PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion on Lecture 16: Chap. 4 (F&W)

Analysis of motion near equilibrium –

Normal Mode Analysis

1. Normal modes of vibration for simple systems
2. Some concepts of linear algebra
3. Normal modes of vibration for more complicated systems

10	Mon, 9/12/2022	Chap. 3 & 6	Lagrangian Mechanics	#8	9/14/2022
11	Wed, 9/14/2022	Chap. 3 & 6	Constants of the motion	#9	9/16/2022
12	Fri, 9/16/2022	Chap. 3 & 6	Hamiltonian equations of motion		
13	Mon, 9/19/2022	Chap. 3 & 6	Liouville theorm	#10	9/21/2022
14	Wed, 9/21/2022	Chap. 3 & 6	Canonical transformations	#11	9/23/2022
15	Fri, 9/23/2022	Chap. 4	Small oscillations about equilibrium	#12	9/26/2022
16	Mon, 9/26/2022	Chap. 4	Normal modes of vibration	#13	9/28/2022
17	Wed, 9/28/2022	Chap. 4	Normal modes of more complicated systems		
18	Fri, 9/30/2022	Chap. 7	Motion of strings		
19	Mon, 10/03/2022	Chap. 7	Sturm-Liouville equations		
20	Wed, 10/05/2022	Chap. 7	Sturm-Liouville equations		
21	Fri, 10/07/2022	Chap. 1-4,6-7	Review		
	Mon, 10/10/2022	No class	Take home exam		
	Wed, 10/12/2022	No class	Take home exam		
	Fri, 10/14/2022	No class	Fall break		
22	Mon, 10/17/2022	Chap. 7	Class resumes		

PHY 711 -- Assignment #13

Sept. 26, 2022

Continue reading Chapter 4. in **Fetter & Walecka**.

1. Consider the example of of an infinite one-dimensional chain of masses *m* and *M* connected with identical springs having constant *k*..
   a. Verify the normal frequencies obtained in the class notes for this system.
   b. Now assume that $k/m \equiv 4\,\omega_0^2$ and $k/M \equiv \omega_0^2$, where ω_0 is a given constant. Plot the normal mode frequencies for this case as a function of *qa*.

Schedule

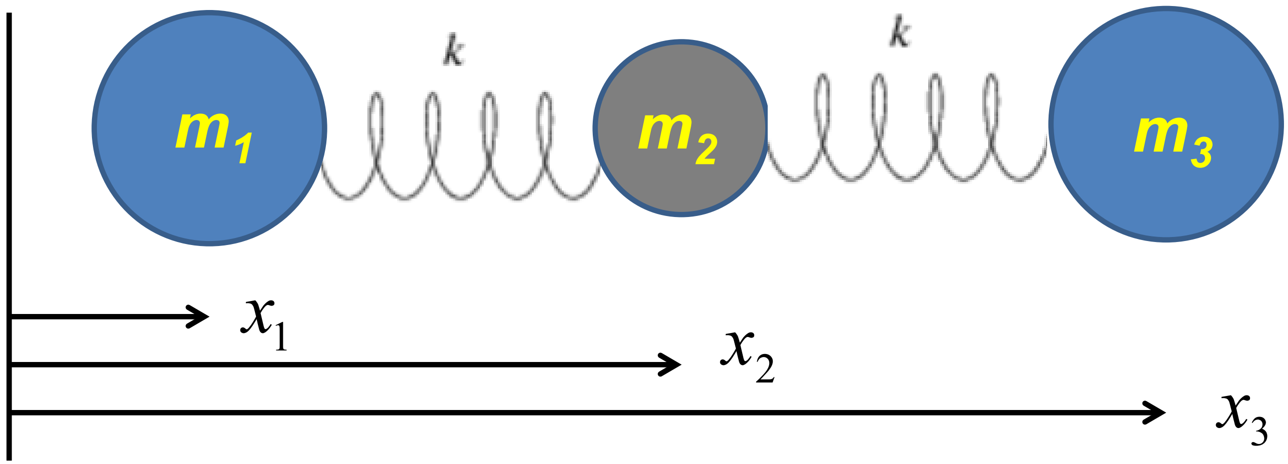
October 2022

S	M	T	W	T	F	S
25	26	27	28	29	30	1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31	1	2	3	4	5

Take-home exam

Fall break

Example – linear molecule



$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k\left(x_2 - x_1 - \ell_{12}\right)^2 - \frac{1}{2}k\left(x_3 - x_2 - \ell_{23}\right)^2$$

Let: $x_1 \to x_1 - x_1^0 \quad x_2 \to x_2 - x_1^0 \; - \ell_{12} \quad x_3 \to x_3 - x_1^0 \; - \ell_{12} - \ell_{23}$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2 - \frac{1}{2} k \left(x_2 - x_1 \right)^2 - \frac{1}{2} k \left(x_3 - x_2 \right)^2$$

