PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion for Lecture 15 – Chap. 4 (F & W)

Analysis of motion near equilibrium

1. **Small oscillations about equilibrium**
2. **Normal modes of vibration**

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/22/2022		Introduction	#1	8/26/2022
2	Wed, 8/24/2022	Chap. 1	Scattering theory		
3	Fri, 8/26/2022	Chap. 1	Scattering theory	#2	8/29/2022
4	Mon, 8/29/2022	Chap. 1	Scattering theory	#3	8/31/2022
5	Wed, 8/31/2022	Chap. 1	Summary of scattering theory	#4	9/02/2022
6	Fri, 9/02/2022	Chap. 2	Non-inertial coordinate systems	#5	9/05/2022
7	Mon, 9/05/2022	Chap. 3	Calculus of Variation	#6	9/7/2022
8	Wed, 9/07/2022	Chap. 3	Calculus of Variation	#7	9/9/2022
9	Fri, 9/09/2022	Chap. 3 & 6	Lagrangian Mechanics		
10	Mon, 9/12/2022	Chap. 3 & 6	Lagrangian Mechanics	#8	9/14/2022
11	Wed, 9/14/2022	Chap. 3 & 6	Constants of the motion	#9	9/16/2022
12	Fri, 9/16/2022	Chap. 3 & 6	Hamiltonian equations of motion		
13	Mon, 9/19/2022	Chap. 3 & 6	Liouville theorm	#10	9/21/2022
14	Wed, 9/21/2022	Chap. 3 & 6	Canonical transformations	#11	9/23/2022
15	Fri, 9/23/2022	Chap. 4	Small oscillations about equilibrium	#12	9/26/2022
16	Mon, 9/26/2022	Chap. 4	Normal modes of vibration		
17	Wed, 9/28/2022	Chap. 4	Normal modes of more complicated systems		
18	Fri, 9/30/2022	Chap. 7	Motion of strings		
19	Mon, 10/03/2022	Chap. 7	Sturm-Liouville equations		

PHY 711 – Assignment #11

September 23, 2022

1. Find the eigenvalues and eigenvectors of the 2×2 matrix

$$M = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}.$$

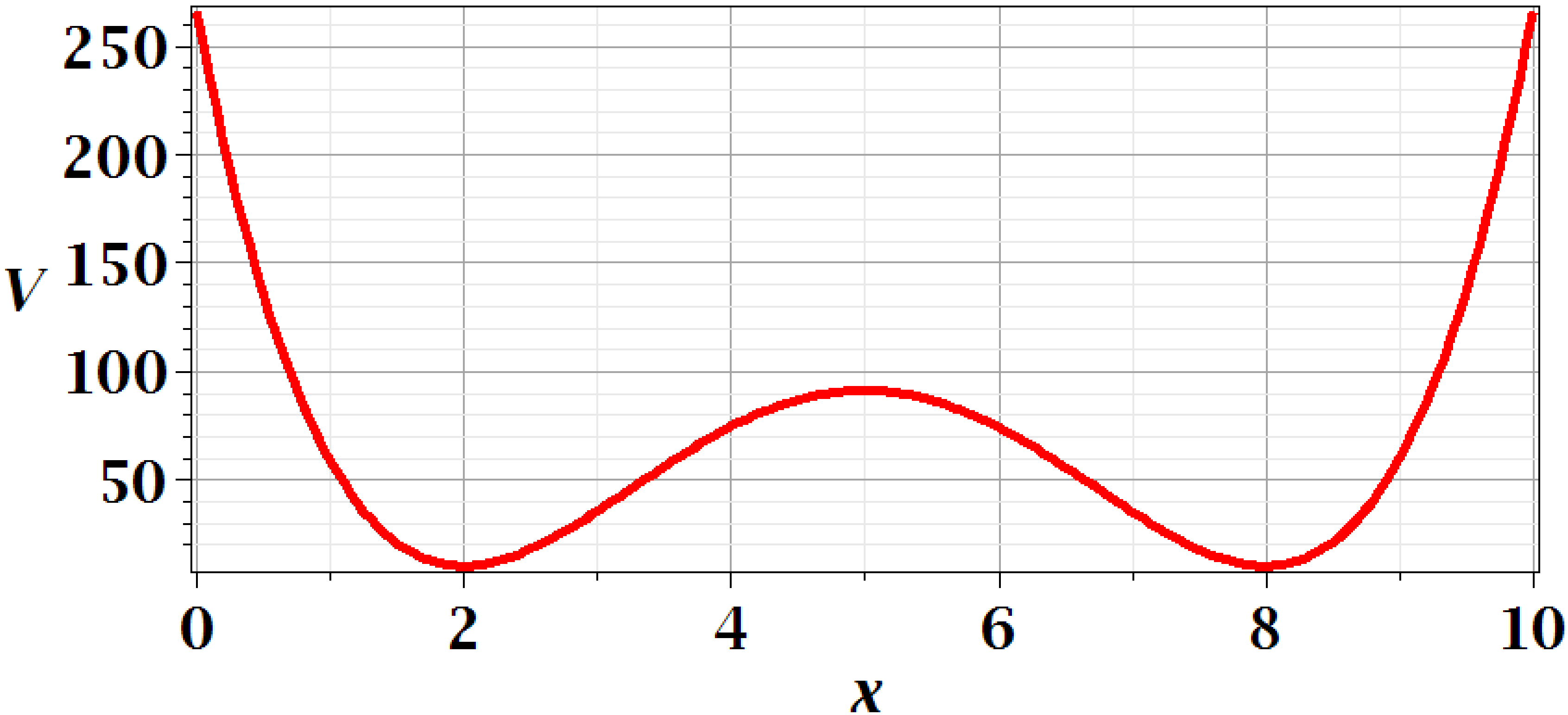
2. Find the eigenvalues and eigenvectors of the 3×3 matrix

