PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion of Lecture 12 – Chap. 3&6 (F&W)

1. Constructing the Hamiltonian
2. Hamilton's canonical equation
3. Examples

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/22/2022		Introduction	#1	8/26/2022
2	Wed, 8/24/2022	Chap. 1	Scattering theory		
3	Fri, 8/26/2022	Chap. 1	Scattering theory	#2	8/29/2022
4	Mon, 8/29/2022	Chap. 1	Scattering theory	#3	8/31/2022
5	Wed, 8/31/2022	Chap. 1	Summary of scattering theory	#4	9/02/2022
6	Fri, 9/02/2022	Chap. 2	Non-inertial coordinate systems	#5	9/05/2022
7	Mon, 9/05/2022	Chap. 3	Calculus of Variation	#6	9/7/2022
8	Wed, 9/07/2022	Chap. 3	Calculus of Variation	#7	9/9/2022
9	Fri, 9/09/2022	Chap. 3 & 6	Lagrangian Mechanics		
10	Mon, 9/12/2022	Chap. 3 & 6	Lagrangian Mechanics	#8	9/14/2022
11	Wed, 9/14/2022	Chap. 3 & 6	Constants of the motion	#9	9/16/2022
12	Fri, 9/16/2022	Chap. 3 & 6	Hamiltonian equations of motion		

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

Application of the Legendre transformation for the Lagrangian and Hamiltonian

$L(q,\dot{q},t)$ and $H(q,p,t)$

suppose $H(q,p,t) = \dot{q}p - L(q,\dot{q},t)$

$$dH = \dot{q}dp + pd\dot{q} - \left(\frac{\partial L}{\partial q}\right)dq - \left(\frac{\partial L}{\partial \dot{q}}\right)d\dot{q} - \left(\frac{\partial L}{\partial t}\right)dt = \left(\frac{\partial H}{\partial q}\right)dq + \left(\frac{\partial H}{\partial p}\right)dp + \left(\frac{\partial H}{\partial t}\right)dt$$

Note that these two terms cancel if $p = \dfrac{\partial L}{\partial \dot{q}}$

The analysis on the following slides is a generalization to multiple dimensions q_σ and p_σ ….

Hamiltonian picture – continued

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

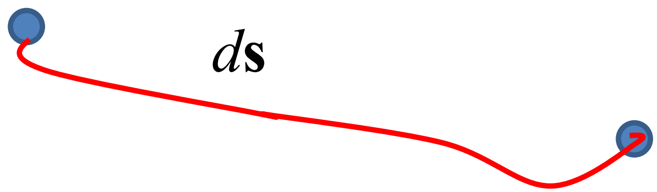
$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + \cancel{p_\sigma d\dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \cancel{\frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma} \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\dot{q}_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$dH = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \qquad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

Direct application of Hamiltonian's principle using the Hamiltonian function --



Generalized coordinates :

$q_\sigma(\{x_i\})$

Define -- Lagrangian: $L \equiv T - U$

$$L = L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right)$$

$$\Rightarrow \text{ Minimization integral: } \quad S = \int_{t_i}^{t_f} L\left(\{q_\sigma\},\{\dot{q}_\sigma\},t\right) dt$$

Expressed in terms of Hamiltonian:

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \qquad \Rightarrow L = \sum_\sigma \dot{q}_\sigma p_\sigma - H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

Hamilton's principle continued:

Minimization integral:

$$S = \int_{t_i}^{t_f} \left(\sum_\sigma \dot{q}_\sigma p_\sigma - H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right) \right) dt$$

$$\delta S = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\dot{q}_\sigma \delta p_\sigma + \delta\dot{q}_\sigma p_\sigma - \frac{\partial H}{\partial q_\sigma}\delta q_\sigma - \frac{\partial H}{\partial p_\sigma}\delta p_\sigma \right)\right) dt = 0$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$\Rightarrow \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Canonical equations

Detail:

$$\int_{t_i}^{t_f} \left(\sum_\sigma \left(\delta\dot{q}_\sigma p_\sigma\right)\right) dt = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d\left(\delta q_\sigma p_\sigma\right)}{dt} - \delta q_\sigma \dot{p}_\sigma \right)\right) dt = \overbrace{\cancel{\sum_\sigma \delta q_\sigma p_\sigma \Big|_{t_i}^{t_f}}}^{0} - \int_{t_i}^{t_f} \left(\sum_\sigma \left(\delta q_\sigma \dot{p}_\sigma\right)\right) dt$$

More comments about "details"

Detail :

$$\int_{t_i}^{t_f}\left(\sum_\sigma\left(\delta\dot{q}_\sigma p_\sigma\right)\right)dt=\int_{t_i}^{t_f}\left(\sum_\sigma\left(\frac{d\left(\delta q_\sigma p_\sigma\right)}{dt}-\delta q_\sigma\dot{p}_\sigma\right)\right)dt=\underbrace{\sum_\sigma\delta q_\sigma p_\sigma\Big|_{t_i}^{t_f}}-\int_{t_i}^{t_f}\left(\sum_\sigma\left(\delta q_\sigma\dot{p}_\sigma\right)\right)dt$$

Vanishes because $\delta q(t_f)=\delta q(t_i)$ due to the premise of Hamilton's principle.

