PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion on Lecture 19 – Chap. 7 (F&W)

Solutions of differential equations

1. The wave equation – traveling wave solutions
2. The wave equation – standing wave solutions
3. The Sturm-Liouville equation

14	Wed, 9/21/2022	Chap. 3 & 6	Canonical transformations	#11	9/23/2022
15	Fri, 9/23/2022	Chap. 4	Small oscillations about equilibrium	#12	9/26/2022
16	Mon, 9/26/2022	Chap. 4	Normal modes of vibration	#13	9/28/2022
17	Wed, 9/28/2022	Chap. 4	Normal modes of more complicated systems	#14	10/03/2022
18	Fri, 9/30/2022	Chap. 7	Motion of strings		
19	Mon, 10/03/2022	Chap. 7	Sturm-Liouville equations	#15	10/05/2022
20	Wed, 10/05/2022	Chap. 7	Sturm-Liouville equations		
21	Fri, 10/07/2022	Chap. 1-4,6-7	Review		
	Mon, 10/10/2022	No class	Take home exam		
	Wed, 10/12/2022	No class	Take home exam		
	Fri, 10/14/2022	No class	Fall break		
22	Mon, 10/17/2022	Chap. 7	Class resumes		

PHY 711 – Assignment #15

10/03/2022

Continue reading Chapter 7 in **Fetter and Walecka**.

Consider a one-dimensional traveling wave characterized by displacement $\mu(x,t)$ as a function of position x for $-\infty \leq x \leq \infty$ and time t for $0 \leq t \leq \infty$, is described by the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0, \tag{1}$$

where c denotes the wave speed. Find the functional form for the traveling wave $\mu(x,t)$ for each of these initial conditions.

1. At $t = 0$,

$$\mu(x,0) = \frac{A}{\cosh(x)} \quad \text{and} \quad \frac{\partial \mu(x,0)}{\partial t} = 0, \tag{2}$$

where A is a given wave amplitude.

2. At $t = 0$,

$$\mu(x,0) = 0 \quad \text{and} \quad \frac{\partial \mu(x,0)}{\partial t} = \frac{A \sinh(x)}{\cosh^2(x)}, \tag{3}$$

where A is a given wave speed amplitude.

One-dimensional wave equation
representing longitudinal or transverse displacements as a function of *x* and *t* , an example of a partial differential equation --

Traveling wave solutions thanks to D'Alembert --

For the displacement function, $\mu(x,t)$, the wave equation has the form:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

Initial value traveling wave solutions $\mu(x,t)$ to the wave equation; attributed to D'Alembert:

These functions would be given

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \varphi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

Assume:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

then: $$\mu(x,0) = \varphi(x) = f(x) + g(x)$$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

Solution -- continued : $\mu(x,t) = f(x-ct) + g(x+ct)$

then : $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\big(\phi(x-ct) + \phi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Checking that D'Alembert's solution solves the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

$$\mu(x,t) = \frac{1}{2}\left(\varphi(x-ct) + \varphi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

$$\frac{\partial \mu(x,t)}{\partial x} = \frac{1}{2}\left(\varphi'(x-ct) + \varphi'(x+ct)\right) + \frac{1}{2c}\left(\psi(x-ct) + \psi(x+ct)\right)$$

$$\frac{\partial^2 \mu(x,t)}{\partial x^2} = \frac{1}{2}\left(\varphi''(x-ct) + \varphi''(x+ct)\right) + \frac{1}{2c}\left(\psi'(x-ct) + \psi'(x+ct)\right)$$

$$\frac{\partial \mu(x,t)}{\partial t} = \frac{c}{2}\left(-\varphi'(x-ct) + \varphi'(x+ct)\right) + \frac{c}{2c}\left(-\psi(x-ct) + \psi(x+ct)\right)$$

$$\frac{\partial^2 \mu(x,t)}{\partial t^2} = \frac{c^2}{2}\left(\varphi''(x-ct) + \varphi''(x+ct)\right) + \frac{c^2}{2c}\left(\psi'(x-ct) + \psi'(x+ct)\right)$$

