



PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF in Olin 103

Discussion of Lecture 10 – Chap. 3&6 in F&W

Lagrangian mechanics

1. Lagrange's equations in the presence of velocity dependent potentials – such as electromagnetic interactions.
2. Effects of constraints

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/22/2022		Introduction	#1	8/26/2022
2	Wed, 8/24/2022	Chap. 1	Scattering theory		
3	Fri, 8/26/2022	Chap. 1	Scattering theory	#2	8/29/2022
4	Mon, 8/29/2022	Chap. 1	Scattering theory	#3	8/31/2022
5	Wed, 8/31/2022	Chap. 1	Summary of scattering theory	#4	9/02/2022
6	Fri, 9/02/2022	Chap. 2	Non-inertial coordinate systems	#5	9/05/2022
7	Mon, 9/05/2022	Chap. 3	Calculus of Variation	#6	9/7/2022
8	Wed, 9/07/2022	Chap. 3	Calculus of Variation	#7	9/9/2022
9	Fri, 9/09/2022	Chap. 3 & 6	Lagrangian Mechanics		
10	Mon, 9/12/2022	Chap. 3 & 6	Lagrangian Mechanics	#8	9/14/2022
11	Wed, 9/14/2022	Chap. 3 & 6	Constants of the motion		



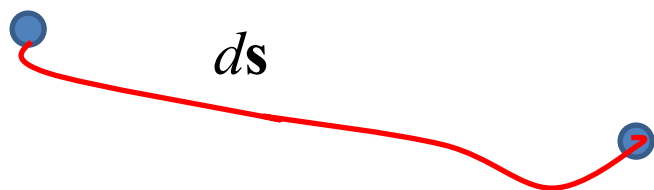
PHY 711 – Assignment #8

September 12, 2022

The material for this exercise is covered in the lecture notes and in Chapters 3 and 6 of Fetter and Walecka.

1. A particle of mass m and charge q is subjected to a vector potential $\mathbf{A}(\mathbf{r}, t) = -(E_0 ct + B_0 x)\hat{\mathbf{z}}$. (Note that we are using the cgs Gaussian units of your text book.) Here E_0 denotes a constant electric field amplitude and B_0 denotes a constant magnetic field amplitude. The initial particle position is $\mathbf{r}(0) = 0$ and the initial particle velocity is $\dot{\mathbf{r}}(0) = 0$.
 - (a) Determine the Lagrangian $L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$ which describes the particle's motion.
 - (b) Write the Euler-Lagrange equations for this system.
 - (c) Find and evaluate the constants of motion for this system.
 - (d) Find the particle trajectories $x(t)$, $y(t)$, $z(t)$ by solving the equations and imposing the given initial conditions.
 - (e) Determine the Hamiltonian for this system and evaluate dH/dt .

Previously derived form for the Lagrangian --



Generalized coordinates :

$$q_{\sigma}(\{x_i\})$$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = -\sum_{\sigma} \frac{\partial U}{\partial q_{\sigma}} \delta q_{\sigma} - \sum_{\sigma} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_{\sigma}} - \frac{\partial T}{\partial q_{\sigma}} \right) \delta q_{\sigma} = 0$$

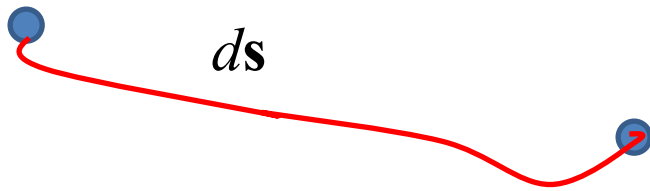
$$= -\sum_{\sigma} \left(\frac{d}{dt} \frac{\partial (T - U)}{\partial \dot{q}_{\sigma}} - \frac{\partial (T - U)}{\partial q_{\sigma}} \right) \delta q_{\sigma} = 0$$

