



PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF in Olin 103

Notes for Lecture 18: Chap. 7 (F&W)

Mechanical motion of a continuous string

1. Comments on linear vs. non-linear differential equations – considering beyond harmonic oscillations
2. Back to linear analyses -- masses coupled by springs \leftrightarrow mass continuum coupled by string
3. Mechanics one-dimensional continuous system
4. The wave equation



13	Mon, 9/19/2022	Chap. 3 & 6	Liouville theorem	#10	9/21/2022
14	Wed, 9/21/2022	Chap. 3 & 6	Canonical transformations	#11	9/23/2022
15	Fri, 9/23/2022	Chap. 4	Small oscillations about equilibrium	#12	9/26/2022
16	Mon, 9/26/2022	Chap. 4	Normal modes of vibration	#13	9/28/2022
17	Wed, 9/28/2022	Chap. 4	Normal modes of more complicated systems	#14	10/03/2022
18	Fri, 9/30/2022	Chap. 7	Motion of strings		
19	Mon, 10/03/2022	Chap. 7	Sturm-Liouville equations		
20	Wed, 10/05/2022	Chap. 7	Sturm-Liouville equations		
21	Fri, 10/07/2022	Chap. 1-4,6-7	Review		
	Mon, 10/10/2022	No class	Take home exam		
	Wed, 10/12/2022	No class	Take home exam		
	Fri, 10/14/2022	No class	Fall break		
22	Mon, 10/17/2022	Chap. 7	Class resumes		



Digression – comment on linear vs non-linear equations

Linear oscillator equations (ODE example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left. \frac{d^2V}{dx^2} \right|_{x_{eq}} + \dots$$

$$\Rightarrow \frac{1}{2}kx^2 \equiv \frac{1}{2}m\omega^2 x^2$$

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

Euler-Lagrange equations: $\ddot{x} = -\omega^2 x$

Superposition property of linear equations: --

Suppose that the functions $x_1(t)$ and $x_2(t)$ are solutions

$\Rightarrow Ax_1(t) + Bx_2(t)$ are also solutions (all A, B)



Non - linear oscillator equations (example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left. \frac{d^2V}{dx^2} \right|_{x_{eq}} + \frac{1}{4!}(x - x_{eq})^4 \left. \frac{d^4V}{dx^4} \right|_{x_{eq}} + \dots$$

$$\Rightarrow \frac{1}{2} m \omega^2 \left(x^2 + \frac{1}{2} \epsilon x^4 \right)$$

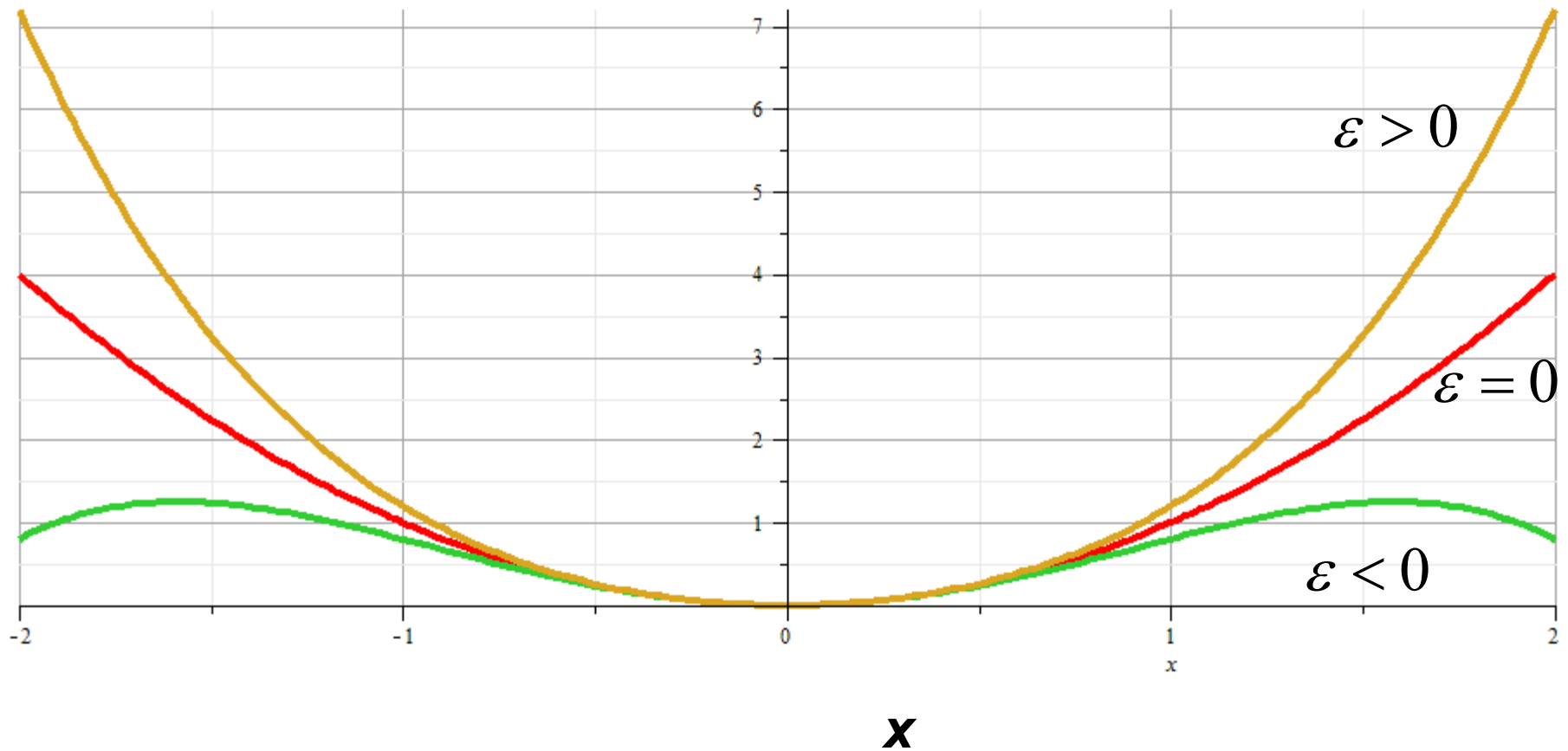
$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 \left(x^2 + \frac{1}{2} \epsilon x^4 \right)$$

