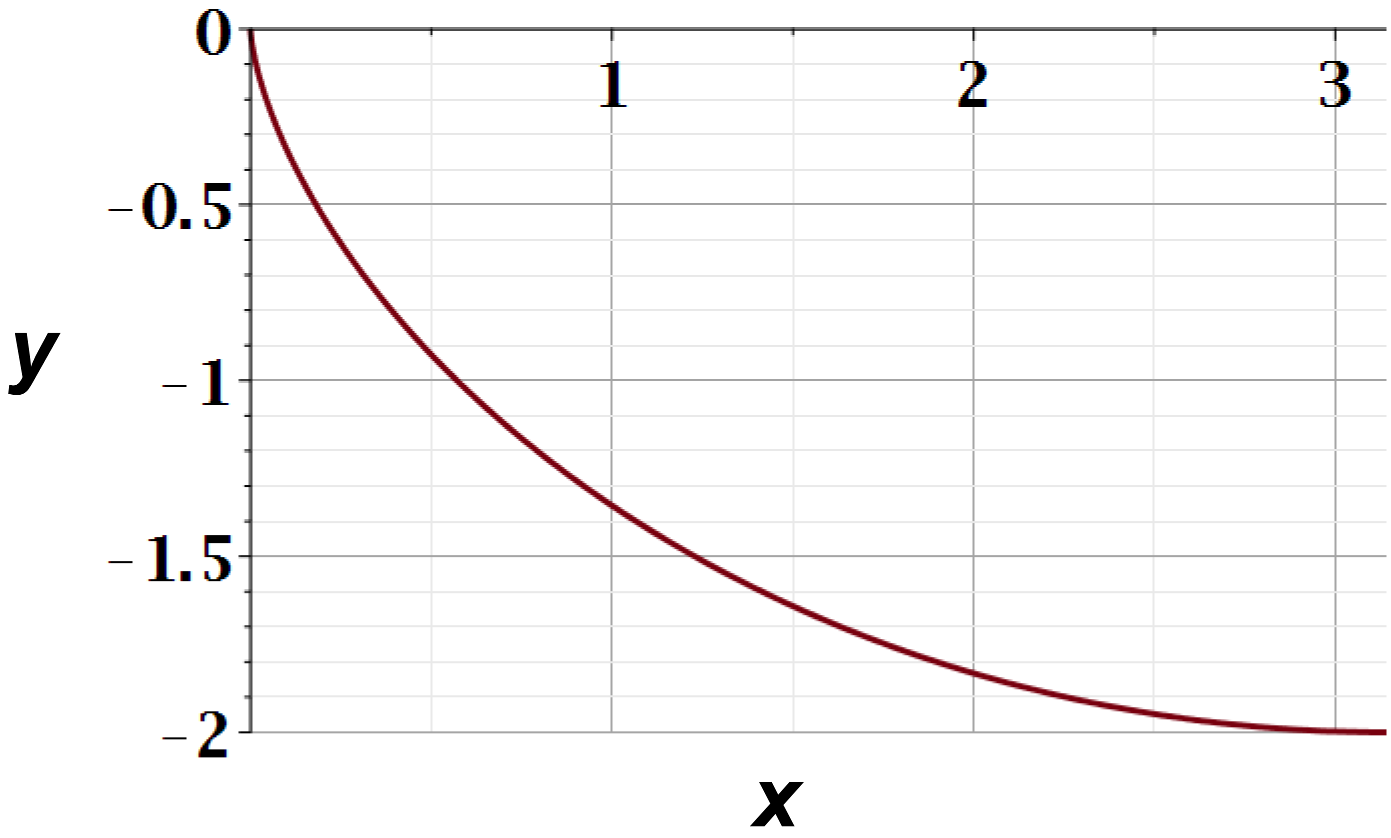
$$x = \int_0^\theta a(1-\cos\theta')d\theta' = a(\theta - \sin\theta)$$

Parametric equations for Brachistochrone:

$$x = a(\theta - \sin\theta)$$
$$y = a(\cos\theta - 1)$$

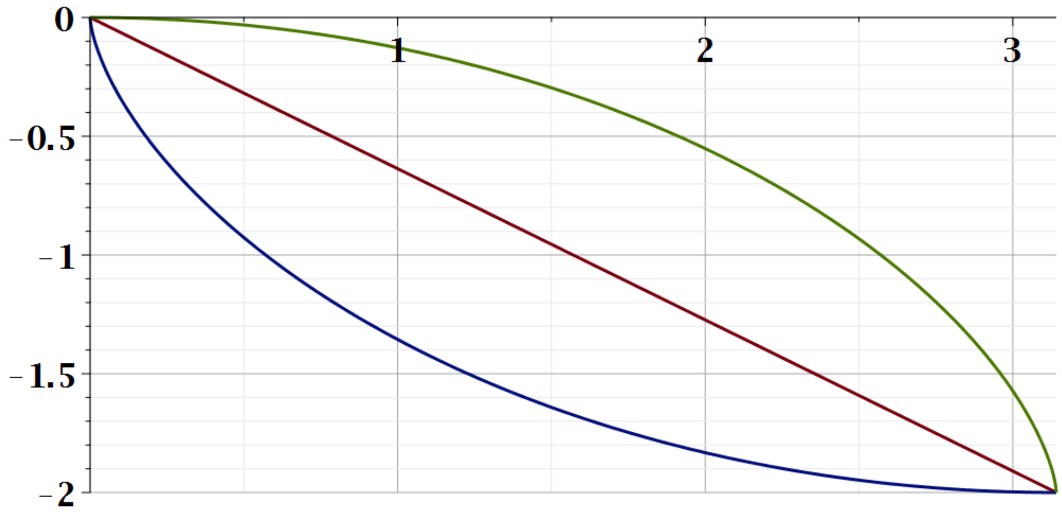
Parametric plot --

plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])



