



PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF in Olin 103

Discussion on Lecture 19 – Chap. 7 (F&W)

Solutions of differential equations

- 1. The wave equation – traveling wave solutions**
- 2. The wave equation – standing wave solutions**
- 3. The Sturm-Liouville equation**



14	Wed, 9/21/2022	Chap. 3 & 6	Canonical transformations	#11	9/23/2022
15	Fri, 9/23/2022	Chap. 4	Small oscillations about equilibrium	#12	9/26/2022
16	Mon, 9/26/2022	Chap. 4	Normal modes of vibration	#13	9/28/2022
17	Wed, 9/28/2022	Chap. 4	Normal modes of more complicated systems	#14	10/03/2022
18	Fri, 9/30/2022	Chap. 7	Motion of strings		
19	Mon, 10/03/2022	Chap. 7	Sturm-Liouville equations	#15	10/05/2022
20	Wed, 10/05/2022	Chap. 7	Sturm-Liouville equations		
21	Fri, 10/07/2022	Chap. 1-4,6-7	Review		
	Mon, 10/10/2022	No class	Take home exam		
	Wed, 10/12/2022	No class	Take home exam		
	Fri, 10/14/2022	No class	Fall break		
22	Mon, 10/17/2022	Chap. 7	Class resumes		



PHY 711 – Assignment #15

10/03/2022

Continue reading Chapter 7 in **Fetter and Walecka**.

Consider a one-dimensional traveling wave characterized by displacement $\mu(x, t)$ as a function of position x for $-\infty \leq x \leq \infty$ and time t for $0 \leq t \leq \infty$, is described by the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0, \quad (1)$$

where c denotes the wave speed. Find the functional form for the traveling wave $\mu(x, t)$ for each of these initial conditions.

1. At $t = 0$,

$$\mu(x, 0) = \frac{A}{\cosh(x)} \quad \text{and} \quad \frac{\partial \mu(x, 0)}{\partial t} = 0, \quad (2)$$

where A is a given wave amplitude.

2. At $t = 0$,

$$\mu(x, 0) = 0 \quad \text{and} \quad \frac{\partial \mu(x, 0)}{\partial t} = \frac{A \sinh(x)}{\cosh^2(x)}, \quad (3)$$

where A is a given wave speed amplitude.



One-dimensional wave equation

representing longitudinal or transverse displacements as a function of x and t , an example of a partial differential equation --

Traveling wave solutions thanks to D'Alembert --

For the displacement function, $\mu(x,t)$, the wave equation has the form:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

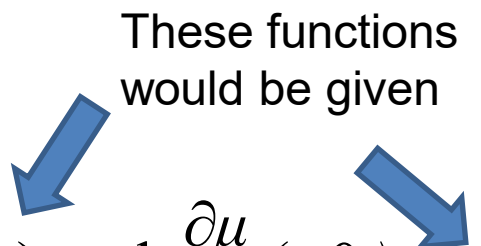
$$\mu(x,t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

Initial value traveling wave solutions $\mu(x,t)$ to the wave equation; attributed to D'Alembert:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = \varphi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

These functions would be given



Assume:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

then: $\mu(x,0) = \varphi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c \left(\frac{df(x)}{dx} - \frac{dg(x)}{dx} \right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c} \int^x \psi(x') dx'$$

Solution -- continued: $\mu(x,t) = f(x-ct) + g(x+ct)$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c \left(\frac{df(x)}{dx} - \frac{dg(x)}{dx} \right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c} \int^x \psi(x') dx'$$

For each x , find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2} \left(\phi(x) - \frac{1}{c} \int^x \psi(x') dx' \right)$$

$$g(x) = \frac{1}{2} \left(\phi(x) + \frac{1}{c} \int^x \psi(x') dx' \right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2} (\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

Checking that D'Alembert's solution solves the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

$$\mu(x, t) = \frac{1}{2} (\varphi(x - ct) + \varphi(x + ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

$$\frac{\partial \mu(x, t)}{\partial x} = \frac{1}{2} (\varphi'(x - ct) + \varphi'(x + ct)) + \frac{1}{2c} (\psi(x - ct) + \psi(x + ct))$$

$$\frac{\partial^2 \mu(x, t)}{\partial x^2} = \frac{1}{2} (\varphi''(x - ct) + \varphi''(x + ct)) + \frac{1}{2c} (\psi'(x - ct) + \psi'(x + ct))$$

$$\frac{\partial \mu(x, t)}{\partial t} = \frac{c}{2} (-\varphi'(x - ct) + \varphi'(x + ct)) + \frac{c}{2c} (-\psi(x - ct) + \psi(x + ct))$$

$$\frac{\partial^2 \mu(x, t)}{\partial t^2} = \frac{c^2}{2} (\varphi''(x - ct) + \varphi''(x + ct)) + \frac{c^2}{2c} (\psi'(x - ct) + \psi'(x + ct))$$