Coupled equations of motion :

$$m_1 \ddot{x}_1 = k\left(x_2 - x_1\right)$$

$$m_2 \ddot{x}_2 = -k\left(x_2 - x_1\right) + k\left(x_3 - x_2\right) = k\left(x_1 - 2x_2 + x_3\right)$$

$$m_3 \ddot{x}_3 = -k\left(x_3 - x_2\right)$$

Let $x_i(t) = X_i^\alpha e^{-i\omega_\alpha t}$

$$-\omega_\alpha^2 m_1 X_1^\alpha = k\left(X_2^\alpha - X_1^\alpha\right)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k\left(X_1^\alpha - 2X_2^\alpha + X_3^\alpha\right)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k\left(X_3^\alpha - X_2^\alpha\right)$$

Coupled linear equations:

$$-\omega_\alpha^2 m_1 X_1^\alpha = k\left(X_2^\alpha - X_1^\alpha\right)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k\left(X_1^\alpha - 2X_2^\alpha + X_3^\alpha\right)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k\left(X_3^\alpha - X_2^\alpha\right)$$

Matrix form:

$$\begin{pmatrix} k-\omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k-\omega_\alpha^2 m_2 & -k \\ 0 & -k & k-\omega_\alpha^2 m_3 \end{pmatrix}\begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

Matrix form:

$$\begin{pmatrix} k-\omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k-\omega_\alpha^2 m_2 & -k \\ 0 & -k & k-\omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

More convenient form:

Let $Y_i \equiv \sqrt{m_i}\, X_i$ Equations for Y_i take the form:

$$\begin{pmatrix} \kappa_{11}-\omega_\alpha^2 & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22}-\omega_\alpha^2 & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33}-\omega_\alpha^2 \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = 0$$

where $\kappa_{ij} = \kappa_{ji} \equiv \dfrac{k}{\sqrt{m_i m_j}}$

Rearranging the equation to an eigenvalue problem:

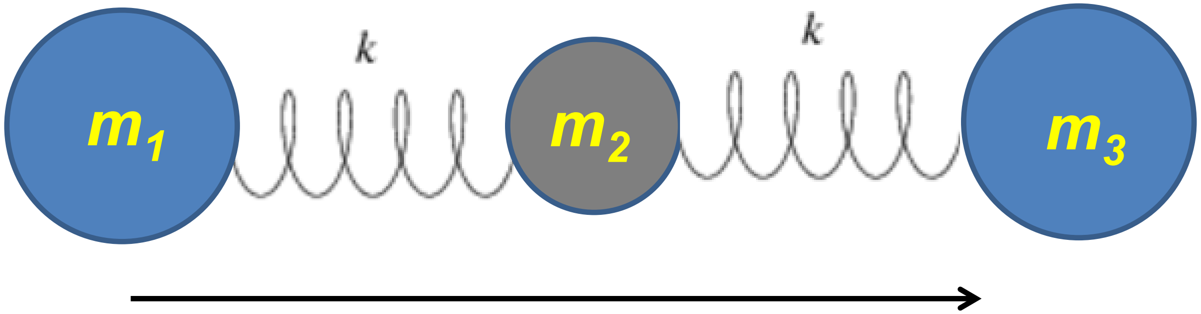
$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

Special case for CO_2 molecule -- $m_1 = m_3 \equiv m_O$ and $m_2 \equiv m_C$

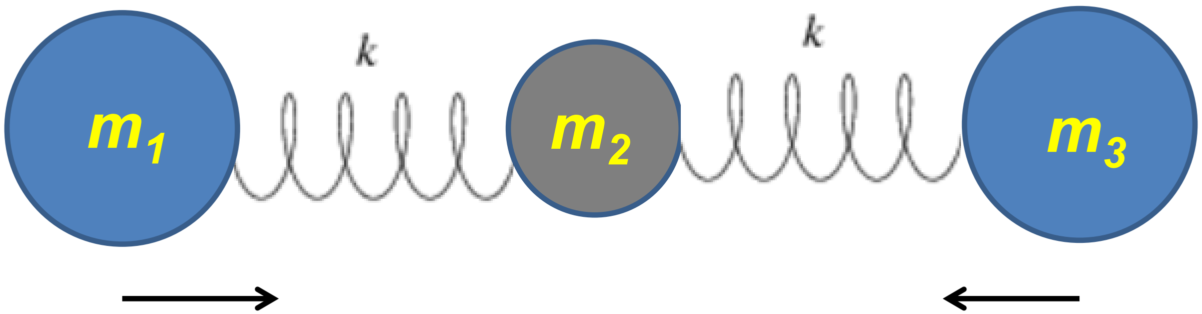
$$\begin{pmatrix} \kappa_{OO} & -\kappa_{OC} & 0 \\ -\kappa_{OC} & 2\kappa_{CC} & -\kappa_{OC} \\ 0 & -\kappa_{OC} & \kappa_{OO} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