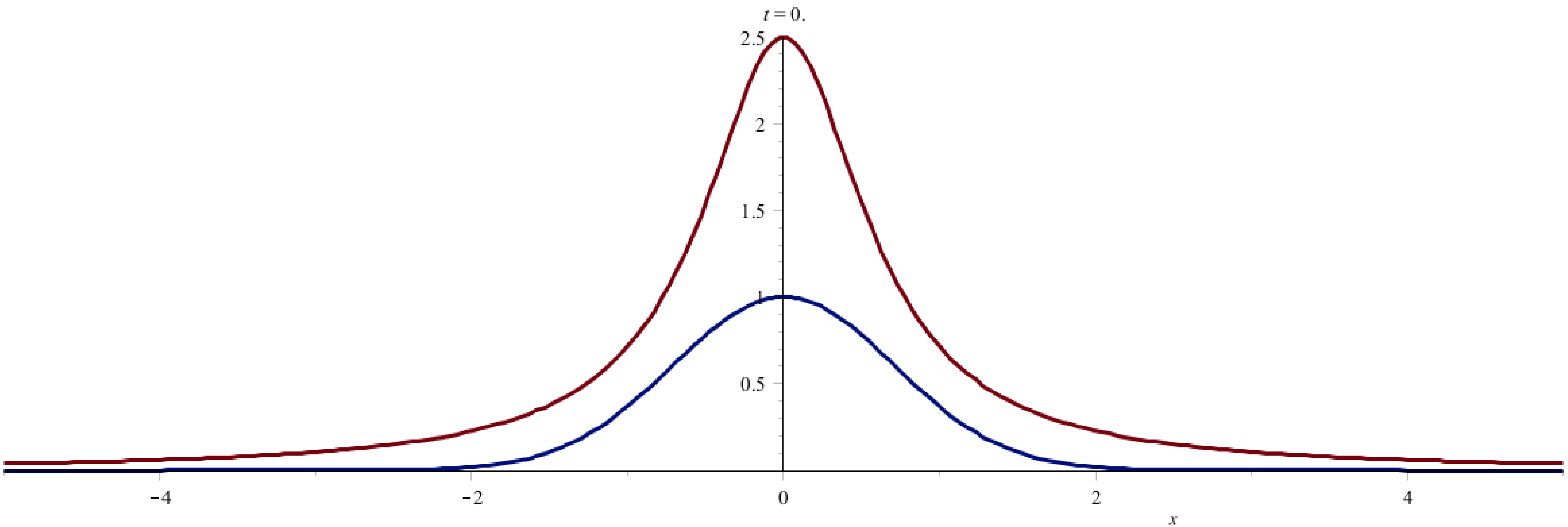
Here we have assumed that $\varphi(u)$ and $\psi(u)$ are continuous functions and

$$\varphi'(u) \equiv \frac{d\varphi(u)}{du}.$$

Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(\mathrm{x},\mathrm{t}) = \frac{1}{2}\left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2}\right)$$