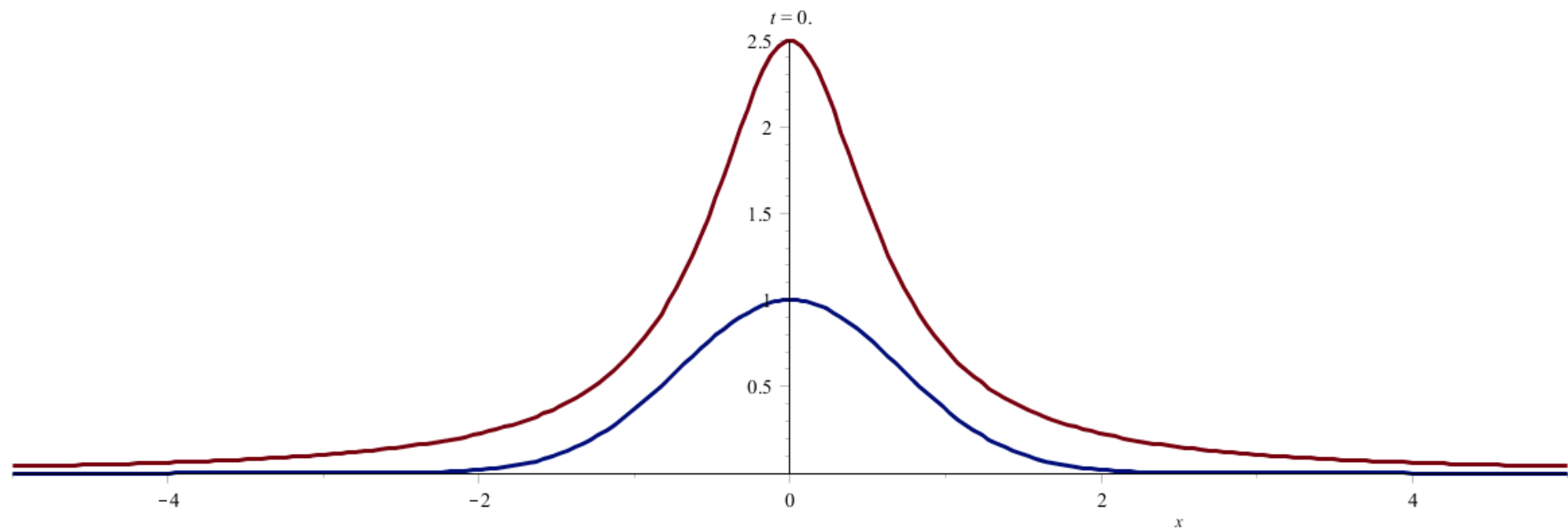
Here we have assumed that $\varphi(u)$ and $\psi(u)$ are continuous functions and

$$\varphi'(u) \equiv \frac{d\varphi(u)}{du}, \quad \varphi''(u) \equiv \frac{d^2\varphi(u)}{du^2}, \quad \text{etc.}$$

Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(x,t) = \frac{1}{2} \left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2} \right)$$