For $m_1 = m_3 \equiv m_O$

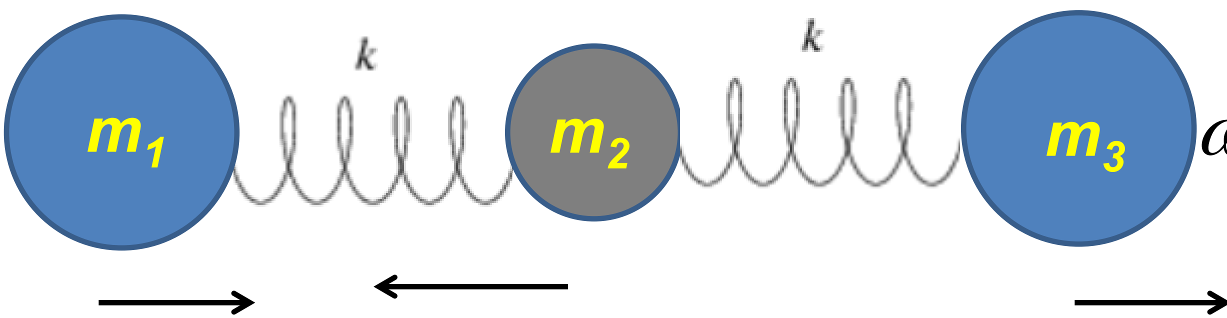
and $m_2 \equiv m_C$



$$\omega_1 = 0$$



$$\omega_2 = \sqrt{\frac{k}{m_O}}$$



$$\omega_3 = \sqrt{\frac{k}{m_O} + \frac{2k}{m_C}}$$

(with help from Maple)

Eigenvalues and eigenvectors:

***N,N'* are normalization constants.**

$$\omega_1^2 = 0 \qquad \begin{pmatrix} Y_1^1 \\ Y_2^1 \\ Y_3^1 \end{pmatrix} = N_1 \begin{pmatrix} \sqrt{\frac{m_O}{m_C}} \\ 1 \\ \sqrt{\frac{m_O}{m_C}} \end{pmatrix}, \qquad \begin{pmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \end{pmatrix} = N'_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\omega_2^2 = \frac{k}{m_O} \qquad \begin{pmatrix} Y_1^2 \\ Y_2^2 \\ Y_3^2 \end{pmatrix} = N_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} X_1^2 \\ X_2^2 \\ X_3^2 \end{pmatrix} = N'_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\omega_3^2 = \frac{k}{m_O} + \frac{2k}{m_C} \qquad \begin{pmatrix} Y_1^3 \\ Y_2^3 \\ Y_3^3 \end{pmatrix} = N_3 \begin{pmatrix} 1 \\ -2\sqrt{\frac{m_O}{m_C}} \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} X_1^3 \\ X_2^3 \\ X_3^3 \end{pmatrix} = N'_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Finding eigenvalues/eigenvectors by hand --

$$\mathbf{M}\mathbf{y}^{\alpha} = \lambda^{\alpha}\mathbf{y}^{\alpha}$$

$$\left(\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right)\mathbf{y}^{\alpha} = 0$$

$$\left|\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right| \equiv \det\left(\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right) = 0 \quad \Rightarrow \text{polynomial for solutions } \lambda^{\alpha}$$

For each α and λ^{α} solve for the eigenvector coefficients $\mathbf{y}^{\alpha}$

Example

$$\mathbf{M} = \begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \qquad A \equiv \frac{k}{m_O} \quad B \equiv \frac{k}{m_C}$$

$$\left|\mathbf{M} - \lambda^{\alpha}\mathbf{I}\right| = \begin{vmatrix} A-\lambda^{\alpha} & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B-\lambda^{\alpha} & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A-\lambda^{\alpha} \end{vmatrix} = \lambda^{\alpha}\left(\lambda^{\alpha} - A\right)\left(\lambda^{\alpha} - (A+2B)\right) = 0$$

Example -- continued

$$\mathbf{M} = \begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \qquad A \equiv \frac{k}{m_O} \quad B \equiv \frac{k}{m_C}$$

$$\left|\mathbf{M} - \lambda^\alpha \mathbf{I}\right| = \begin{vmatrix} A-\lambda^\alpha & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B-\lambda^\alpha & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A-\lambda^\alpha \end{vmatrix} = \lambda^\alpha \left(\lambda^\alpha - A\right)\left(\lambda^\alpha - (A+2B)\right)$$

Solving for eigenvector corresponding to $\lambda^\alpha \equiv \lambda^1 = 0$

$$\begin{pmatrix} A & -\sqrt{AB} & 0 \\ -\sqrt{AB} & 2B & -\sqrt{AB} \\ 0 & -\sqrt{AB} & A \end{pmatrix} \begin{pmatrix} y_{O1}^1 \\ y_C^1 \\ y_{O2}^1 \end{pmatrix} = 0 \qquad \Rightarrow \frac{y_{O1}^1}{y_C^1} = \frac{y_{O2}^1}{y_C^1} = \sqrt{\frac{B}{A}}$$

Note that the normalization of the eigenvector is arbitrary.