In the Hamiltonian formulation --

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$\Rightarrow \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Why are these equations known as the "canonical equations"?

a. Because they are beautiful.
b. The term is meant to elevate their importance to the level of the music of J. S. Bach
c. To help you remember them
d. No good reason; it is just a name

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$

2. Compute generalized momenta : $p_\sigma \equiv \dfrac{\partial L}{\partial \dot{q}_\sigma}$

3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$

4. Form Hamiltonian function : $H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$

5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

What happens when you miss a step in the recipe?

a. No big deal
b. Big deal – can lead to shame and humiliation (or at least wrong analysis)

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \Rightarrow \quad \text{Second order differential equations for } q_\sigma(t)$$

Hamiltonian picture

For independent generalized coordinates $q_\sigma(t)$ and momenta $p_\sigma(t)$:

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \quad \Rightarrow \quad \text{Two first order differential equations}$$

Constants of the motion in Hamiltonian formalism

$$H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \Rightarrow \text{constant } \ q_\sigma \ \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \qquad \Rightarrow \text{constant } \ p_\sigma \ \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \sum_\sigma \left(-\dot{p}_\sigma \dot{q}_\sigma + \dot{q}_\sigma \dot{p}_\sigma \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{ constant } \ H \ \text{ if } \ \frac{\partial H}{\partial t} = 0$$

What is the physical meaning of a constant H?

Comment -- Whenever you find a constant of the motion, it is helpful for analyzing the trajectory. In this case, H often represents the mechanical energy of the system so that constant H implies that energy is conserved.

Example 1: one-dimensional potential:

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)$$

$$p_x = m\dot{x} \qquad p_y = m\dot{y} \qquad p_z = m\dot{z}$$

$$H = m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \left(\tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(z)\right)$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(z)$$

Constants: $\bar{p}_x, \bar{p}_y, \bar{H}$ (using bar to indicate constant)

Equations of motion: $\displaystyle \frac{dz}{dt} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \frac{dp_z}{dt} = -\frac{dV}{dz}$

Example 2: Motion in a central potential

$$L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)$$

$$p_r = m\dot{r} \qquad\qquad p_\varphi = mr^2\dot{\varphi}$$

$$H = m\dot{r}^2 + mr^2\dot{\varphi}^2 - \left(\tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)\right)$$

$$= \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V(r)$$

$$H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + V(r)$$

Constants: $\bar{p}_\varphi, \bar{H}$

Equations of motion:

$$\frac{dr}{dt} = \frac{p_r}{m} \qquad\qquad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = \frac{\bar{p}_\varphi^2}{mr^3} - \frac{\partial V}{\partial r}$$

Other examples

Lagrangian for symmetric top with Euler angles α, β, γ:

$$L = L(\alpha, \beta, \gamma, \dot\alpha, \dot\beta, \dot\gamma) = \tfrac{1}{2} I_1 \left(\dot\alpha^2 \sin^2 \beta + \dot\beta^2 \right) + \tfrac{1}{2} I_3 \left(\dot\alpha \cos\beta + \dot\gamma \right)^2$$

$$- Mgh \cos\beta$$

$$p_\alpha = I_1 \dot\alpha \sin^2 \beta + I_3 \left(\dot\alpha \cos\beta + \dot\gamma \right) \cos\beta$$

$$p_\beta = I_1 \dot\beta$$

$$p_\gamma = I_3 \left(\dot\alpha \cos\beta + \dot\gamma \right)$$

$$H = \tfrac{1}{2} I_1 \left(\dot\alpha^2 \sin^2 \beta + \dot\beta^2 \right) + \tfrac{1}{2} I_3 \left(\dot\alpha \cos\beta + \dot\gamma \right)^2 + Mgh \cos\beta$$

$$H = \frac{\left(p_\alpha - p_\gamma \cos\beta \right)^2}{2 I_1 \sin^2 \beta} + \frac{p_\beta^2}{2 I_1} + \frac{p_\gamma^2}{2 I_3} + Mgh \cos\beta$$

Constants: $\bar{p}_\alpha, \bar{p}_\gamma, \bar{H}$

Other examples

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0 \left(-\dot{x}y + \dot{y}x\right)$$

$$p_x = m\dot{x} - \frac{q}{2c} B_0 y$$

$$p_y = m\dot{y} + \frac{q}{2c} B_0 x$$

$$p_z = m\dot{z}$$

$$H = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Canonical form

$$H = \frac{\left(p_x + \frac{q}{2c} B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c} B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants: $\bar{p}_z, \bar{H}$

Canonical equations of motion for constant magnetic field:

$$H = \frac{\left(p_x + \frac{q}{2c}B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c}B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants: $\quad \overline{p}_z, \overline{H}$