Euler - Lagrange equations :

$$\ddot{x} = -\omega^2 (x + \epsilon x^3)$$

Superposition-- no longer applies

$$V(x) \approx \frac{1}{2} m \omega^2 \left(x^2 + \frac{1}{2} \varepsilon x^4 \right)$$





Non - linear example -- continued

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 \left(x^2 + \frac{1}{2} \epsilon x^4 \right)$$

Euler - Lagrange equations :

$$\ddot{x} + \omega^2 (x + \epsilon x^3) = 0$$

Perturbation expansion:

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots$$

Euler-Lagrange equations:

$$\text{zero order: } \ddot{x}_0 + \omega^2 x_0 = 0$$

$$\text{first order: } \ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$$

Non - linear example - - continued

$$\ddot{x} + \omega^2 (x + \varepsilon x^3) = 0$$

Initial conditions :

Perturbation expansion :

$$x(0) = X_0 \quad \dot{x}(0) = 0$$

$$x(t) = x_0(t) + \varepsilon x_1(t) + \dots$$

Euler - Lagrange equations :

$$\text{zero order : } \ddot{x}_0 + \omega^2 x_0 = 0$$

$$\Rightarrow x_0(t) = X_0 \cos(\omega t)$$

$$\text{first order : } \ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$$

$$\Rightarrow \ddot{x}_1(t) + \omega^2 x_1(t) = -X_0^3 \cos^3(\omega t) = -\frac{X_0^3}{4} (3\cos(\omega t) + \cos(3\omega t))$$

$$\Rightarrow x_1(t) = -\frac{X_0^3}{8\omega^2} \left\{ 3\omega t \sin(\omega t) + \frac{1}{4} [\cos(\omega t) - \cos(3\omega t)] \right\}$$

$$x(t) = X_0 \cos(\omega t) - \varepsilon \frac{X_0^3}{8\omega^2} \left\{ 3\omega t \sin(\omega t) + \frac{1}{4} [\cos(\omega t) - \cos(3\omega t)] \right\} + O(\varepsilon^2)$$

Non - linear example - - continued

$$\ddot{x} + \omega^2 (x + \varepsilon x^3) = 0$$

Initial conditions :

$$x(0) = X_0 \quad \dot{x}(0) = 0$$

Perturbation expansion:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \dots$$

Previous result (blows up at large t):