Digression on matrices -- continued

Eigenvalues of a matrix are "invariant" under a similarity transformation

Eigenvalue properties of matrix: $\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

Transformed matrix: $\mathbf{M}'\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

If $\mathbf{M}' = \mathbf{S}\mathbf{M}\mathbf{S}^{-1}$ then $\lambda'_\alpha = \lambda_\alpha$ and $\mathbf{S}^{-1}\mathbf{y}'_\alpha = \mathbf{y}_\alpha$

Proof $\mathbf{S}\mathbf{M}\mathbf{S}^{-1}\mathbf{y}'_\alpha = \lambda'_\alpha \mathbf{y}'_\alpha$

$$\mathbf{M}\left(\mathbf{S}^{-1}\mathbf{y}'_\alpha\right) = \lambda'_\alpha \left(\mathbf{S}^{-1}\mathbf{y}'_\alpha\right)$$

This means that if a matrix is "similar" to a Hermitian matrix, it has the same eigenvalues. The corresponding eigenvectors of **M** and **M'** are not the same but $\mathbf{y}_\alpha = \mathbf{S}^{-1}\mathbf{y}'_\alpha$

Example of a similarity transformation:

Original problem written in eigenvalue form:

$$\begin{pmatrix} k/m_1 & -k/m_1 & 0 \\ -k/m_2 & 2k/m_2 & -k/m_2 \\ 0 & -k/m_3 & k/m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix}$$

Note that this matrix is not symmetric

Let $\mathbf{S} = \begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix}$; $\quad \mathbf{SMS}^{-1} = \begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix}$

Let $\mathbf{Y} \equiv \mathbf{SX}$

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

Note that this matrix is symmetric

where $\quad \kappa_{ij} = \kappa_{ji} \equiv \dfrac{k}{\sqrt{m_i m_j}}$

Note, here we have defined **S** as a transformation matrix (often called a similarity transformation matrix)

Sometimes, the similarity transformation is also unitary so that

$$\mathbf{U}^{-1} = \mathbf{U}^{H}$$

Example for 2x2 case --

$$\mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \mathbf{U}^{-1} = \mathbf{U}^{H} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

How can you find a unitary transformation that also diagonalizes a matrix?

Example -- $\mathbf{M} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \qquad \mathbf{M}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Example -- $\mathbf{M} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ $\mathbf{M}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\mathbf{M}' = \mathbf{U}\mathbf{M}\mathbf{U}^H$ for $\mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\mathbf{M}' = \begin{pmatrix} A\cos^2\theta + C\sin^2\theta + B\sin 2\theta & -B\cos 2\theta - \frac{1}{2}(C-A)\sin 2\theta \\ -B\cos 2\theta - \frac{1}{2}(C-A)\sin 2\theta & A\sin^2\theta + C\cos^2\theta - B\sin 2\theta \end{pmatrix}$$

$\Rightarrow$ choose $\theta = \frac{1}{2}\tan^{-1}\left(\frac{-2B}{C-A}\right)$

$\Rightarrow \lambda_1 = A\cos^2\theta + C\sin^2\theta + B\sin 2\theta$

$\Rightarrow \lambda_2 = A\sin^2\theta + C\cos^2\theta - B\sin 2\theta$

Note that this “trick” is special for 2x2 matrices, but numerical extensions based on the trick are possible.

Note that transformations using unitary matrices are often convenient and they can be easily constructed from the eigenvalues of a matrix.

Suppose you have an $N \times N$ matrix $\mathbf{M}$ and find all N eigenvalues/vectors:

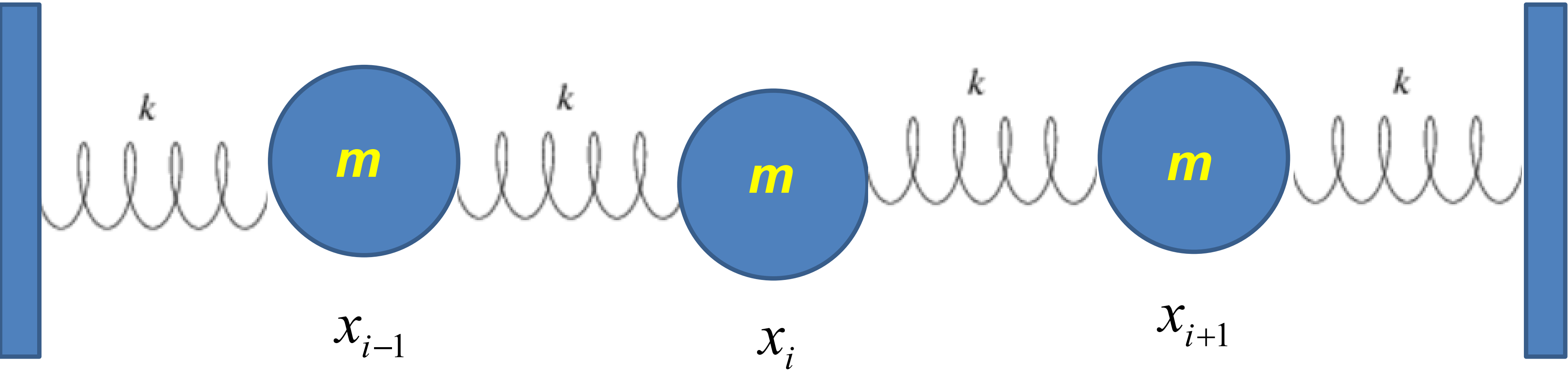
$\mathbf{M}\mathbf{y}^{\alpha} = \lambda^{\alpha}\mathbf{y}^{\alpha}$ orthonormalized so that $\left\langle \mathbf{y}^{\alpha} \middle| \mathbf{y}^{\beta} \right\rangle = \delta_{\alpha\beta}$