$$= -\sum_{\sigma} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} - \frac{\partial L}{\partial q_{\sigma}} \right) \delta q_{\sigma} = 0$$

$$L(q_{\sigma}, \dot{q}_{\sigma}; t) = T - U$$

Note: This is only true if

$$\frac{\partial U}{\partial \dot{q}_{\sigma}} = 0$$



Generalized coordinates :
 $q_\sigma(\{x_i\})$

Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

$$(\mathbf{F}-m\mathbf{a}) \cdot d\mathbf{s} = - \sum_\sigma \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

$$\Rightarrow \text{Minimization integral: } S = \int_{t_i}^{t_f} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$$

→ Hamilton's principle from the "backwards" application of the Euler-Lagrange equations to

Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$



Summary –

Hamilton's principle:

Given the Lagrangian function: $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U,$

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$ are those which minimize the action: $S = \int L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$

Euler-Lagrange equations:

$$\sum_{\sigma} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} - \frac{\partial L}{\partial q_{\sigma}} \right) \delta q_{\sigma} = 0 \quad \Rightarrow \text{for each } \sigma : \quad \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} - \frac{\partial L}{\partial q_{\sigma}} \right) = 0$$

Note: in “proof” of Hamilton’s principle:

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0 \quad \text{for} \quad L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$$

It was necessary to assume that :

$\frac{d}{dt} \frac{\partial U}{\partial \dot{q}_\sigma}$ does not contribute to the result.

⇒ How can we represent velocity-dependent forces?

Why do we need velocity dependent forces?

- a. Friction is sometimes represented as a velocity dependent force. (difficult to treat with Lagrangian mechanics.)
- b. Lorentz force on a moving charged particle in the presence of a magnetic field.

Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B}(\mathbf{r}, t)$:

$$\text{Lorentz force: } \mathbf{F} = q\left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}\right)$$

$$x\text{-component: } F_x = q\left(E_x + \frac{1}{c} (\mathbf{v} \times \mathbf{B})_x\right)$$

In this case, it is convenient to use cartesian coordinates

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$x\text{-component: } \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \right) = 0$$

$$\text{Apparently: } F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}$$

$$\text{Answer: } U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Note: Here we are using cartesian coordinates for convenience.

$$\text{where } \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

Units for electromagnetic fields and forces

cgs Gaussian units -- (as used your textbook)

E and **B** fields as related to vector and scalar potentials:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

Corresponding Lagrangian potential:

$$U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

SI units --

E and **B** fields as related to vector and scalar potentials:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

Corresponding Lagrangian potential:

$$U = q\Phi(\mathbf{r}, t) - q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Lorentz forces, continued:

x – component of Lorentz force: $F_x = q\left(E_x + \frac{1}{c}(\mathbf{v} \times \mathbf{B})_x\right)$

Suppose: $U = q\Phi(\mathbf{r}, t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$

Consider: $F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}$

$$-\frac{\partial U}{\partial x} = -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \left(\dot{x} \frac{\partial A_x(\mathbf{r}, t)}{\partial x} + \dot{y} \frac{\partial A_y(\mathbf{r}, t)}{\partial x} + \dot{z} \frac{\partial A_z(\mathbf{r}, t)}{\partial x} \right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} A_x(\mathbf{r}, t)$$

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} \frac{dA_x(\mathbf{r}, t)}{dt} = -\frac{q}{c} \left(\frac{\partial A_x(\mathbf{r}, t)}{\partial x} \dot{x} + \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \dot{y} + \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \dot{z} + \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \right)$$

Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \left(\dot{x} \frac{\partial A_x(\mathbf{r}, t)}{\partial x} + \dot{y} \frac{\partial A_y(\mathbf{r}, t)}{\partial x} + \dot{z} \frac{\partial A_z(\mathbf{r}, t)}{\partial x} \right)$$