$$M = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}.$$

Note that you may compute these by hand and/or using Maple or Mathematica.

Motivation for studying small oscillations – many interacting systems have stable and meta-stable configurations which are well approximated by:

$$V(x) \approx V(x_{eq}) + \tfrac{1}{2}\left(x - x_{eq}\right)^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} = V(x_{eq}) + \tfrac{1}{2}k\left(x - x_{eq}\right)^2$$



Equations of motion for a single oscillator:

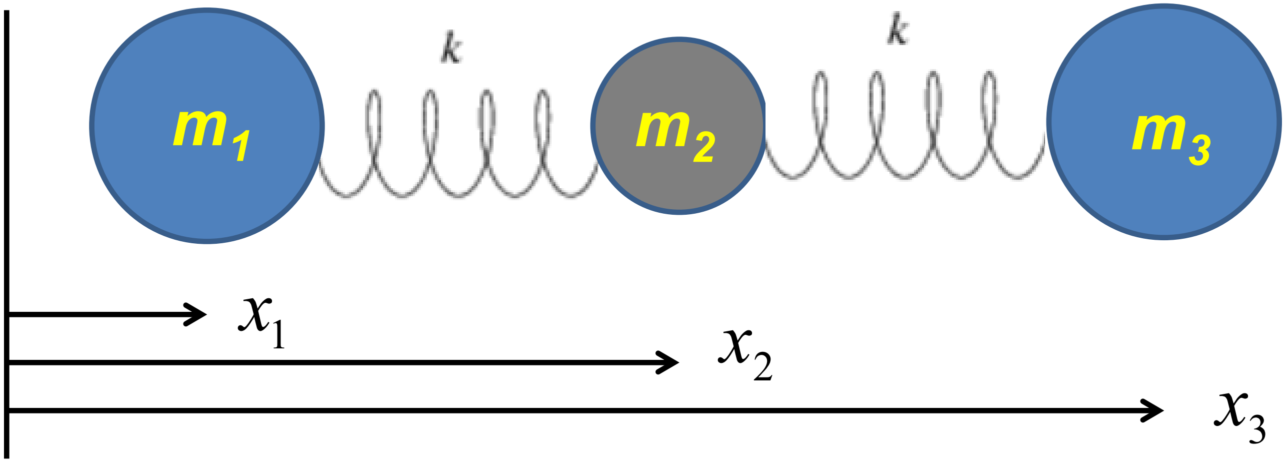
Let $k \equiv m\omega^2$

$$L(x,\dot{x},t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \qquad \Rightarrow m\ddot{x} = -m\omega^2 x$$

$$x(t) = A\sin(\omega t + \varphi)$$

Example – linear molecule



$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$- \frac{1}{2} k \left(x_2 - x_1 - \ell_{12} \right)^2 - \frac{1}{2} k \left(x_3 - x_2 - \ell_{23} \right)^2$$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2$$

$$-\frac{1}{2}k\left(x_2 - x_1 - \ell_{12}\right)^2 - \frac{1}{2}k\left(x_3 - x_2 - \ell_{23}\right)^2$$