Now construct an $N \times N$ matrix $\mathbf{U}$ by listing the eigenvector columns:

$$\mathbf{U} \equiv \begin{pmatrix} y_1^1 & y_1^2 & \cdots & y_1^N \\ y_2^1 & y_2^2 & \cdots & y_2^N \\ \vdots & \vdots & \cdots & \vdots \\ y_N^1 & y_N^2 & \cdots & y_N^N \end{pmatrix} \qquad \mathbf{U}^{-1} \equiv \begin{pmatrix} y_1^{1*} & y_2^{1*} & \cdots & y_N^{1*} \\ y_1^{2*} & y_2^{2*} & \cdots & y_N^{N*} \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{N*} & y_2^{N*} & \cdots & y_N^{N*} \end{pmatrix} \qquad \Rightarrow \text{ by construction } \mathbf{U}^{-1}\mathbf{U} = \mathbf{I}$$

Also by construction $\mathbf{U}^{-1}\mathbf{M}\mathbf{U} = \begin{pmatrix} \lambda^1 & 0 & \cdots & 0 \\ 0 & \lambda^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda^N \end{pmatrix}$

Consider an extended system of masses and springs:



x_{i-1} x_i x_{i+1}

Note : each mass coordinate is measured relative to its equilibrium position x_i^0

$$L = T - V = \frac{1}{2} m \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} \left(x_{i+1} - x_i \right)^2$$

Note: In fact, we have N masses; x_0 and x_{N+1} will be treated using boundary conditions.

$$L = T - V = \frac{1}{2} m \sum_{i=1}^{N} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} \left(x_{i+1} - x_i \right)^2$$

$$x_0 \equiv 0 \text{ and } x_{N+1} \equiv 0$$

From Euler - Lagrange equations :

$$m\ddot{x}_1 = k\left(x_2 - 2x_1\right)$$

$$m\ddot{x}_2 = k\left(x_3 - 2x_2 + x_1\right)$$

....

$$m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$$

....

$$m\ddot{x}_N = k\left(x_{N-1} - 2x_N\right)$$

Matrix formulation --

Assume $x_i(t) = X_i e^{-i\omega t}$

$$\frac{m}{k}\omega^2 \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & -1 & 2 & -1 \\ \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix}$$

Can solve as an eigenvalue problem –

(Why did we not have to transform the equations as we did in the previous example?)

Because of its very regular form, this example also has an algebraic solution --

From Euler - Lagrange equations :

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{with } x_0 = 0 = x_{N+1}$$

Try : $\quad x_j(t) = Ae^{-i\omega t + iqaj}$

Here "*a*" is the equilibrium length of a spring and *q* has the units of 1/length.

$$-\omega^2 Ae^{-i\omega t+iqaj} = \frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right)Ae^{-i\omega t+iqaj}$$

$$-\omega^2 = \frac{k}{m}\left(2\cos(qa) - 2\right)$$

$$\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$$

Is this treatment cheating?

a. Yes.
b. No cheating, but we are not done.

From Euler - Lagrange equations - - continued :

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{with } x_0 = 0 = x_{N+1}$$

Try : $\quad x_j(t) = Ae^{-i\omega t + iqaj} \qquad \Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$

Note that : $\quad x_j(t) = Be^{-i\omega t - iqaj} \qquad \Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$

General solution :

$$x_j(t) = \Re\left(Ae^{-i\omega t + iqaj} + Be^{-i\omega t - iqaj}\right)$$

Impose boundary conditions :

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t}\right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)}\right) = 0$$

Impose boundary conditions -- continued:

$$x_0(t) = \Re\left(Ae^{-i\omega t} + Be^{-i\omega t} \right) = 0$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t+iqa(N+1)} + Be^{-i\omega t-iqa(N+1)} \right) = 0$$

$$\Rightarrow B = -A$$

$$x_{N+1}(t) = \Re\left(Ae^{-i\omega t}\left(e^{iqa(N+1)} - e^{-iqa(N+1)} \right)\right) = 0$$

$$\Rightarrow \sin\left(qa\left(N+1 \right)\right) = 0$$

$$\Rightarrow \; qa\left(N+1 \right) = \nu\pi \quad \text{where } \nu = 1,2\cdots N$$

$$qa = \frac{\nu\pi}{N+1}$$

Recap -- solution for integer parameter ν

$$x_j(t) = \Re\left(2iAe^{-i\omega_\nu t} \sin\left(\frac{\nu\pi j}{N+1} \right) \right)$$