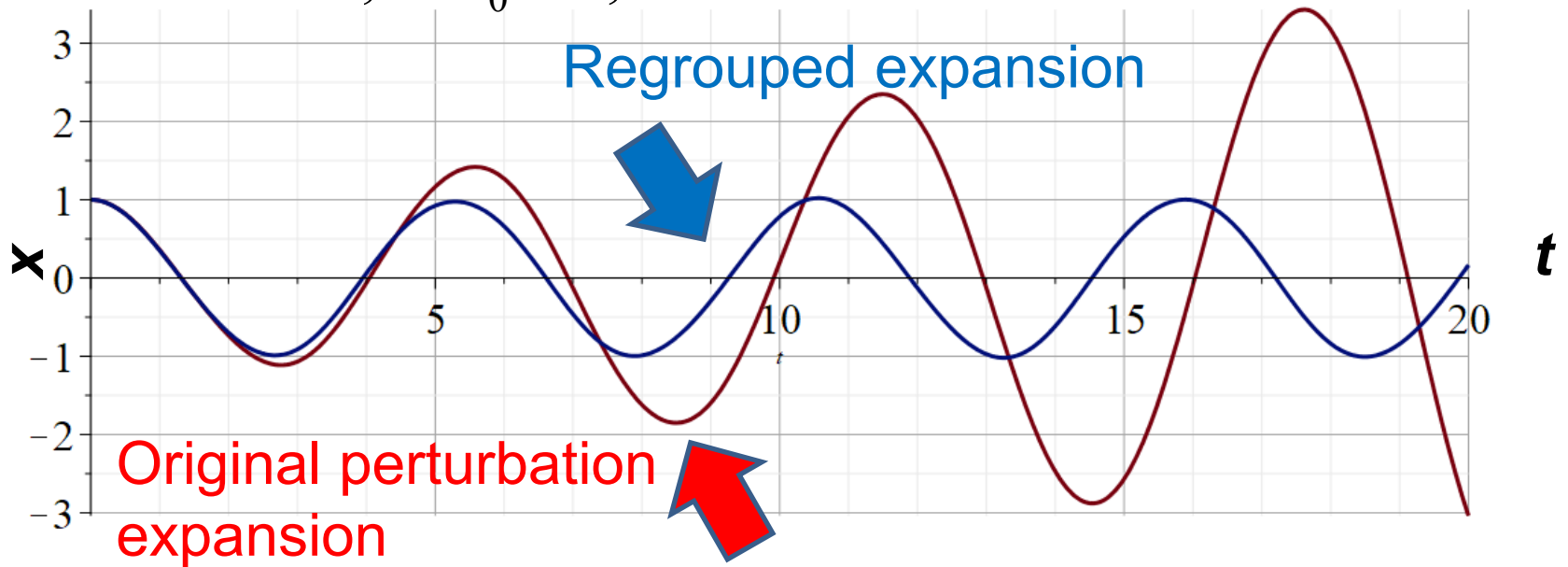
$$x(t) = X_0 \cos(\omega t) - \varepsilon \frac{X_0^3}{8\omega^2} \left\{ 3\omega t \sin(\omega t) + \frac{1}{4} [\cos(\omega t) - \cos(3\omega t)] \right\} + O(\varepsilon^2)$$

By rearranging terms (allowing effective frequency to vary):

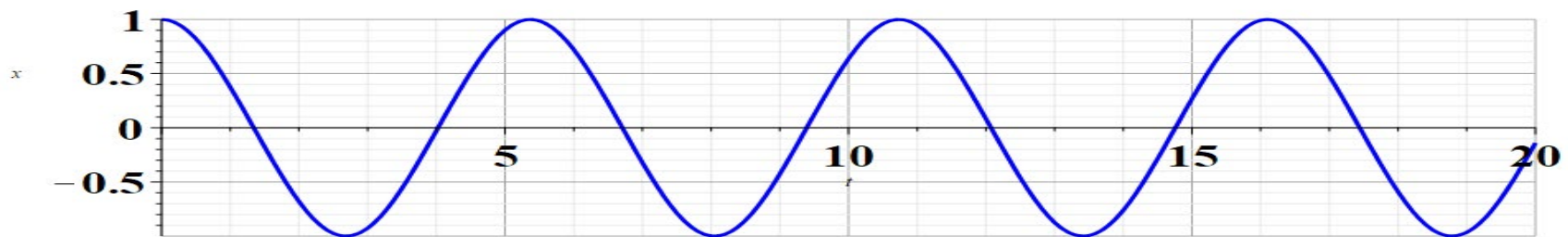
$$x(t) = X_0 \cos \left(\omega \left(1 + \varepsilon \frac{3X_0^2}{8\omega} \right) t \right) - \varepsilon \frac{X_0^3}{32\omega^2} \{ \cos(\omega t) - \cos(3\omega t) \} + O(\varepsilon^2)$$



