PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion of Lecture 11-- Chap. 3 & 6 (F &W)

Details and extensions of Lagrangian mechanics

1. **Constants of the motion**
2. **Conserved quantities**
3. **Legendre transformations**

Physics Colloquium

Thursday

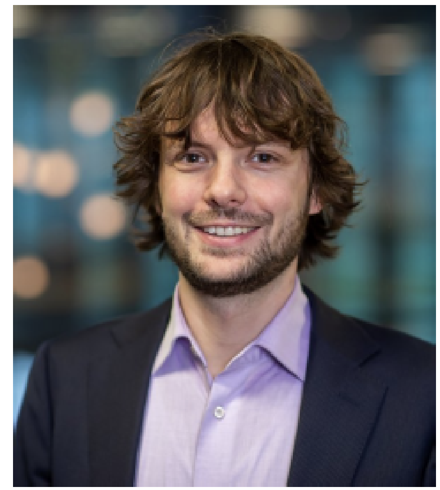
4PM in Olin 101
and by zoom

September 15, 2022

Organic Neuromorphic Electronics and Biohybrid Systems

Neuromorphic computing could address the inherent limitations of conventional silicon technology in dedicated machine learning applications. Recent work on large crossbar arrays of two-terminal memristive devices has led to the development of promising neuromorphic systems. However, delivering a compact and efficient parallel computing technology that is capable of embedding artificial neural networks in hardware remains a significant challenge.

Organic electronic materials have shown potential to overcome some of these limitations. This talk describes state-of-the-art organic neuromorphic devices and provides an overview of the current challenges in the field and attempts to address them. I demonstrate a concept based on novel organic mixed-ionic



Yoeri van de Burgt

Professor,
Eindhoven University of Technology,
The Netherlands

4:00 pm - Olin 101*

*Link provided for those unable to attend in person.

Note: For additional information on the seminar

Your questions –

From Zezong -- What are the differences in physical meanings of Langrangian and Hamiltonian pictures?

From Lee --What are the mathematical rules for treating a system with a non-conservative potential while using the Euler-Lagrange equations and Hamilton's Principle? In Newtonian (non-variational) language, I understand "conservative" for a potential to mean spatial dependence only. However, from the formalism in class and the book using generalized coordinates, it seems that "conservative" narrows to mean no velocity dependence. Are these two notions one and the same, only appearing different after the change to generalized coordinates? Is the more general setup that would allow treatment of all non-conservative potentials Eq. 15.17 and 15.3 from the book? Or would this approach be exactly the same as if we simply construct the potential U from nonconservative forces then build the Lagrangian and use Euler-Lagrange?