Let: $x_1 \rightarrow x_1 - x_1^0 \quad x_2 \rightarrow x_2 - x_1^0 - \ell_{12} \quad x_3 \rightarrow x_3 - x_1^0 - \ell_{12} - \ell_{23}$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k\left(x_2 - x_1\right)^2 - \frac{1}{2}k\left(x_3 - x_2\right)^2$$

Coupled equations of motion:

$$m_1\ddot{x}_1 = k\left(x_2 - x_1\right)$$

$$m_2\ddot{x}_2 = -k\left(x_2 - x_1\right) + k\left(x_3 - x_2\right) = k\left(x_1 - 2x_2 + x_3\right)$$

$$m_3\ddot{x}_3 = -k\left(x_3 - x_2\right)$$

Coupled equations of motion :

$$m_1\ddot{x}_1 = k(x_2 - x_1)$$

$$m_2\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 - 2x_2 + x_3)$$

$$m_3\ddot{x}_3 = -k(x_3 - x_2)$$

Let $x_i(t) = X_i^\alpha e^{-i\omega_\alpha t}$ where X_i^α and ω_α are to be determined

$$-\omega_\alpha^2 m_1 X_1^\alpha = k\left(X_2^\alpha - X_1^\alpha\right)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k\left(X_1^\alpha - 2X_2^\alpha + X_3^\alpha\right)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k\left(X_3^\alpha - X_2^\alpha\right)$$

Coupled linear equations:

$$-\omega_\alpha^2 m_1 X_1^\alpha = k\left(X_2^\alpha - X_1^\alpha\right)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k\left(X_1^\alpha - 2X_2^\alpha + X_3^\alpha\right)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k\left(X_3^\alpha - X_2^\alpha\right)$$

Matrix form:

$$\begin{pmatrix} k-\omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k-\omega_\alpha^2 m_2 & -k \\ 0 & -k & k-\omega_\alpha^2 m_3 \end{pmatrix}\begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

Matrix form:

$$\begin{pmatrix} k-\omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k-\omega_\alpha^2 m_2 & -k \\ 0 & -k & k-\omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

More convenient form:

Let $Y_i \equiv \sqrt{m_i} X_i$ Equations for Y_i take the form:

$$\begin{pmatrix} \kappa_{11}-\omega_\alpha^2 & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22}-\omega_\alpha^2 & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33}-\omega_\alpha^2 \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = 0$$

where $\kappa_{ij} = \kappa_{ji} \equiv \dfrac{k}{\sqrt{m_i m_j}}$

Digression:

Eigenvalue properties of matrices $\quad \mathbf{M}\mathbf{y}_{\alpha} = \lambda_{\alpha}\mathbf{y}_{\alpha}$

Hermitian matrix: $\quad H_{ij} = H^{*}_{ji}$

Theorem for Hermitian matrices:

$$\lambda_{\alpha} \text{ have real values and } \mathbf{y}_{\alpha}^{H} \cdot \mathbf{y}_{\beta} = \delta_{\alpha\beta}$$

Unitary matrix: $\quad \mathbf{U}\mathbf{U}^{H} = \mathbf{I}$

$$|\lambda_{\alpha}| = 1 \text{ and } \mathbf{y}_{\alpha}^{H} \cdot \mathbf{y}_{\beta} = \delta_{\alpha\beta}$$

Digression on matrices -- continued

Eigenvalues of a matrix are "invariant" under a similarity transformation

Eigenvalue properties of matrix: $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Transformed matrix: $\mathbf{M}'\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

If $\mathbf{M}' = \mathbf{S}\mathbf{M}\mathbf{S}^{-1}$ then $\lambda'_\alpha = \lambda_\alpha$ and $\mathbf{S}^{-1}\mathbf{y}'_\alpha = \mathbf{y}_\alpha$

Proof $\mathbf{S}\mathbf{M}\mathbf{S}^{-1}\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

$$\mathbf{M}\left(\mathbf{S}^{-1}\mathbf{y}'_\alpha\right) = \lambda'_\alpha \left(\mathbf{S}^{-1}\mathbf{y}'_\alpha\right)$$