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} \left(\frac{\partial A_x(\mathbf{r}, t)}{\partial x} \dot{x} + \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \dot{y} + \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \dot{z} + \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \right)$$

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}} \\ &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \dot{y} \left(\frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left(\frac{\partial A_z(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \right) - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \\ &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} + \frac{q}{c} \dot{y} \left(\frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left(\frac{\partial A_z(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \right) \\ &= qE_x(\mathbf{r}, t) + \frac{q}{c} (\dot{y}B_z(\mathbf{r}, t) - \dot{z}B_y(\mathbf{r}, t)) = qE_x(\mathbf{r}, t) + \frac{q}{c} (\mathbf{v} \times \mathbf{B}(\mathbf{r}, t))_x \end{aligned}$$

Some details on last step:

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}} \\ &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} + \frac{q}{c} \dot{y} \left(\frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left(\frac{\partial A_z(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} \\ &= -q \frac{\partial \Phi(\mathbf{r}, t)}{\partial x} - \frac{q}{c} \frac{\partial A_x(\mathbf{r}, t)}{\partial t} + \frac{q}{c} \dot{y} \left(\frac{\partial A_y(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial y} \right) + \frac{q}{c} \dot{z} \left(\frac{\partial A_z(\mathbf{r}, t)}{\partial x} - \frac{\partial A_x(\mathbf{r}, t)}{\partial z} \right) \end{aligned}$$

Note that: $\mathbf{E}(\mathbf{r}, t) = -\nabla \Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}$ $\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$

So that:

$$F_x(\mathbf{r}, t) = qE_x(\mathbf{r}, t) + \frac{q}{c} (\dot{y}B_z(\mathbf{r}, t) - \dot{z}B_y(\mathbf{r}, t)) = qE_x(\mathbf{r}, t) + \frac{q}{c} (\mathbf{v} \times \mathbf{B}(\mathbf{r}, t))_x$$

Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad U = q\Phi(\mathbf{r}, t) - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{where } \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Example Lorentz force

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Suppose $\mathbf{E}(\mathbf{r}, t) \equiv 0$, $\mathbf{B}(\mathbf{r}, t) \equiv B_0 \hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{2} B_0 (-y \hat{\mathbf{x}} + x \hat{\mathbf{y}})$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \frac{d}{dt} \left(m\dot{x} - \frac{q}{2c} B_0 y \right) - \frac{q}{2c} B_0 \dot{y} = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \quad \Rightarrow \quad \frac{d}{dt} \left(m\dot{y} + \frac{q}{2c} B_0 x \right) + \frac{q}{2c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \quad \Rightarrow \quad \frac{d}{dt} m\dot{z} = 0$$

Example Lorentz force -- continued

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$\frac{d}{dt} \left(m\dot{x} - \frac{q}{2c} B_0 y \right) - \frac{q}{2c} B_0 \dot{y} = 0 \quad \Rightarrow \quad m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt} \left(m\dot{y} + \frac{q}{2c} B_0 x \right) + \frac{q}{2c} B_0 \dot{x} = 0 \quad \Rightarrow \quad m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \quad \Rightarrow \quad m\ddot{z} = 0$$



Example Lorentz force -- continued

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$m\ddot{x} = +\frac{q}{c} B_0 \dot{y}$$

$$m\ddot{y} = -\frac{q}{c} B_0 \dot{x}$$

$$m\ddot{z} = 0$$

Note that same equations are obtained from direct application of Newton's laws :

$$m\ddot{\mathbf{r}} = \frac{q}{c} \dot{\mathbf{r}} \times B_0 \hat{\mathbf{z}}$$