Course schedule

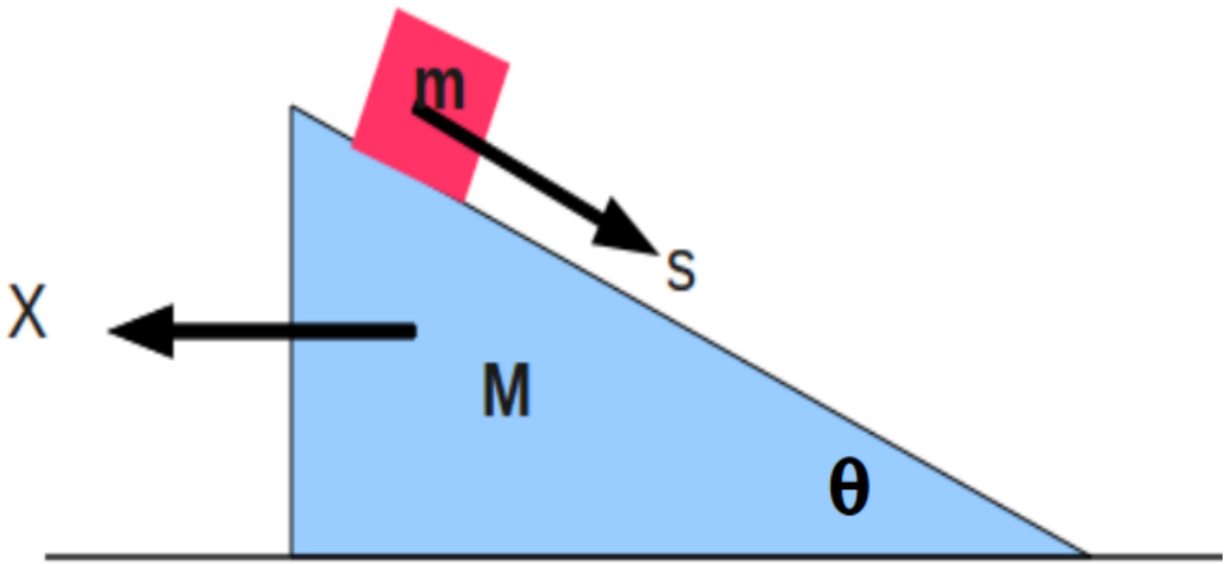
(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/22/2022		Introduction	#1	8/26/2022
2	Wed, 8/24/2022	Chap. 1	Scattering theory		
3	Fri, 8/26/2022	Chap. 1	Scattering theory	#2	8/29/2022
4	Mon, 8/29/2022	Chap. 1	Scattering theory	#3	8/31/2022
5	Wed, 8/31/2022	Chap. 1	Summary of scattering theory	#4	9/02/2022
6	Fri, 9/02/2022	Chap. 2	Non-inertial coordinate systems	#5	9/05/2022
7	Mon, 9/05/2022	Chap. 3	Calculus of Variation	#6	9/7/2022
8	Wed, 9/07/2022	Chap. 3	Calculus of Variation	#7	9/9/2022
9	Fri, 9/09/2022	Chap. 3 & 6	Lagrangian Mechanics		
10	Mon, 9/12/2022	Chap. 3 & 6	Lagrangian Mechanics	#8	9/14/2022
11	Wed, 9/14/2022	Chap. 3 & 6	Constants of the motion	#9	9/16/2022
12	Fri, 9/16/2022	Chap. 3 & 6	Hamiltonian equations of motion		

PHY 711 - Assignment #9

Sept. 14, 2022

Continue reading Chapters 3 and 6 in **Fetter and Walecka**.



1. The figure above shows a box of mass m sliding on the frictionless surface of an inclined plane (angle θ). The inclined plane itself has a mass M and is supported on a horizontal frictionless surface. Write down the Lagrangian for this system in terms of the generalized coordinates X and s and the fixed constants of the system (θ, m, M, etc.) and solve for the equations of motion, assuming that the system is initially at rest. (Note that X and s represent components of vectors whose directions are related by the angle θ.)

Summary of Lagrangian formalism (without constraints)

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} = \text{(constant)}$$

Comment -- Note that in deriving these equations we have assumed that there are only conservative forces (no friction) acting on this system. The equations are easily modified to take constraints, including those from static friction, into account. What is harder to treat is dynamic friction which is typically modeled by velocity dependent dissipative forces. However, some tricks for this have been developed such as described in the textbook by Herbert Goldstein.

Examples of constants of the motion:

Example 1: one-dimensional potential:

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$\Rightarrow \frac{d}{dt} m\dot{x} = 0 \qquad \Rightarrow m\dot{x} \equiv p_x \quad \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{y} = 0 \qquad \Rightarrow m\dot{y} \equiv p_y \quad \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{z} = -\frac{\partial V}{\partial z}$$

Examples of constants of the motion:

Example 2: Motion in a central potential

$$L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)$$

$$\Rightarrow \frac{d}{dt} m r^2 \dot{\varphi} = 0 \qquad \Rightarrow m r^2 \dot{\varphi} \equiv p_\varphi \quad \text{(constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{r} = m r \dot{\varphi}^2 - \frac{\partial V}{\partial r} = \frac{{p_\varphi}^2}{m r^3} - \frac{\partial V}{\partial r}$$