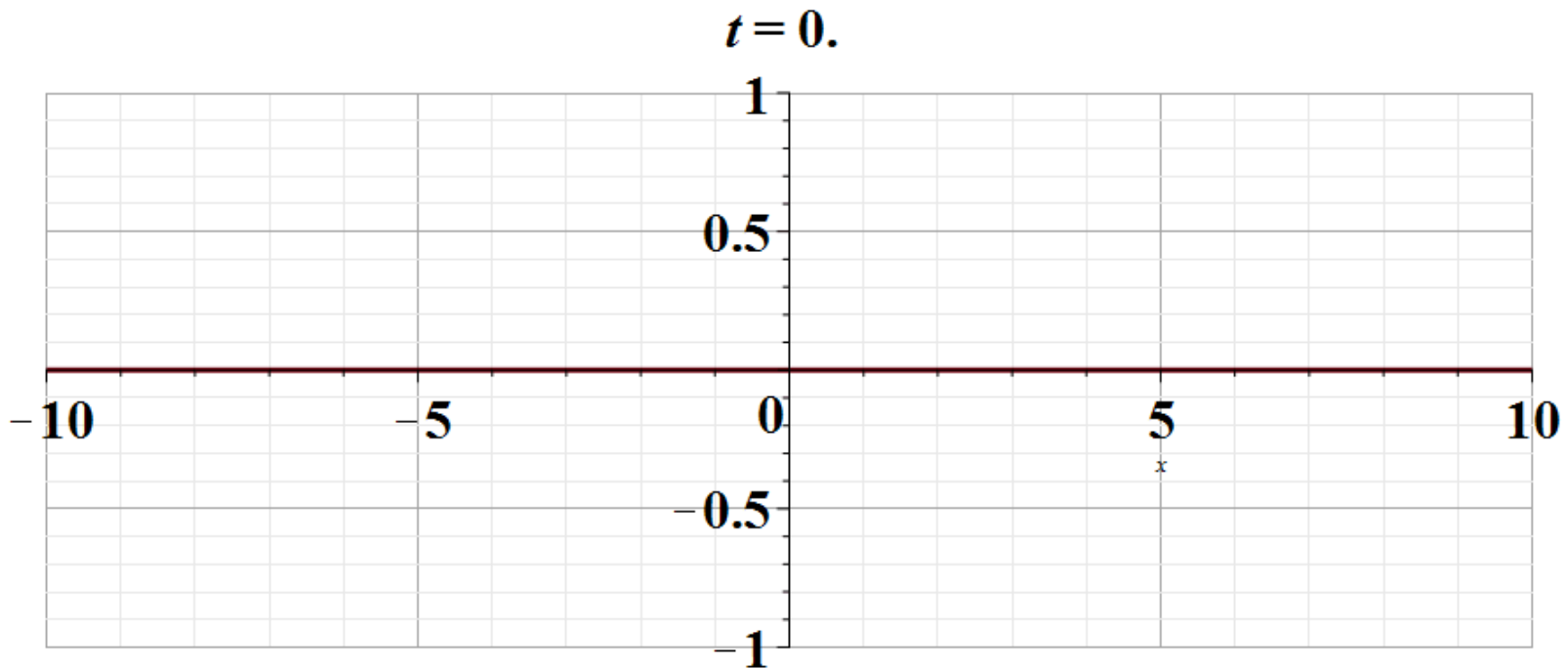


Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \quad \text{and} \quad \frac{\partial \mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2} e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c} \left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2} \right)$$

$$\text{Note that } \frac{\partial \mu(x,t)}{\partial t} = -\frac{1}{\sigma^2} \left((x+ct)e^{-(x+ct)^2/\sigma^2} + (x-ct)e^{-(x-ct)^2/\sigma^2} \right)$$



Other types of solutions to the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that because of the way that the equation is written, it is possible to find "separable" solutions of the form

$$\mu(x, t) = X(x)T(t)$$

or more generally, a linear combination of separable solutions:

$$\mu(x, t) = \sum_n X_n(x)T_n(t)$$

Separable solutions to the wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{for} \quad \mu(x, t) = X(x)T(t)$$

$$\Rightarrow \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \frac{1}{c^2 T(t)} \frac{d^2 T(t)}{dt^2}$$

For example, suppose the time function is harmonic in time with frequency ω : $T(t) = \cos(\omega t + \eta)$

Then the spacial function must satisfy the ordinary differential equation:

$$\frac{d^2 X(x)}{dx^2} = -\frac{\omega^2}{c^2} X(x)$$

$$\Rightarrow X(x) = A \sin(kx + \nu) \quad \text{where} \quad k = \frac{\omega}{c}$$

It is often the case, there are boundary values specified for $X(x)$.