Example of transformation:

Original problem written in eigenvalue form:

$$\begin{pmatrix} k/m_1 & -k/m_1 & 0 \\ -k/m_2 & 2k/m_2 & -k/m_2 \\ 0 & -k/m_3 & k/m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix}$$

Let $\mathbf{S} = \begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix}$; $\mathbf{SMS}^{-1} = \begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix}$

Let $\mathbf{Y} \equiv \mathbf{SX}$

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

where $\kappa_{ij} = \kappa_{ji} \equiv \dfrac{k}{\sqrt{m_i m_j}}$

In our case :

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

for $m_1 = m_3 \equiv m_O$ and $m_2 \equiv m_C$ $\quad (CO_2)$

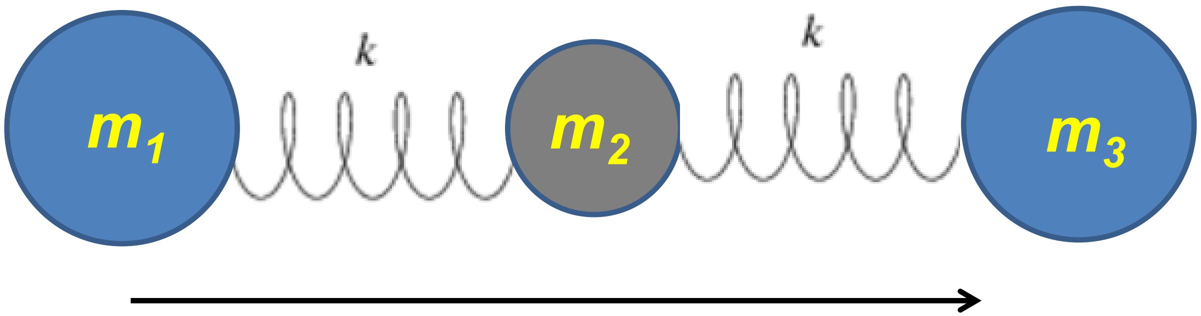
$$\begin{pmatrix} \kappa_{OO} & -\kappa_{OC} & 0 \\ -\kappa_{OC} & 2\kappa_{CC} & -\kappa_{OC} \\ 0 & -\kappa_{OC} & \kappa_{OO} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

Eigenvalues and eigenvectors :

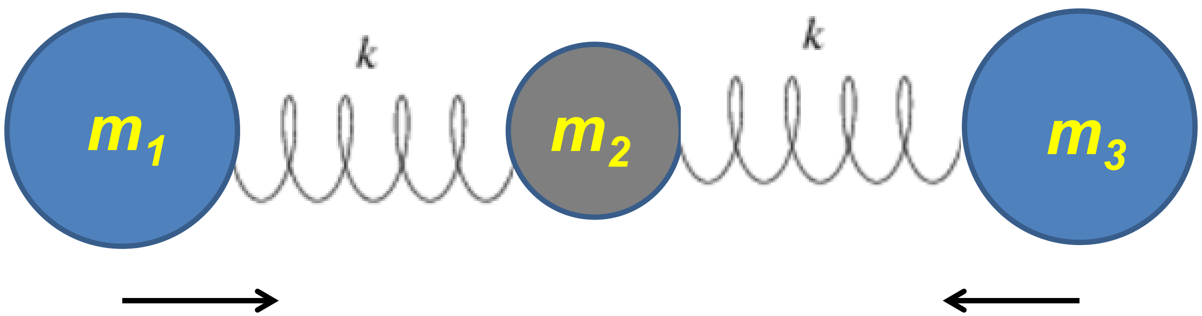
$$\omega_1^2 = 0 \qquad \begin{pmatrix} Y_1^1 \\ Y_2^1 \\ Y_3^1 \end{pmatrix} = N_1 \begin{pmatrix} \sqrt{\frac{m_O}{m_C}} \\ 1 \\ \sqrt{\frac{m_O}{m_C}} \end{pmatrix}, \quad \begin{pmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \end{pmatrix} = N'_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\omega_2^2 = \frac{k}{m_O} \qquad \begin{pmatrix} Y_1^2 \\ Y_2^2 \\ Y_3^2 \end{pmatrix} = N_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} X_1^2 \\ X_2^2 \\ X_3^2 \end{pmatrix} = N'_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