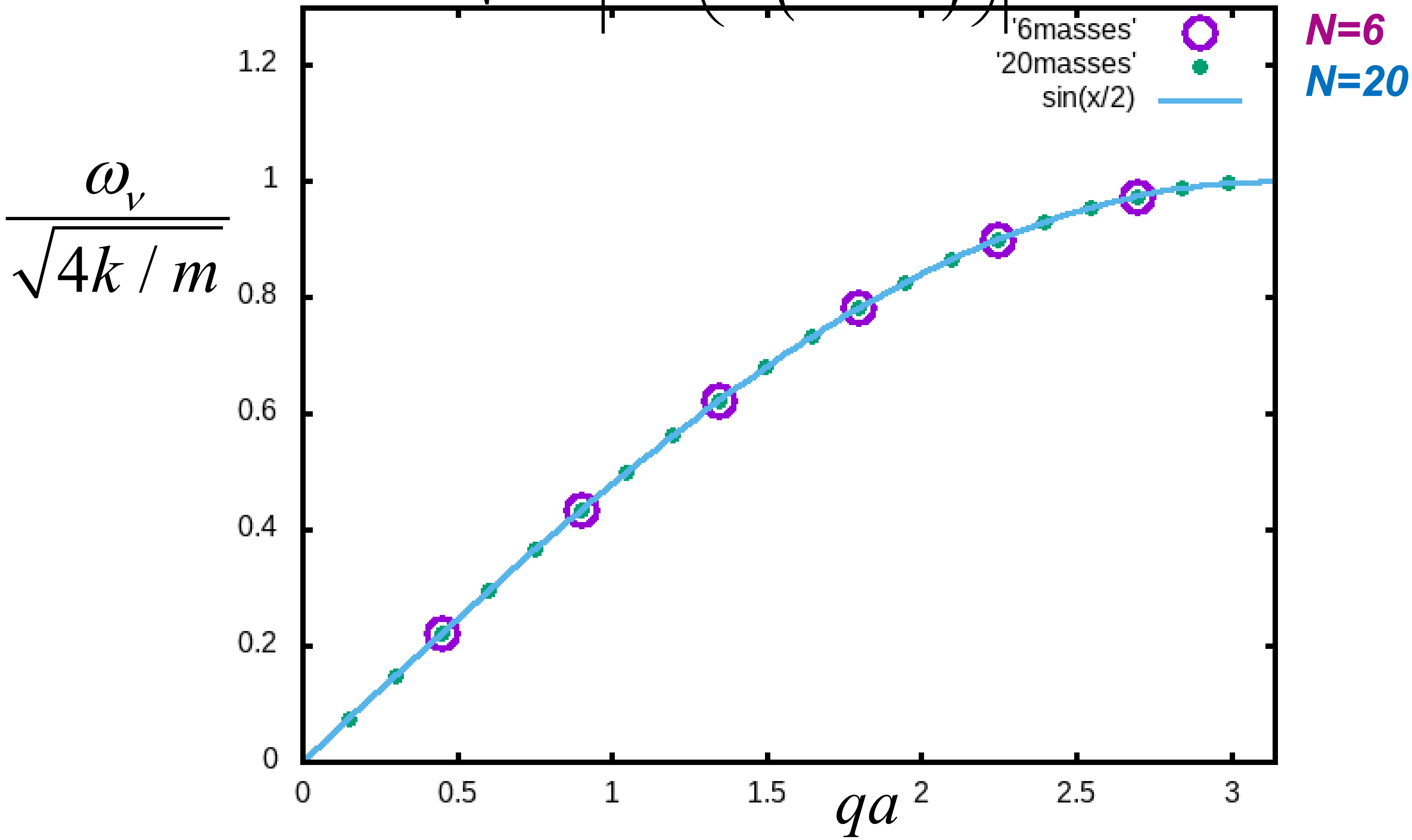
$$\omega_\nu{}^2 = \frac{4k}{m} \sin^2\left(\frac{\nu\pi}{2(N+1)} \right)$$

Note that non-trivial, unique values are

$$\nu = 1,2,\cdots N$$

Examples

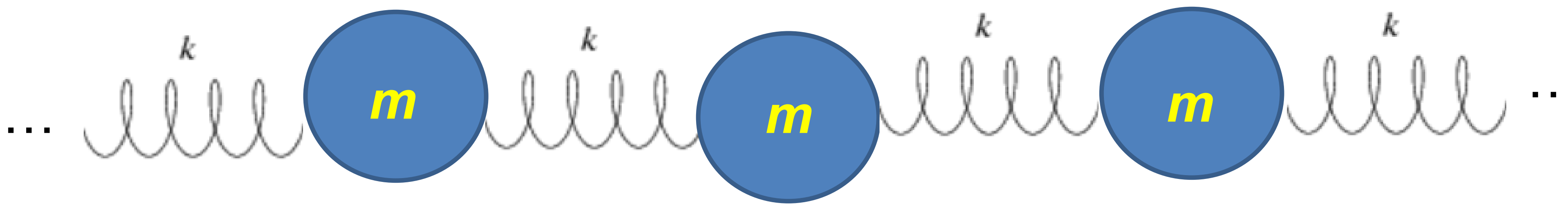
$$\omega_\nu = \sqrt{\frac{4k}{m}}\left|\sin\left(\frac{\nu\pi}{2(N+1)}\right)\right|$$