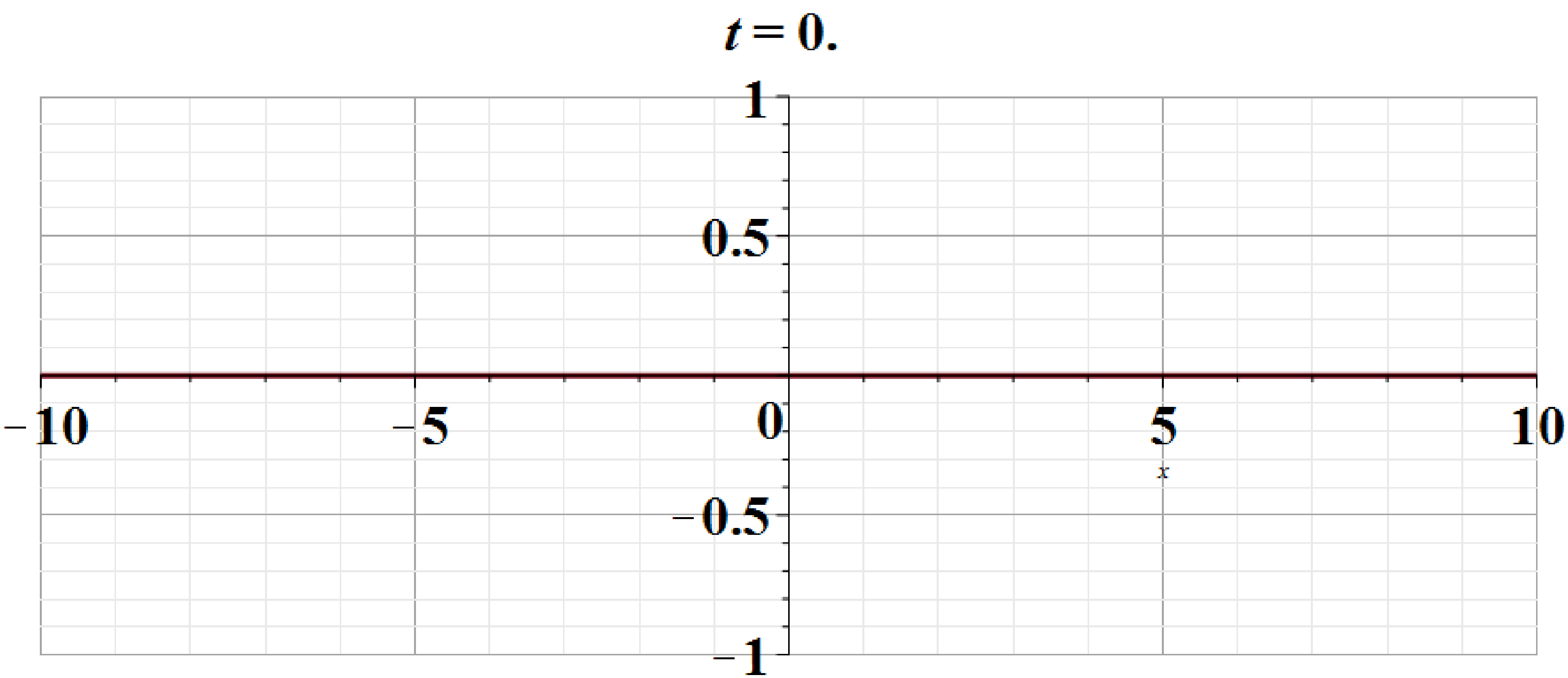


Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \text{ and } \frac{\partial \mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2} e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c}\left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2}\right)$$

Note that $\displaystyle \frac{\partial \mu(x,t)}{\partial t} = -\frac{1}{\sigma^2}\left((x+ct)e^{-(x+ct)^2/\sigma^2} + (x-ct)e^{-(x-ct)^2/\sigma^2}\right)$



Other types of solutions to the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that because of the way that the equation is written, it is possible to find "separable" solutions of the form

$$\mu(x,t) = X(x)T(t)$$

or more generally, a linear combination of separable solutions:

$$\mu(x,t) = \sum_n X_n(x)T_n(t)$$

Other types of solutions to the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{for} \qquad \mu(x,t) = X(x)T(t)$$

$$\Rightarrow \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \frac{1}{c^2 T(t)} \frac{d^2 T(t)}{dt^2}$$

For example, suppose the time function is harmonic in time with frequency ω: $\quad T(t) = \cos(\omega t + \eta)$

Then the spacial function must statisfy the ordinary differential equation:

$$\frac{d^2 X(x)}{dx^2} = -\frac{\omega^2}{c^2} X(x)$$

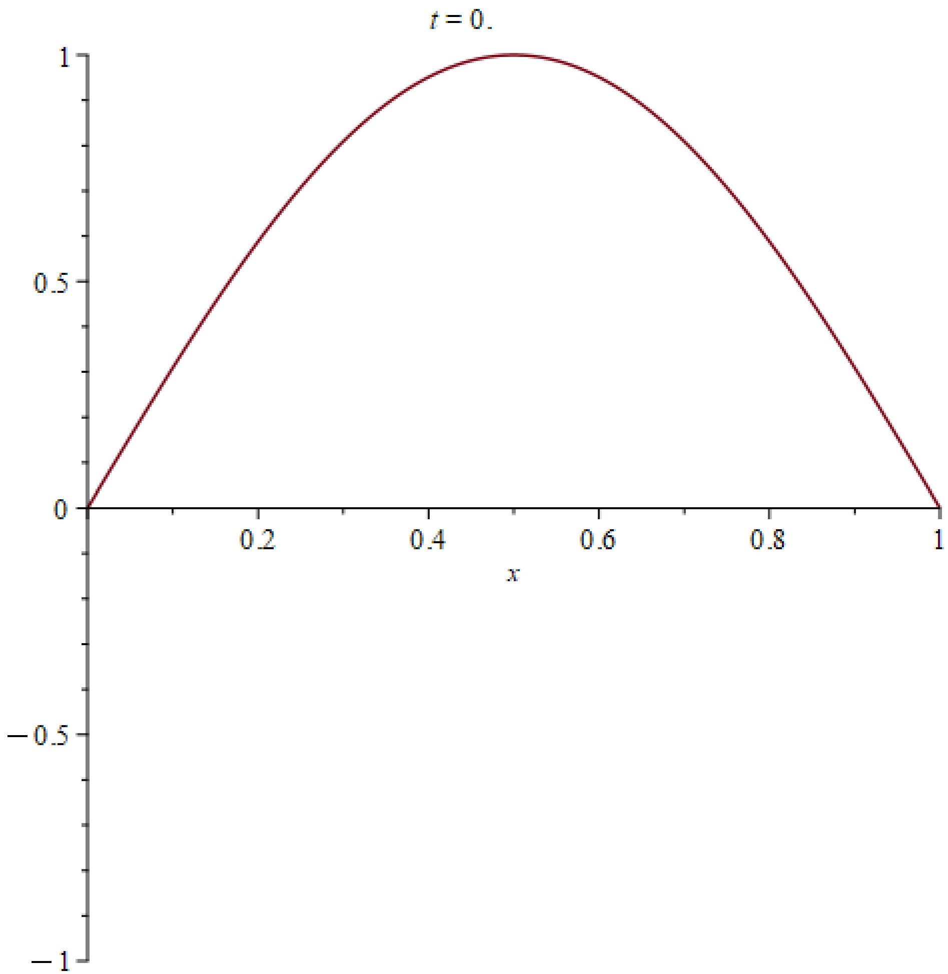
$$\Rightarrow X(x) = A \sin(kx + \nu) \quad \text{where} \quad k = \frac{\omega}{c}$$

It is often the case, there are boundary values specified for *X(x).*

For example, suppose $X(0)=0$ and $X(a)=0$ --

$$\Rightarrow X(x) = A\sin\left(\frac{n\pi x}{a}\right) \quad \text{and } \omega = \frac{n\pi c}{a}$$

Standing wave -- $$\mu(x,t) = A \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi c t}{a}\right)$$



How are the traveling wave and standing wave solutions to the wave equations related?

A. They are exactly the same
B. They are not related
C. ???

The wave equation and related linear PDE's

One dimensional wave equation for $\mu(x,t)$:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Generalization for spacially dependent tension and mass density plus an extra potential energy density:

$$\sigma(x)\frac{\partial^2 \mu(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left(\tau(x)\frac{\partial \mu(x,t)}{\partial x}\right) + v(x)\mu(x,t) = 0$$

Factoring time and spatial variables:

$$\mu(x,t) = \phi(x)\ \cos(\omega t + \alpha)$$

Sturm-Liouville equation for spatial function $\phi(x)$:

$$-\frac{d}{dx}\left(\tau(x)\frac{d\phi(x)}{dx}\right) + v(x)\phi(x) = \omega^2\sigma(x)\phi(x)$$

Linear second-order ordinary differential equations
Sturm-Liouville equations

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$

given functions

solution to be determined

applied force

When applicable, it is assumed that the form of the applied force is known.