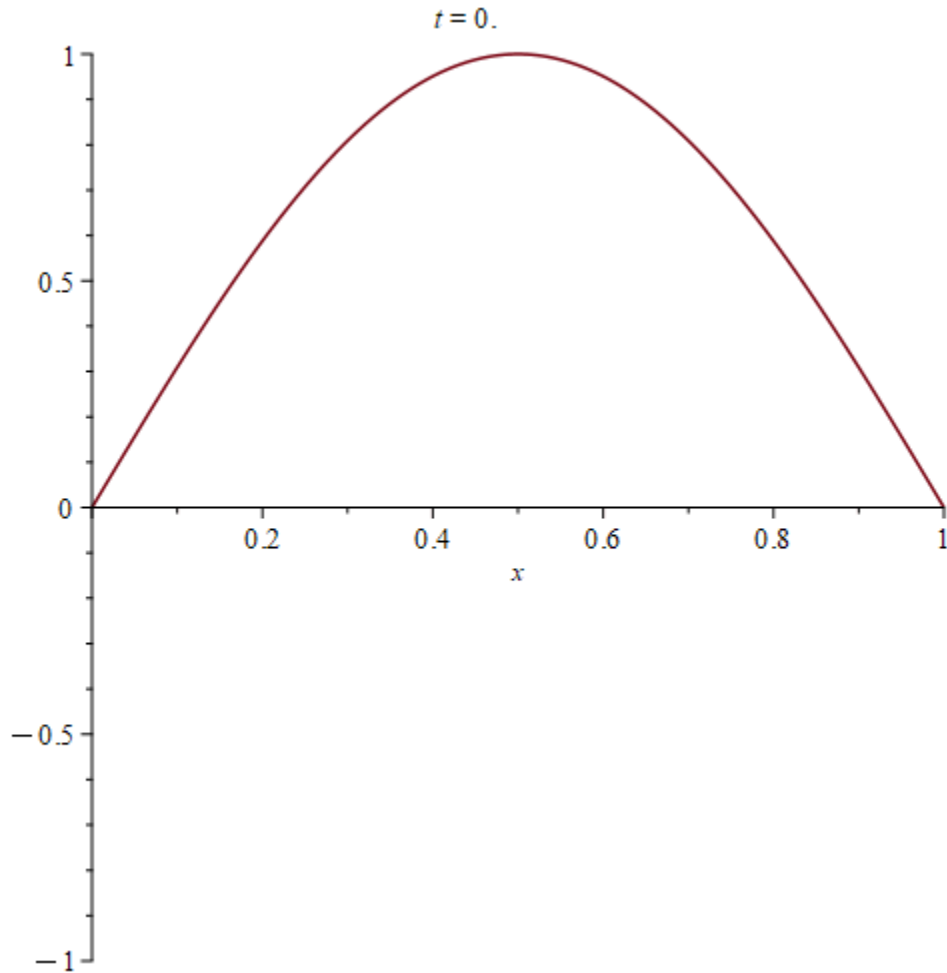
For example, suppose $X(0) = 0$ and $X(a) = 0$ — —

$$A \sin(kx) \Big|_{x=0} = 0 \quad A \sin(kx) \Big|_{x=a} = 0$$

$$\Rightarrow k = \frac{n\pi x}{a} \quad \text{for } n = 0, 1, 2, \dots$$

$$\Rightarrow X(x) = A \sin\left(\frac{n\pi x}{a}\right) \quad \text{and } \omega = \frac{n\pi c}{a}$$

Standing wave --
$$\mu(x,t) = A \sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi ct}{a}\right)$$



How are the traveling wave and standing wave solutions to the wave equations related?

- A. They are exactly the same
- B. They are not related
- C. ???

The wave equation and related linear PDE's

One dimensional wave equation for $\mu(x,t)$:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Generalization for spatially dependent tension and mass density plus an extra potential energy density:

$$\sigma(x) \frac{\partial^2 \mu(x,t)}{\partial t^2} - \frac{\partial}{\partial x} \left(\tau(x) \frac{\partial \mu(x,t)}{\partial x} \right) + v(x) \mu(x,t) = 0$$

Factoring time and spatial variables:

$$\mu(x,t) = \phi(x) \cos(\omega t + \alpha)$$

Sturm-Liouville equation for spatial function $\phi(x)$:

$$-\frac{d}{dx} \left(\tau(x) \frac{d\phi(x)}{dx} \right) + v(x) \phi(x) = \omega^2 \sigma(x) \phi(x)$$



Linear second-order ordinary differential equations

Sturm-Liouville equations

Inhomogenous problem: $\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \varphi(x) = F(x)$

given functions

applied force

solution to be determined

When applicable, it is assumed that the form of the applied force is known.