Example Lorentz force -- continued

Evaluation of equations :

$$m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0 \quad \dot{x}(t) = V_0 \sin\left(\frac{qB_0}{mc}t + \phi\right)$$

$$m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0 \quad \dot{y}(t) = V_0 \cos\left(\frac{qB_0}{mc}t + \phi\right)$$

$$m\ddot{z} = 0 \quad \dot{z}(t) = V_{0z}$$

$$x(t) = x_0 - \frac{mc}{qB_0} V_0 \cos\left(\frac{qB_0}{mc}t + \phi\right)$$

$$y(t) = y_0 + \frac{mc}{qB_0} V_0 \sin\left(\frac{qB_0}{mc}t + \phi\right)$$

$$z(t) = z_0 + V_{0z}t$$

Example Lorentz force -- continued

Consider formulation with different Gauge: $\mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c} B_0 \dot{x} y$$

$$\frac{d}{dt} \left(m \dot{x} - \frac{q}{c} B_0 y \right) = 0 \quad \Rightarrow \quad m \ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt} (m \dot{y}) + \frac{q}{c} B_0 \dot{x} = 0 \quad \Rightarrow \quad m \ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m \dot{z} = 0 \quad \Rightarrow \quad m \ddot{z} = 0$$

Does it surprise you that the same equations of motion are obtained with a different Gauge?

How do these two different forms of \mathbf{A} correspond to the same \mathbf{B} ?

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

Consider $\mathbf{A}'(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) + \nabla f(\mathbf{r}, t)$

Note that $\nabla \times \mathbf{A}(\mathbf{r}, t) = \nabla \times \mathbf{A}'(\mathbf{r}, t)$

In our case, $\mathbf{A}(\mathbf{r}, t) = \frac{1}{2} B_0 (-y \hat{\mathbf{x}} + x \hat{\mathbf{y}})$

$$\mathbf{A}'(\mathbf{r}, t) = -B_0 y \hat{\mathbf{x}}$$

What is $f(\mathbf{r}, t)$?

Now consider formulation of motion with constraints --
Comments on generalized coordinates:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Here we have assumed that the generalized coordinates q_σ are independent. Now consider the possibility that the coordinates are related through constraint equations of the form:

Lagrangian: $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$

Constraints: $f_j = f_j(\{q_\sigma(t)\}, t) = 0$

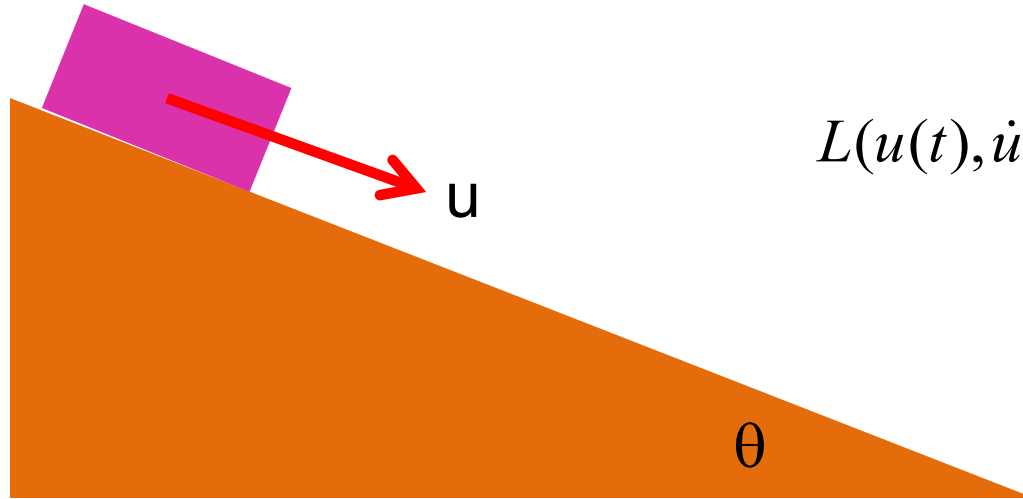
Modified Euler - Lagrange equations: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