For $\omega = 1$, $X_0 = 1$, $\epsilon = 0.5$

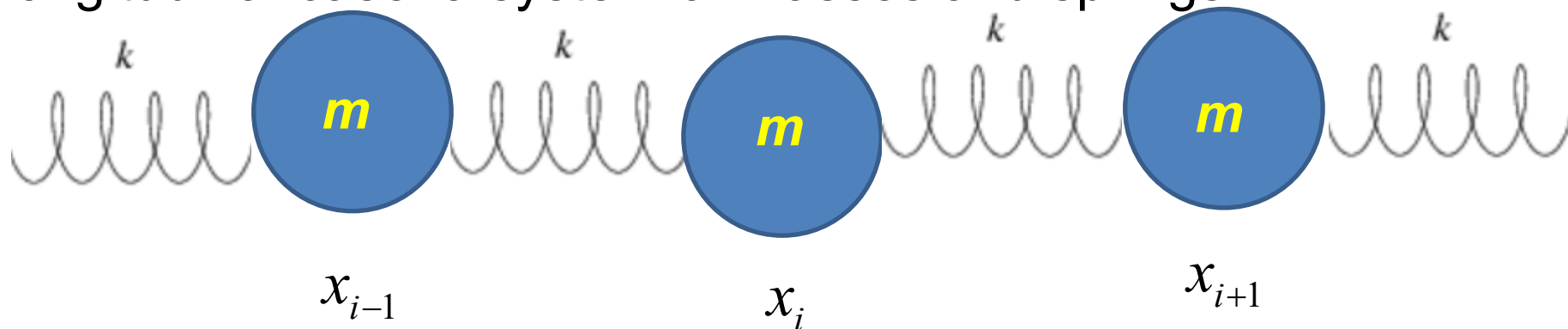


Numerical solution according to Maple



Back to linear equations –

Longitudinal case: a system of masses and springs:



$$L = T - V = \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} (x_{i+1} - x_i)^2$$
$$\Rightarrow m \ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1})$$

Now imagine the continuum version of this system:

$$x_i(t) \Rightarrow \mu(x, t) \quad \ddot{x}_i \Rightarrow \frac{\partial^2 \mu}{\partial t^2}$$

$$x_{i+1} - 2x_i + x_{i-1} \Rightarrow \frac{\partial^2 \mu}{\partial x^2} (\Delta x)^2$$

Discrete equation : $m\ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1}))$

Continuum equation : $m \frac{\partial^2 \mu}{\partial t^2} = k(\Delta x)^2 \frac{\partial^2 \mu}{\partial x^2}$

$$\frac{\partial^2 \mu}{\partial t^2} = \left(\frac{k\Delta x}{m / \Delta x} \right) \frac{\partial^2 \mu}{\partial x^2}$$



system parameter with
units of (velocity)²

For transverse oscillations on a string
with tension τ and mass/length σ :

$$\left(\frac{k\Delta x}{m / \Delta x} \right) \Rightarrow \frac{\tau}{\sigma}$$

More details

Longitudinal case

Consider Taylor's series (focussing on x -dependence)

$$\mu(x + \Delta x) = \mu(x) + \Delta x \left. \frac{d\mu}{dx} \right|_x + \frac{1}{2} (\Delta x)^2 \left. \frac{d^2\mu}{dx^2} \right|_x + \frac{1}{6} (\Delta x)^3 \left. \frac{d^3\mu}{dx^3} \right|_x + \frac{1}{24} (\Delta x)^4 \left. \frac{d^4\mu}{dx^4} \right|_x + \dots$$