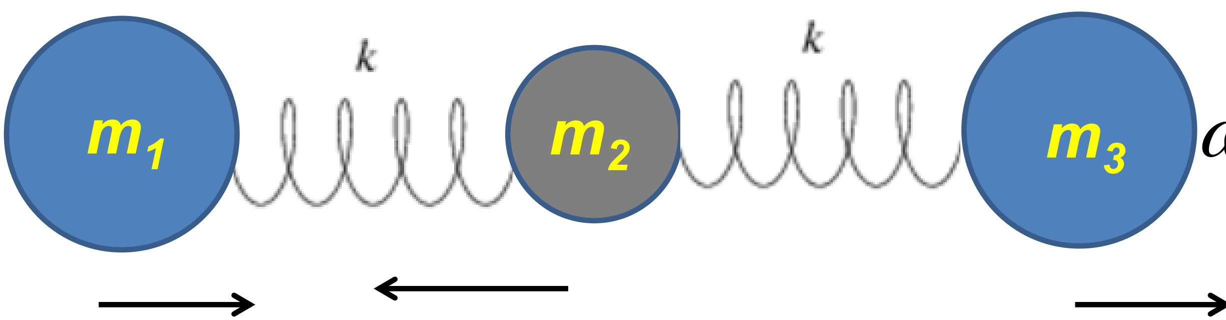
$$\omega_3^2 = \frac{k}{m_O} + \frac{2k}{m_C} \qquad \begin{pmatrix} Y_1^3 \\ Y_2^3 \\ Y_3^3 \end{pmatrix} = N_3 \begin{pmatrix} 1 \\ -2\sqrt{\frac{m_O}{m_C}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} X_1^3 \\ X_2^3 \\ X_3^3 \end{pmatrix} = N'_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$



$$\omega_1 = 0$$



$$\omega_2 = \sqrt{\frac{k}{m_O}}$$



$$\omega_3 = \sqrt{\frac{k}{m_O} + \frac{2k}{m_C}}$$

General solution :

$$x_i(t) = \Re\left(\sum_{\alpha} C^{\alpha} X_i^{\alpha} e^{-i\omega_{\alpha} t} \right)$$

For example, normal mode amplitudes

C^{α} can be determined from initial conditions

Comment on solving for eigenvalues and eigenvectors – while it is reasonable to find these analytically for 2x2 or 3x3 matrices, it is prudent to use Maple or Mathematica for larger systems.

Maple example

Mathematica example

Additional digression on matrix properties

Singular value decomposition

It is possible to factor any real matrix $\mathbf{A}$ into unitary matrices $\mathbf{V}$ and $\mathbf{U}$ together with positive diagonal matrix $\mathbf{\Sigma}$:

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathbf{H}}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{pmatrix}$$

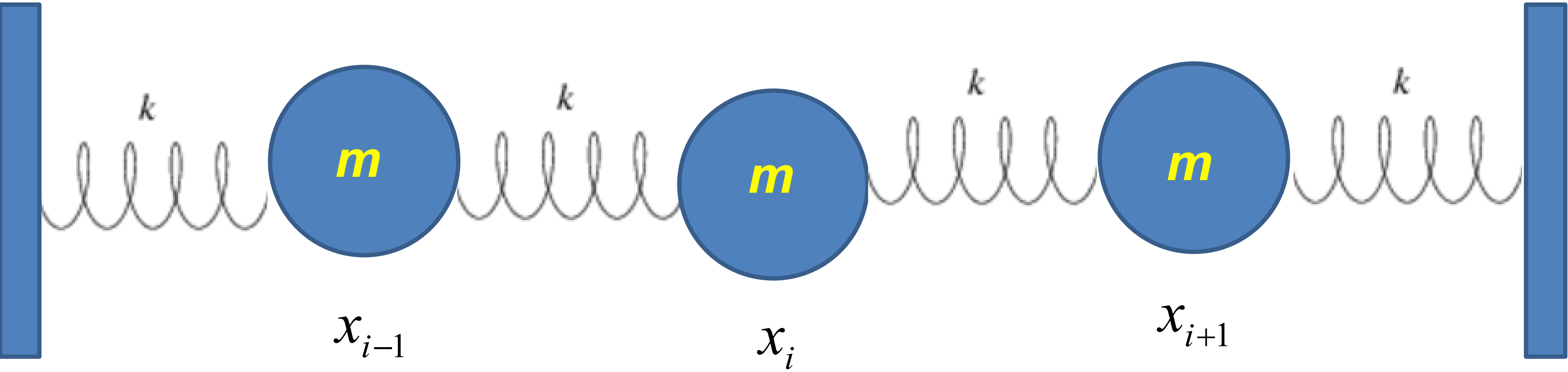
Singular value decomposition -- continued