Lagrange
multipliers

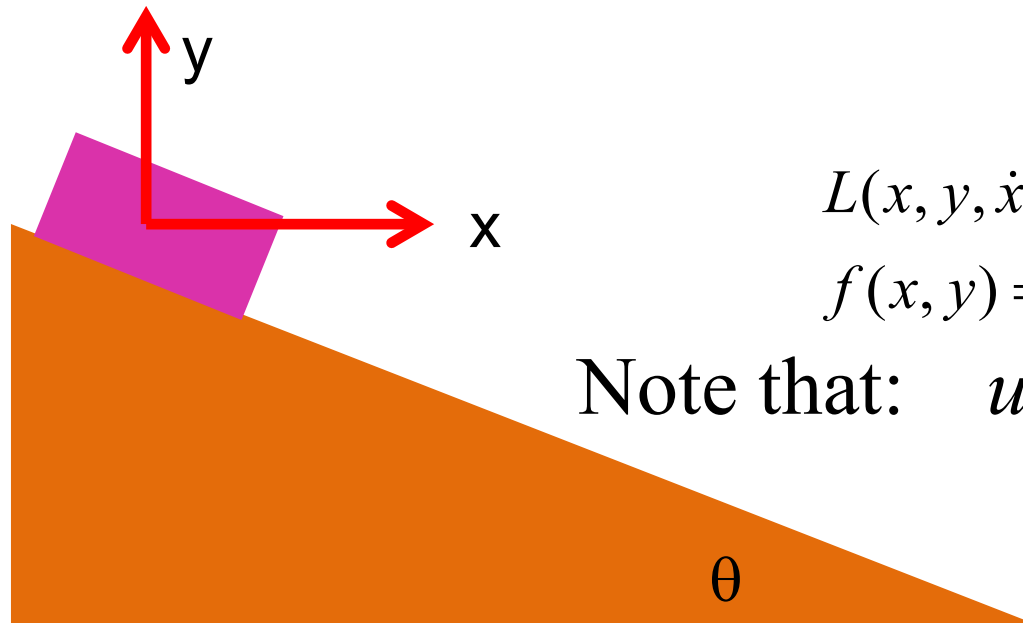




Simple example:



$$L(u(t), \dot{u}(t)) = \frac{1}{2} m \dot{u}^2 + m g u \sin \theta$$



$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - m g y$$

$$f(x, y) = \sin \theta x + \cos \theta y = 0$$

Note that: $u = x \cos \theta - y \sin \theta$

Case 1:

$$L(u(t), \dot{u}(t)) = \frac{1}{2} m \dot{u}^2 + m g u \sin \theta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 = m \ddot{u} - m g \sin \theta = 0$$

$$\Rightarrow \ddot{u} = g \sin \theta$$

Case 2:

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - m g y$$

$$f(x, y) = \sin \theta x + \cos \theta y = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x} = 0 = m \ddot{x} + \lambda \sin \theta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda \frac{\partial f}{\partial y} = 0 = m \ddot{y} + m g + \lambda \cos \theta$$

$$\sin \theta \ddot{x} + \cos \theta \ddot{y} = 0$$

$$\Rightarrow \lambda = -m g \cos \theta$$

$$(\cos \theta \ddot{x} - \sin \theta \ddot{y}) = g \sin \theta$$

Which method would you use to solve the problem?

Case 1

Case 2

Force of constraint;
normal to incline

Rational for Lagrange multipliers

Recall Hamilton's principle:

$$S = \int_{t_i}^{t_f} L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t) dt$$

$$\delta S = 0 = \int_{t_i}^{t_f} \left(\sum_{\sigma} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma \right) dt$$

With constraints: $f_j = f_j(\{q_\sigma(t)\}, t) = 0$

Variations δq_σ are no longer independent.

$$\delta f_j = 0 = \sum_{\sigma} \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma \quad \text{at each } t$$

\Rightarrow Add 0 to Euler-Lagrange equations in the form:

$$\sum_j \lambda_j \sum_{\sigma} \frac{\partial f_j}{\partial q_\sigma} \delta q_\sigma$$