Checking the results

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}\, dx$$



T=infinite
T=5.2668
T=4.4429

$$\text{(units of } \frac{1}{\sqrt{(2g)}} \text{)}$$

Summary of the method of calculus of variation --

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Find the function $y(x)$ which extremizes $I\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

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$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Euler-Lagrange equation:

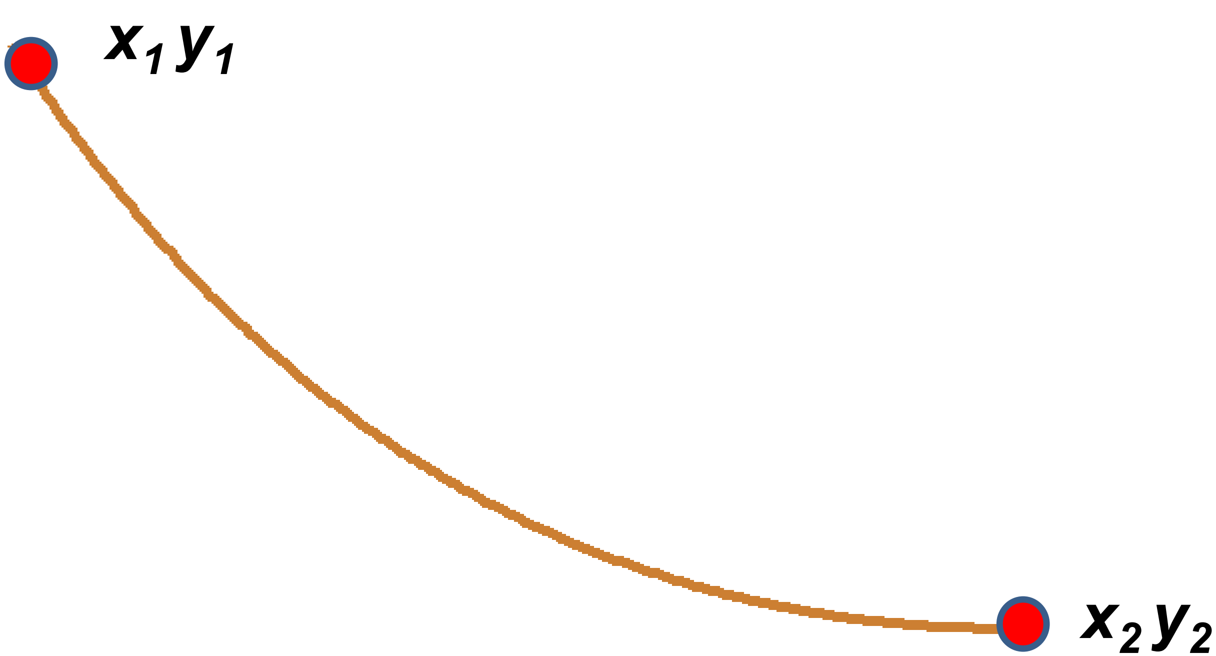
$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Alternate Euler-Lagrange equation:

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Another example optimization problem:

Determine the shape *y(x)* of a rope of length L and mass density ρ hanging between two points



Example from internet --

Potential energy of hanging rope :

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Length of rope :

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Define a composite function to minimize :

$$W \equiv E + \lambda L$$

Lagrange multiplier

$$\delta W = 0 = \delta E + \lambda \delta L \text{ for fixed } \lambda$$

is a very clever mathematical trick to help solve the minimization and constraint at the same time.

$$W = \int_{x_1}^{x_2} (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$f\left(\left\{y, \frac{dy}{dx}\right\}\right) = (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x} \right)$$

$$\Rightarrow (\rho g y + \lambda) \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) = K$$

$$(\rho g y+\lambda)\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}-\frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$(\rho g y+\lambda)\left(\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=K$$

$$y(x)=-\frac{1}{\rho g}\left(\lambda+K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

$$y(x) = -\frac{1}{\rho g}\left(\lambda + K\cosh\left(\frac{x-a}{K/\rho g}\right)\right)$$

Integration constants : K, a, λ

Constraints : $y(x_1) = y_1$

$$y(x_2) = y_2$$

$$\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = L$$

Summary of results

For the class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx \qquad \delta I = 0$$

A necessary condition is the Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial\left(dy/dx\right)}\right)\right] = 0$$

or $$\frac{d}{dx}\left(f - \frac{\partial f}{\partial\left(dy/dx\right)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Application to particle dynamics – next time --

$$x \to t \qquad \text{(time)}$$

$$y \to q \qquad \text{(generalized coordinate)}$$

$$f \to L \qquad \text{(Lagrangian)}$$

$$I \to A \text{ or } S \qquad \text{(action)}$$

Denote: $\dot{q} \equiv \frac{dq}{dt}$

$$A = \int_{t_1}^{t_2} L\left(\{q, \dot{q}\}; t\right) dt$$