Homogenous problem: $F(x)=0$

Examples of Sturm-Liouville eigenvalue equations --

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=0$$

Bessel functions: $0 \le x \le \infty$

$$\tau(x)=-x \quad v(x)=x \quad \sigma(x)=\frac{1}{x} \quad \lambda=\nu^2 \quad \varphi(x)=J_\nu(x)$$

Legendre functions: $-1 \le x \le 1$

$$\tau(x)=-\left(1-x^2\right) \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=l(l+1) \quad \varphi(x)=P_l(x)$$

Fourier functions: $0 \le x \le 1$

$$\tau(x)=1 \quad v(x)=0 \quad \sigma(x)=1 \quad \lambda=n^2\pi^2 \quad \varphi(x)=\sin(n\pi x)$$

Solution methods of Sturm-Liouville equations (assume all functions and constants are real):

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Orthogonality of eigenfunctions: $\int_a^b \sigma(x) f_n(x) f_m(x) dx = \delta_{nm} N_n,$

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2 dx.$

Completeness of eigenfunctions:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

Why all of the fuss about eigenvalues and eigenvectors?

a. They are sometimes useful in finding solutions to differential equations
b. Not all eigenfunctions have analytic forms.
c. It is possible to solve a differential equation without the use of eigenfunctions.
d. Eigenfunctions have some useful properties.

Comment on orthogonality of eigenfunctions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_m(x)=\lambda_m\sigma(x)f_m(x)$$

$$f_m(x)\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)-f_n(x)\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_m(x)$$

$$=\left(\lambda_n-\lambda_m\right)\sigma(x)f_n(x)f_m(x)$$

$$-\frac{d}{dx}\left(f_m(x)\tau(x)\frac{df_n(x)}{dx}-f_n(x)\tau(x)\frac{df_m(x)}{dx}\right)=\left(\lambda_n-\lambda_m\right)\sigma(x)f_n(x)f_m(x)$$

Comment on orthogonality of eigenfunctions -- continued

$$-\frac{d}{dx}\left(f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right) = \left(\lambda_n - \lambda_m\right)\sigma(x) f_n(x) f_m(x)$$

Now consider integrating both sides of the equation in the interval

$a \le x \le b$:

$$-\left(f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right)\Bigg|_a^b = \left(\lambda_n - \lambda_m\right)\int_a^b dx\,\sigma(x) f_n(x) f_m(x)$$

Vanishes for various boundary conditions at *x=a and x=b*

Comment on orthogonality of eigenfunctions -- continued

$$-\left(f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right)\Bigg|_a^b = (\lambda_n - \lambda_m)\int_a^b dx\sigma(x) f_n(x) f_m(x)$$

Possible boundary values for Sturm-Liouville equations:

1. $f_m(a) = f_m(b) = 0$

2. $\tau(x)\frac{df_m(x)}{dx}\Big|_a = \tau(x)\frac{df_m(x)}{dx}\Big|_b = 0$

3. $f_m(a) = f_m(b)$ and $\frac{df_m(a)}{dx} = \frac{df_m(b)}{dx}$

In any of these cases, we can conclude that:

$$\int_a^b dx\sigma(x) f_n(x) f_m(x) = 0 \text{ for } \lambda_n \neq \lambda_m$$

Comment on "completeness"

It can be shown that for any reasonable function *h(x)*, defined within the interval $a < x < b$, we can expand that function as a linear combination of the eigenfunctions $f_n(x)$

$$h(x) \approx \sum_n C_n f_n(x),$$

$$\text{where} \quad C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$$

These ideas lead to the notion that the set of eigenfunctions $f_n(x)$ form a ``complete'' set in the sense of ``spanning'' the space of all functions in the interval $a < x < b$, as summarized by the statement:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x - x').$$

Comment on "completeness" -- continued

$$h(x) \approx \sum_n C_n f_n(x),$$

where $$C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$$

Consider the squared error of the expansion:

$$\epsilon^2 = \int_a^b dx\sigma(x)\left(h(x) - \sum_n C_n f_n(x)\right)^2$$

ϵ^2 can be minimized:

$$\frac{\partial \epsilon^2}{\partial C_m} = 0 = -2\int_a^b dx\sigma(x)\left(h(x) - \sum_n C_n f_n(x)\right)f_m(x)$$

$$\Rightarrow C_m = \frac{1}{N_m}\int_a^b dx\sigma(x)h(x)f_m(x)$$