Note that solution form remains correct for $N \rightarrow \infty$

$$\omega(qa) = \sqrt{4k/m}\left|\sin\left(\frac{qa}{2}\right)\right|$$

For extended (infinite) chain without boundaries:



x_{i-1} x_i x_{i+1}

From Euler-Lagrange equations:

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{for all } x_j$$

Try: $x_j(t) = Ae^{-i\omega t + iqaj}$

$$-\omega^2 Ae^{-i\omega t+iqaj} = \frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right)Ae^{-i\omega t+iqaj}$$

$$-\omega^2 = \frac{k}{m}\left(2\cos(qa) - 2\right)$$

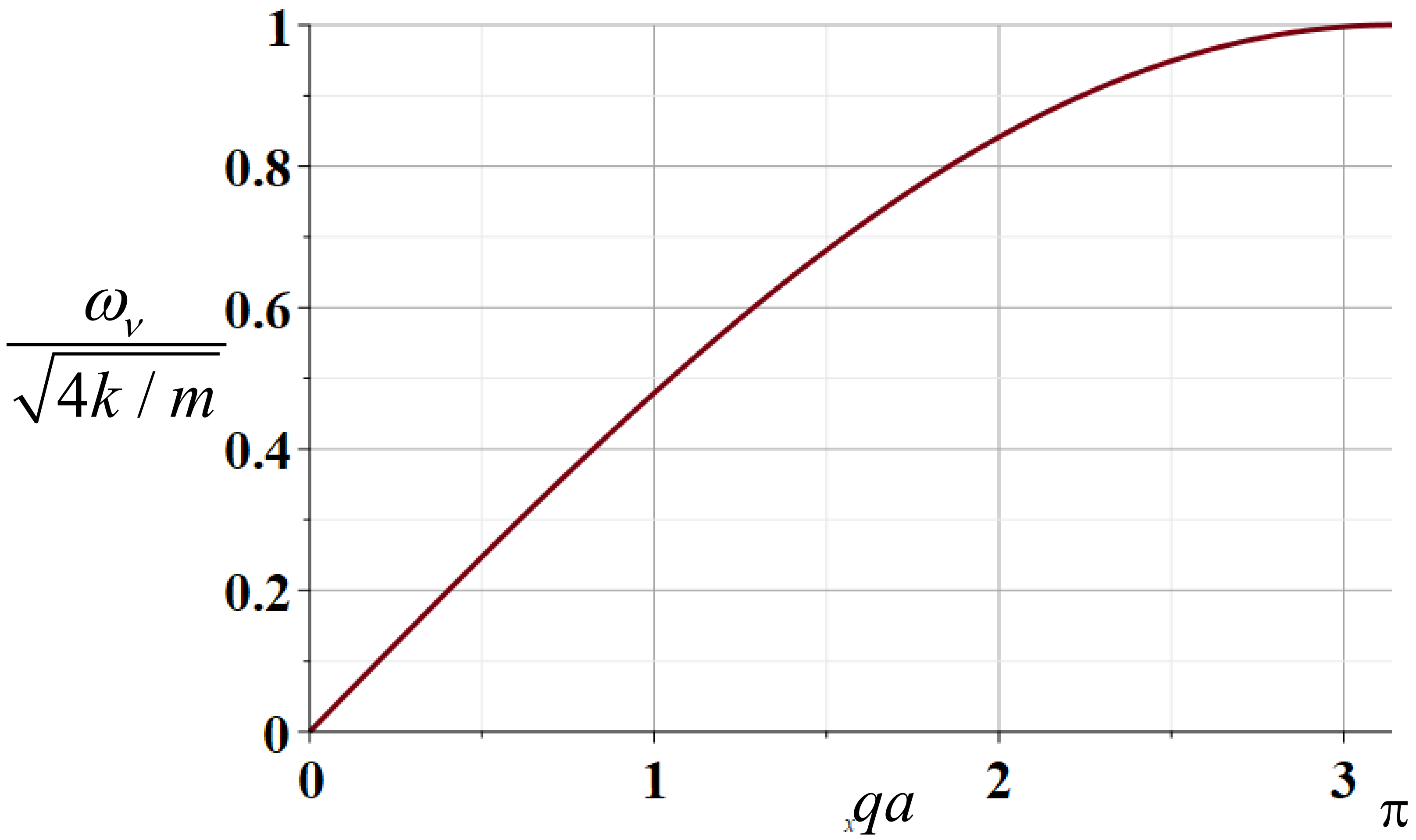
$$\Rightarrow \omega^2 = \frac{4k}{m}\sin^2\left(\frac{qa}{2}\right)$$

Note that we are assuming that all masses and springs are identical here.