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c}B_0 y}{m} \qquad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c}B_0 x}{m}$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = \frac{qB_0}{2mc}\left(p_y - \frac{q}{2c}B_0 x\right)$$

$$\frac{dp_y}{dt} = -\frac{\partial H}{\partial y} = -\frac{qB_0}{2mc}\left(p_x + \frac{q}{2c}B_0 y\right)$$

Canonical equations of motion for constant magnetic field -- continued:

$$\frac{dx}{dt}=\frac{p_x+\frac{q}{2c}B_0 y}{m} \qquad \frac{dy}{dt}=\frac{p_y-\frac{q}{2c}B_0 x}{m}$$

$$\frac{dp_x}{dt}=\frac{qB_0}{2mc}\left(p_y-\frac{q}{2c}B_0 x\right)=\frac{qB_0}{2c}\frac{dy}{dt}$$

$$\frac{dp_y}{dt}=-\frac{qB_0}{2mc}\left(p_x+\frac{q}{2c}B_0 y\right)=-\frac{qB_0}{2c}\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2}=\frac{\dot{p}_x}{m}+\frac{q}{2mc}B_0\dot{y}=\frac{qB_0}{mc}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2}=\frac{\dot{p}_y}{m}-\frac{q}{2mc}B_0\dot{x}=-\frac{qB_0}{mc}\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{qB_0}{mc}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = -\frac{qB_0}{mc}\frac{dx}{dt}$$

Are these results equivalent to the results of the Lagrangian analysis?

a. Yes
b. No

General treatment of particle of mass m and charge q moving in 3 dimensions in an potential $U(\mathbf{r})$ as well as electromagnetic scalar and vector potentials $\Phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$:

Lagrangian: $$L(\mathbf{r},\dot{\mathbf{r}},t)=\frac{1}{2}m\dot{\mathbf{r}}^2-U(\mathbf{r})-q\Phi(\mathbf{r},t)+\frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

Hamiltonian: $$\mathbf{p}=\frac{\partial L}{\partial\dot{\mathbf{r}}}=m\dot{\mathbf{r}}+\frac{q}{c}\mathbf{A}(\mathbf{r},t)$$

$$H(\mathbf{r},\mathbf{p},t)=\mathbf{p}\cdot\dot{\mathbf{r}}-L(\mathbf{r},\dot{\mathbf{r}},t)$$

$$=\frac{1}{2m}\left(\mathbf{p}-\frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)^2+U(\mathbf{r})+q\Phi(\mathbf{r},t)$$

Some details: $L(\mathbf{r},\dot{\mathbf{r}},t)=\frac{1}{2}m\dot{\mathbf{r}}^2-U(\mathbf{r})-q\Phi(\mathbf{r},t)+\frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$

Hamiltonian: $\mathbf{p}=\frac{\partial L}{\partial\dot{\mathbf{r}}}=m\dot{\mathbf{r}}+\frac{q}{c}\mathbf{A}(\mathbf{r},t)$

$$H(\mathbf{r},\mathbf{p},t)=\mathbf{p}\cdot\dot{\mathbf{r}}-L(\mathbf{r},\dot{\mathbf{r}},t)$$

$$=\left(m\dot{\mathbf{r}}+\frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)\cdot\dot{\mathbf{r}}-\left(\frac{1}{2}m\dot{\mathbf{r}}^2-U(\mathbf{r})-q\Phi(\mathbf{r},t)+\frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)\right)$$

$$=\frac{1}{2}m\dot{\mathbf{r}}^2+U(\mathbf{r})+q\Phi(\mathbf{r},t)$$

$$H(\mathbf{r},\mathbf{p},t)=\frac{1}{2m}\left(\mathbf{p}-\frac{q}{c}\mathbf{A}(\mathbf{r},t)\right)^2+U(\mathbf{r})+q\Phi(\mathbf{r},t)$$

Canonical form

Other properties of Hamiltonian formalism – Poisson brackets:

$$H = H\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \quad \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

Similarly for an arbitrary function: $F = F\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Poisson brackets -- continued:

For an arbitrary function: $F = F\left(\{q_\sigma(t)\},\{p_\sigma(t)\},t\right)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

Define:

$$[F,G]_{PB} \equiv \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma} \right) = -[G,F]_{PB}$$

So that: $\displaystyle \frac{dF}{dt} = [F,H]_{PB} + \frac{\partial F}{\partial t}$

Poisson brackets -- continued:

$$[F,G]_{PB} \equiv \sum_{\sigma}\left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\sigma}\right) = -[G,F]_{PB}$$

Examples:

$$[x,x]_{PB} = 0 \qquad [x,p_x]_{PB} = 1 \qquad [x,p_y]_{PB} = 0$$

$$[L_x,L_y]_{PB} = L_z$$

Liouville theorem

Let $D \equiv$ density of particles in phase space:

$$\frac{dD}{dt} = [D,H]_{PB} + \frac{\partial D}{\partial t} = 0$$

$\Rightarrow$ For next time