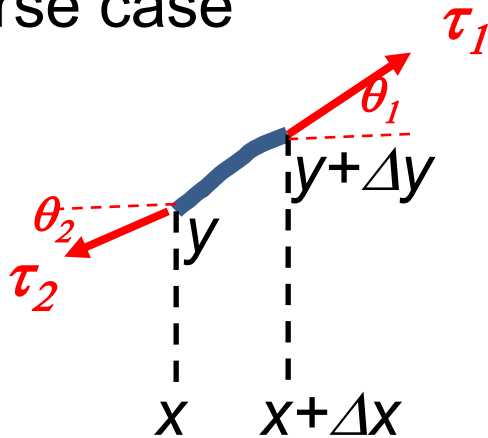
$$\mu(x - \Delta x) = \mu(x) - \Delta x \left. \frac{d\mu}{dx} \right|_x + \frac{1}{2} (\Delta x)^2 \left. \frac{d^2\mu}{dx^2} \right|_x - \frac{1}{6} (\Delta x)^3 \left. \frac{d^3\mu}{dx^3} \right|_x + \frac{1}{24} (\Delta x)^4 \left. \frac{d^4\mu}{dx^4} \right|_x + \dots$$

$$\text{Therefore } (\Delta x)^2 \left. \frac{d^2\mu}{dx^2} \right|_x = \mu(x + \Delta x) + \mu(x - \Delta x) - 2\mu(x) - \frac{1}{12} (\Delta x)^4 \left. \frac{d^4\mu}{dx^4} \right|_x + \dots$$

$$\Rightarrow \left. \frac{d^2\mu}{dx^2} \right|_x \approx \frac{\mu(x + \Delta x) + \mu(x - \Delta x) - 2\mu(x)}{(\Delta x)^2}$$

More details

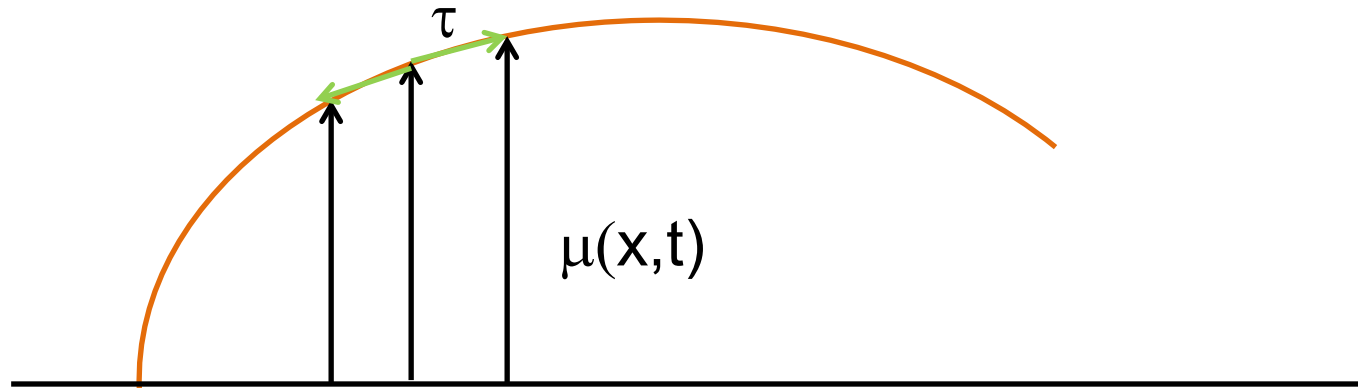
Transverse case



Net vertical force on increment of string:

$$\begin{aligned}\tau_1 \sin \theta_1 - \tau_2 \sin \theta_2 &\approx \tau_1 \tan \theta_1 - \tau_2 \tan \theta_2 \\ &\approx \tau \left(\left. \frac{dy}{dx} \right|_{x+\Delta x} - \left. \frac{dy}{dx} \right|_x \right) \\ &= \tau \Delta x \frac{d}{dx} \left(\frac{dy}{dx} \right) \\ &= \tau \left(\Delta x \frac{d^2 y}{dx^2} \right)\end{aligned}$$

Transverse displacement:



$$\frac{\partial^2 \mu}{\partial t^2} = \frac{\tau}{\sigma} \frac{\partial^2 \mu}{\partial x^2}$$

Wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} = c^2 \frac{\partial^2 \mu}{\partial x^2}$$

Lagrangian for continuous system :

Denote the generalized displacement by $\mu(x, t)$:

$$L = L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right)$$

Hamilton's principle :

$$\delta \int_{t_i}^{t_f} dt \int_{x_i}^{x_f} dx L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x} \frac{\partial L}{\partial(\partial \mu / \partial x)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial(\partial \mu / \partial t)} = 0$$

Euler - Lagrange equations for continuous system :

$$\frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x} \frac{\partial L}{\partial (\partial \mu / \partial x)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial (\partial \mu / \partial t)} = 0$$

Example :

$$L = \frac{\sigma}{2} \left(\frac{\partial \mu}{\partial t} \right)^2 - \frac{\tau}{2} \left(\frac{\partial \mu}{\partial x} \right)^2$$

$$\Rightarrow \sigma \frac{\partial^2 \mu}{\partial t^2} - \tau \frac{\partial^2 \mu}{\partial x^2} = 0$$

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{for} \quad c^2 = \frac{\tau}{\sigma}$$



Note that this is an example of a **partial** differential equation

General solutions $\mu(x, t)$ to the wave equation :

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x, t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.