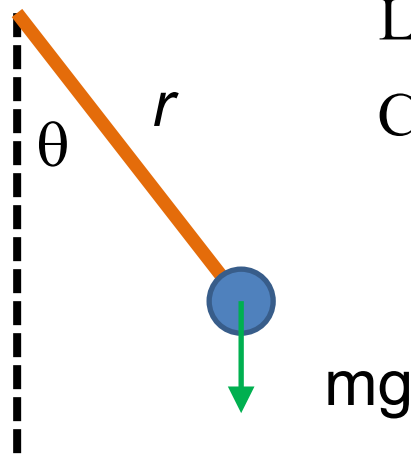
Euler-Lagrange equations with constraints:

Lagrangian : $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$

Constraints : $f_j = f_j(\{q_\sigma(t)\}, t) = 0$

Modified Euler - Lagrange equations : $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

Example:



Lagrangian : $L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + mgr \cos \theta$

Constraints : $f = r - \ell = 0$

Example continued:

$$\text{Lagrangian: } L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + mgr \cos \theta$$

$$\text{Constraints: } f = r - \ell = 0$$

$$\frac{d}{dt} m \dot{r} - m r \dot{\theta}^2 - mg \cos \theta + \lambda = 0$$

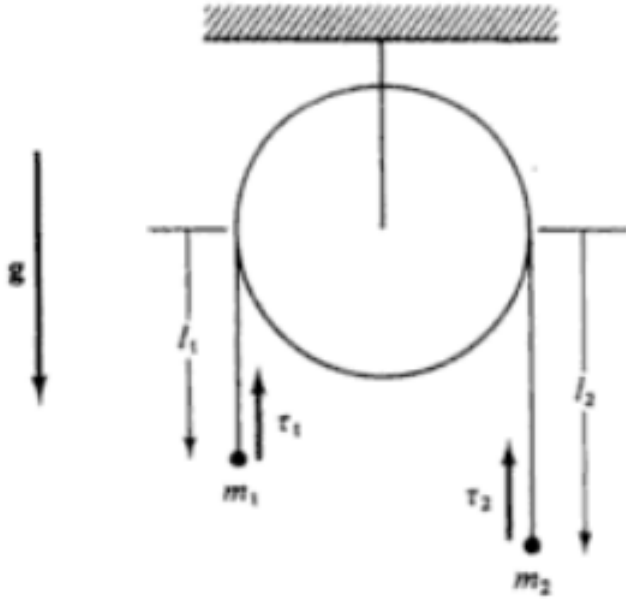
$$\frac{d}{dt} m r^2 \dot{\theta} + mgr \sin \theta = 0$$

$$\dot{r} = 0 = \ddot{r} \quad r = \ell$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{\ell} \sin \theta$$

$$\Rightarrow \lambda = m \ell \dot{\theta}^2 + mg \cos \theta$$

Another example:



Lagrangian: $L = \frac{1}{2} m_1 \dot{\ell}_1^2 + \frac{1}{2} m_2 \dot{\ell}_2^2 + m_1 g \ell_1 + m_2 g \ell_2$

Constraints: $f = \ell_1 + \ell_2 - \ell = 0$

$$\frac{d}{dt} m_1 \dot{\ell}_1 - m_1 g + \lambda = 0$$

$$\frac{d}{dt} m_2 \dot{\ell}_2 - m_2 g + \lambda = 0$$

$$\dot{\ell}_1 + \dot{\ell}_2 = 0 = \ddot{\ell}_1 + \ddot{\ell}_2$$

Figure 19.1 Atwood's machine.

$$\Rightarrow \lambda = \frac{2m_1 m_2}{m_1 + m_2} g$$

$$\ddot{\ell}_1 = -\ddot{\ell}_2 = \frac{m_1 - m_2}{m_1 + m_2} g$$