Recall alternative form of Euler-Lagrange equations:

Starting from:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Also note that: $\displaystyle \frac{dL}{dt} = \sum_\sigma \frac{\partial L}{\partial q_\sigma}\dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\ddot{q}_\sigma + \frac{\partial L}{\partial t}$

$$= \sum_\sigma \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma + \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\ddot{q}_\sigma + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) = \frac{\partial L}{\partial t}$$

Additional constant of the motion:

$$\text{If} \quad \frac{\partial L}{\partial t} = 0;$$

$$\text{then:} \quad \frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) = \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma = -E \quad \text{(constant)}$$

Example 1: one-dimensional potential:

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$\Rightarrow \frac{d}{dt}\left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z) - m\dot{x}^2 - m\dot{y}^2 - m\dot{z}^2\right) = 0$$

$$\Rightarrow -\left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + V(z)\right) = -E \quad \text{(constant)}$$

For this case, we also have $m\dot{x} \equiv p_x$ and $m\dot{y} \equiv p_y$

$$\Rightarrow E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \tfrac{1}{2} m\dot{z}^2 + V(z)$$

Summary from previous slide

$$L = \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - V(z)$$

➔3 variable functions

$$E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \tfrac{1}{2} m\dot{z}^2 + V(z)$$

p_x, p_y, E constant

➔1 variable function

Why might this be useful?

Additional constant of the motion -- continued:

If $\frac{\partial L}{\partial t} = 0;$

$$\text{then: } \frac{d}{dt}\left(L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right) = \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow L - \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma = -E \quad \text{(constant)}$$

Example 2: Motion in a central potential

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$

➔2 variable functions

$$\Rightarrow \frac{d}{dt}\left(\tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r) - m\dot{r}^2 - mr^2\dot{\phi}^2\right) = 0$$

$$\Rightarrow -\left(\tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)\right) = -E \quad \text{(constant)}$$

For this case, we also have $mr^2\dot{\varphi} \equiv p_\varphi$

$$\Rightarrow E = \frac{p_\varphi^2}{2mr^2} + \tfrac{1}{2}m\dot{r}^2 + V(r)$$

➔1 variable function

Other examples

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c}B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{\partial L}{\partial z} = 0 \qquad \Rightarrow m\dot{z} = p_z \quad \text{(constant)}$$

$$E = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\dot{q}_\sigma - L$$

$$= m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c}B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$-\tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{2c}B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\Rightarrow E = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{p_z^2}{2m}$$

Other examples

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c} B_0 \dot{x} y$$

$$\frac{\partial L}{\partial z} = 0 \qquad \Rightarrow m\dot{z} = p_z \quad \text{(constant)}$$

$$\frac{\partial L}{\partial x} = 0 \qquad \Rightarrow m\dot{x} = p_x \quad \text{(constant)}$$

$$E = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma - L$$

$$= m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q}{c} B_0 \dot{x} y$$

$$- \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{c} B_0 \dot{x} y$$

$$\Rightarrow E = \tfrac{1}{2} m\dot{y}^2 + \frac{p_x^2}{2m} + \frac{p_z^2}{2m}$$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Mathematical transformations for continuous functions of several variables & Legendre transforms:

Simple change of variables:

$$z(x,y) \quad \Leftrightarrow \quad x(y,z)???$$

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\text{But:} \quad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\left(\partial z / \partial y\right)_x}{\left(\partial z / \partial x\right)_y}$$

Assuming $dz=0$.

Note on notation for partial derivatives

$$z(x,y) \quad \Rightarrow \quad dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

hold *y* fixed.

hold *x* fixed.