Initial value solutions $\mu(x,t)$ to the wave equation;
 attributed to D' Alembert :

These functions
 would be given

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = \phi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

Assume :

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

then : $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c \left(\frac{df(x)}{dx} - \frac{dg(x)}{dx} \right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c} \int^x \psi(x') dx'$$

Solution -- continued: $\mu(x,t) = f(x-ct) + g(x+ct)$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c \left(\frac{df(x)}{dx} - \frac{dg(x)}{dx} \right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c} \int^x \psi(x') dx'$$

For each x , find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2} \left(\phi(x) - \frac{1}{c} \int^x \psi(x') dx' \right)$$

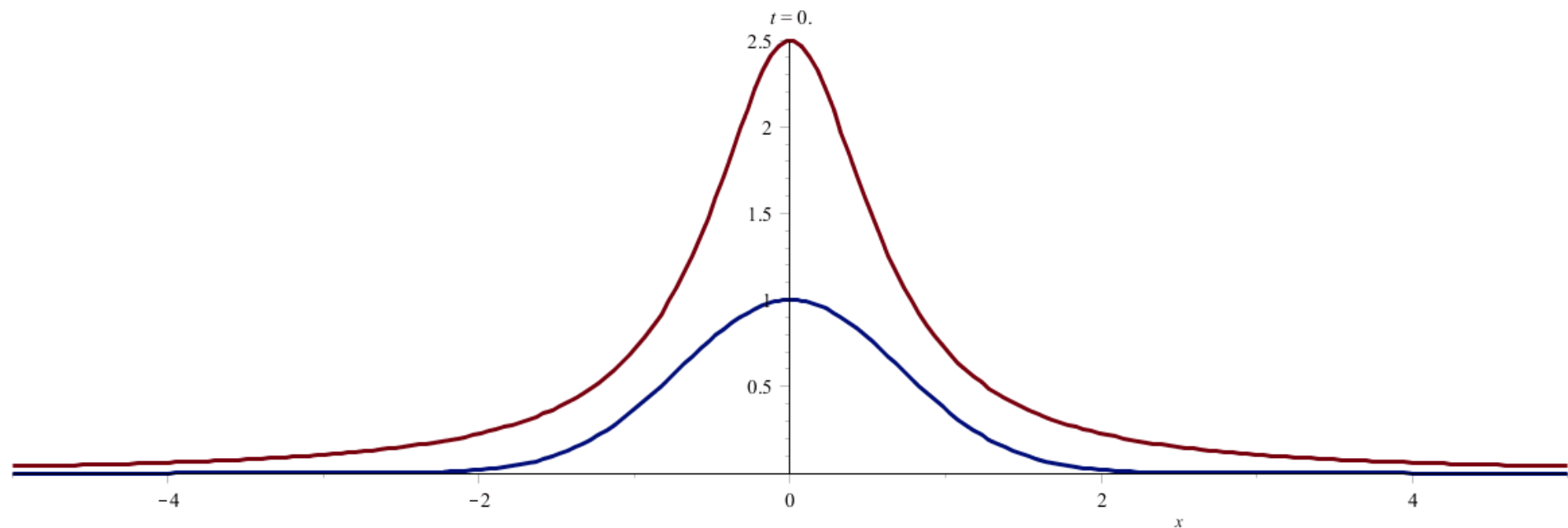
$$g(x) = \frac{1}{2} \left(\phi(x) + \frac{1}{c} \int^x \psi(x') dx' \right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2} (\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(x,t) = \frac{1}{2} \left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2} \right)$$



Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \quad \text{and} \quad \frac{\partial \mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2} e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c} \left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2} \right)$$

$$\text{Note that } \frac{\partial \mu(x,t)}{\partial t} = -\frac{1}{\sigma^2} \left((x+ct)e^{-(x+ct)^2/\sigma^2} + (x-ct)e^{-(x-ct)^2/\sigma^2} \right)$$