Consider using SVD to solve a singular linear algebra problem $\mathbf{AX} = \mathbf{B}$

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{H}$$

$$\mathbf{X} = \sum_{i \text{ for } \sigma_i > \varepsilon} \mathbf{v}_i \frac{\langle \mathbf{u}_i^H \mid \mathbf{B} \rangle}{\sigma_i}$$

Consider an extended system of masses and springs:



Note: each mass coordinate is measured relative to its equilibrium position x_i^0

$$L = T - V = \frac{1}{2} m \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} \left(x_{i+1} - x_i\right)^2$$

Note: In fact, we have N masses; x_0 and x_{N+1} will be treated using boundary conditions.

$$L = T - V = \frac{1}{2} m \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} \left(x_{i+1} - x_i \right)^2$$

$$x_0 \equiv 0 \text{ and } x_{N+1} \equiv 0$$

From Euler - Lagrange equations :

$$m\ddot{x}_1 = k\left(x_2 - 2x_1\right)$$

$$m\ddot{x}_2 = k\left(x_3 - 2x_2 + x_1\right)$$

....

$$m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$$

....

$$m\ddot{x}_N = k\left(x_{N-1} - 2x_N\right)$$

Matrix formulation --

Assume $x_i(t) = X_i e^{-i\omega t}$

$$\frac{m}{k}\omega^2 \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & -1 & 2 & -1 \\ \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix}$$

Can solve as an eigenvalue problem --

> with(LinearAlgebra);

=

$$A := \begin{bmatrix} 5 & -1 & 0 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 \\ 0 & -1 & 5 & -1 & 0 \\ 0 & 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & -1 & 5 \end{bmatrix};$$

> Eigenvalues(A);

$$\begin{bmatrix} 5 \\ 6 \\ 4 \\ 5-\sqrt{3} \\ 5+\sqrt{3} \end{bmatrix}$$

This example also has an algebraic solution --

From Euler - Lagrange equations :

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{with } x_0 = 0 = x_{N+1}$$

Try : $\quad x_j(t) = Ae^{-i\omega t + iqaj}$

$$-\omega^2 Ae^{-i\omega t+iqaj} = \frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right)Ae^{-i\omega t+iqaj}$$

$$-\omega^2 = \frac{k}{m}\left(2\cos\left(qa\right) - 2\right)$$

$$\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$$

From Euler-Lagrange equations -- continued:

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{with } x_0 = 0 = x_{N+1}$$

Try: $x_j(t) = Ae^{-i\omega t + iqaj}$ $\qquad \Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$

Note that: $x_j(t) = Be^{-i\omega t - iqaj}$ $\qquad \Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$

General solution:

$$x_j(t) = \Re\left(Ae^{-i\omega t + iqaj} + Be^{-i\omega t - iqaj}\right)$$

Impose boundary conditions:

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t}\right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)}\right) = 0$$

Impose boundary conditions -- continued:

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t} \right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)} \right) = 0$$

$$\Rightarrow B = -A$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t}\left(e^{iqa(N+1)} - e^{-iqa(N+1)} \right) \right) = 0$$

$$\Rightarrow \sin\left(qa(N+1) \right) = 0$$

$$\Rightarrow \quad qa(N+1) = \nu\pi \quad \text{where } \nu = 0,1,2\cdots$$

$$qa = \frac{\nu\pi}{N+1}$$

Summary of results:

$$\Rightarrow \omega_\nu^2 = \frac{4k}{m}\sin^2\left(\frac{\nu\pi}{2(N+1)}\right) \qquad x_n = \Re\left(2iA\sin\left(\frac{\nu\pi n}{N+1}\right)\right)$$

$$\nu = 0,1,\ldots N \qquad n = 1,2,\ldots\ldots N$$