Here "*a*" is the equilibrium length of a spring and *q* has the units of 1/length.

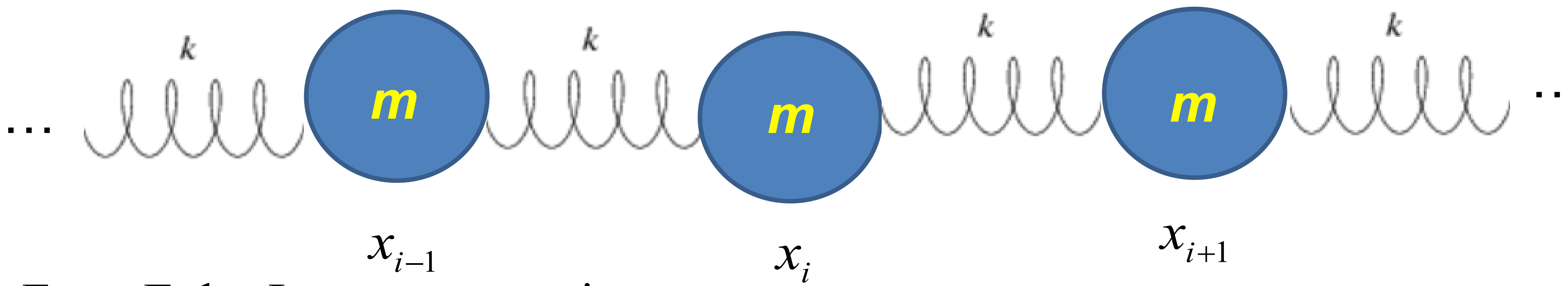
distinct values for $0 \le qa \le \pi$

Plot of distinct values of $\omega_\nu(q)$



Note that for $N \rightarrow \infty$, q becomes a continuous variable within the range $0 < qa < \pi$.

For extended (infinite) chain without boundaries:



From Euler-Lagrange equations:

$$m\ddot{x}_j = k\left(x_{j+1} - 2x_j + x_{j-1}\right) \qquad \text{for all } x_j \qquad \text{Try:} \quad x_j(t) = Ae^{-i\omega t + iqaj}$$

$$-\omega^2 Ae^{-i\omega t+iqaj} = \frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right)Ae^{-i\omega t+iqaj} \qquad -\omega^2 = \frac{k}{m}\left(2\cos(qa) - 2\right)$$

$$\Rightarrow \omega = \sqrt{\frac{4k}{m}}\left|\sin\left(\frac{qa}{2}\right)\right| \qquad \text{distinct values for } 0 \le qa \le \pi$$

Note that there are an infinite number of normal mode frequencies!

Does this make sense?

(A) Yes (B) No

Consider an infinite system of masses and springs now with two kinds of masses:

x_i y_i x_{i+1} y_{i+1} x_{i+2}

Note: each mass coordinate is measured relative to its equilibrium position $x_i^0 \equiv 0, y_i^0 \equiv 0, \cdots$

$$L = T - V$$

$$= \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 + \frac{1}{2} M \sum_{i=0}^{\infty} \dot{y}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - y_i\right)^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(y_i - x_i\right)^2$$

$$L = T - V$$

$$= \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 + \frac{1}{2} M \sum_{i=0}^{\infty} \dot{y}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - y_i\right)^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(y_i - x_i\right)^2$$

Euler - Lagrange equations :

$$m\ddot{x}_j = k\left(y_{j-1} - 2x_j + y_j\right)$$

$$M\ddot{y}_j = k\left(x_j - 2y_j + x_{j+1}\right)$$

Trial solution :

$$x_j(t) = Ae^{-i\omega t + i2qaj}$$

$$y_j(t) = Be^{-i\omega t + i2qaj}$$

Note that 2qa is an unknown parameter.

Does this form seem reasonable?

$$\begin{pmatrix} m\omega^2 - 2k & k\left(e^{-i2qa} + 1\right) \\ k\left(e^{i2qa} + 1\right) & M\omega^2 - 2k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Comment on notation --

x_i y_i x_{i+1} y_{i+1} x_{i+2}

Trial solution:

$$x_j(t) = Ae^{-i\omega t + i2qaj}$$

$$y_j(t) = Be^{-i\omega t + i2qaj}$$

Using 2qa as our unknown parameter is a convenient choice so that we can easily relate our solution to the m=M case.

$$\begin{pmatrix} m\omega^2 - 2k & k\left(e^{-i2qa}+1\right) \\ k\left(e^{i2qa}+1\right) & M\omega^2 - 2k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Solutions:

$$\omega_{\pm}^{\ 2} = \frac{k}{m} + \frac{k}{M} \pm k\sqrt{\frac{1}{m^2} + \frac{1}{M^2} + \frac{2\cos(2qa)}{mM}}$$