Simple change of variables -- continued:

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\left(\partial z / \partial y\right)_x}{\left(\partial z / \partial x\right)_y} \qquad \Rightarrow \left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{\left(\partial z / \partial x\right)_y}$$

Simple change of variables -- continued:

Example:

$$z(x,y)=e^{x^2+y}$$

$$x(y,z)=\left(\ln z-y\right)^{1/2}$$

$$z(x,y)\Rightarrow dz=\left(\frac{\partial z}{\partial x}\right)_y dx+\left(\frac{\partial z}{\partial y}\right)_x dy$$

$$x(y,z)\Rightarrow dx=\left(\frac{\partial x}{\partial y}\right)_z dy+\left(\frac{\partial x}{\partial z}\right)_y dz$$

$$\left(\frac{\partial x}{\partial y}\right)_z \overset{?}{=} -\frac{\left(\partial z/\partial y\right)_x}{\left(\partial z/\partial x\right)_y} \qquad \left(\frac{\partial x}{\partial z}\right)_y \overset{?}{=} \frac{1}{\left(\partial z/\partial x\right)_y}$$

$$-\frac{1}{2\left(\ln z-y\right)^{1/2}} \overset{\checkmark}{=} -\frac{e^{x^2+y}}{2xe^{x^2+y}} \qquad \frac{1}{2z\left(\ln z-y\right)^{1/2}} \overset{\checkmark}{=} \frac{1}{2xe^{x^2+y}}$$

Now that we see that these transformations are possible, we should ask the question why we might want to do this?

An example comes from thermodynamics where we have various interdependent variables such as temperature T, pressure P, volume V, etc. etc. Often a measurable property can be specified as a function of two of those, while the other variables are also dependent on those two. For example we might specify T and P while the volume will be V(T,P). Or we might specify T and V while the pressure will be P(T,V).

Other examples from thermo --
For thermodynamic functions:

$$\text{Internal energy:} \qquad U = U(S,V)$$

$$dU = TdS - PdV$$

$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$

$$\Rightarrow T = \left(\frac{\partial U}{\partial S}\right)_V \qquad P = -\left(\frac{\partial U}{\partial V}\right)_S$$

$$\text{Enthalpy:} \qquad H = H(S,P) = U + PV$$

$$dH = dU + PdV + VdP = TdS + VdP = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP$$

$$\Rightarrow T = \left(\frac{\partial H}{\partial S}\right)_P \qquad V = \left(\frac{\partial H}{\partial P}\right)_S$$

Name	Potential	Differential Form
Internal energy	$E(S,V,N)$	$dE = TdS - PdV + \mu dN$
Entropy	$S(E,V,N)$	$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$
Enthalpy	$H(S,P,N) = E + PV$	$dH = TdS + VdP + \mu dN$
Helmholtz free energy	$F(T,V,N) = E - TS$	$dF = -SdT - PdV + \mu dN$
Gibbs free energy	$G(T,P,N) = F + PV$	$dG = -SdT + VdP + \mu dN$
Landau potential	$\Omega(T,V,\mu) = F - \mu N$	$d\Omega = -SdT - PdV - Nd\mu$

Mathematical transformations for continuous functions of several variables & Legendre transforms continued:

$$z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

$$\text{Let} \quad u \equiv \left(\frac{\partial z}{\partial x}\right)_y \quad \text{and} \quad v \equiv \left(\frac{\partial z}{\partial y}\right)_x$$

Define new function

$$w(u,y) \Rightarrow dw = \left(\frac{\partial w}{\partial u}\right)_y du + \left(\frac{\partial w}{\partial y}\right)_u dy$$

$$\text{For} \quad w = z - ux, \qquad dw = dz - udx - xdu = udx + vdy - udx - xdu$$

$$dw = -xdu + vdy$$

$$\Rightarrow \left(\frac{\partial w}{\partial u}\right)_y = -x \qquad \left(\frac{\partial w}{\partial y}\right)_u = \left(\frac{\partial z}{\partial y}\right)_x = v$$

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Hamiltonian picture – continued

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$