Homogenous problem: $F(x)=0$

Examples of Sturm-Liouville eigenvalue equations --

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \varphi(x) = 0$$

Bessel functions: $0 \leq x < \infty$

$$\tau(x) = -x \quad v(x) = x \quad \sigma(x) = \frac{1}{x} \quad \lambda = \nu^2 \quad \varphi(x) = J_\nu(x)$$

Legendre functions: $-1 \leq x \leq 1$

$$\tau(x) = -(1-x^2) \quad v(x) = 0 \quad \sigma(x) = 1 \quad \lambda = l(l+1) \quad \varphi(x) = P_l(x)$$

Fourier functions: $0 \leq x \leq 1$

$$\tau(x) = 1 \quad v(x) = 0 \quad \sigma(x) = 1 \quad \lambda = n^2 \pi^2 \quad \varphi(x) = \sin(n\pi x)$$

Solution methods of Sturm-Liouville equations

(assume all functions and constants are real):

$$\text{Homogenous problem: } \left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi_0(x) = 0$$

$$\text{Inhomogenous problem: } \left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi(x) = F(x)$$

Eigenfunctions:

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x)$$

$$\text{Orthogonality of eigenfunctions: } \int_a^b \sigma(x) f_n(x) f_m(x) dx = \delta_{nm} N_n,$$

$$\text{where } N_n \equiv \int_a^b \sigma(x) (f_n(x))^2 dx.$$

Completeness of eigenfunctions:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x')$$

Why all of the fuss about eigenvalues and eigenvectors?

- a. They are sometimes useful in finding solutions to differential equations
- b. Not all eigenfunctions have analytic forms.
- c. It is possible to solve a differential equation without the use of eigenfunctions.
- d. Eigenfunctions have some useful properties.

Comment on orthogonality of eigenfunctions

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x)$$

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_m(x) = \lambda_m \sigma(x) f_m(x)$$

$$\begin{aligned} f_m(x) \left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_n(x) - f_n(x) \left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_m(x) \\ = (\lambda_n - \lambda_m) \sigma(x) f_n(x) f_m(x) \end{aligned}$$

$$-\frac{d}{dx} \left(f_m(x) \tau(x) \frac{df_n(x)}{dx} - f_n(x) \tau(x) \frac{df_m(x)}{dx} \right) = (\lambda_n - \lambda_m) \sigma(x) f_n(x) f_m(x)$$

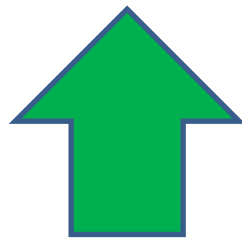
Comment on orthogonality of eigenfunctions -- continued

$$-\frac{d}{dx} \left(f_m(x) \tau(x) \frac{df_n(x)}{dx} - f_n(x) \tau(x) \frac{df_m(x)}{dx} \right) = (\lambda_n - \lambda_m) \sigma(x) f_n(x) f_m(x)$$

Now consider integrating both sides of the equation in the interval

$a \leq x \leq b$:

$$-\left(f_m(x) \tau(x) \frac{df_n(x)}{dx} - f_n(x) \tau(x) \frac{df_m(x)}{dx} \right) \Big|_a^b = (\lambda_n - \lambda_m) \int_a^b dx \sigma(x) f_n(x) f_m(x)$$



Vanishes for various boundary conditions
at $x=a$ and $x=b$



Comment on orthogonality of eigenfunctions -- continued

$$-\left(f_m(x)\tau(x)\frac{df_n(x)}{dx} - f_n(x)\tau(x)\frac{df_m(x)}{dx} \right) \Big|_a^b = (\lambda_n - \lambda_m) \int_a^b dx \sigma(x) f_n(x) f_m(x)$$

Possible boundary values for Sturm-Liouville equations:

1. $f_m(a) = f_m(b) = 0$

2. $\tau(x)\frac{df_m(x)}{dx} \Big|_a = \tau(x)\frac{df_m(x)}{dx} \Big|_b = 0$

3. $f_m(a) = f_m(b)$ and $\frac{df_m(a)}{dx} = \frac{df_m(b)}{dx}$

In any of these cases, we can conclude that:

$$\int_a^b dx \sigma(x) f_n(x) f_m(x) = 0 \text{ for } \lambda_n \neq \lambda_m$$



Comment on "completeness"

It can be shown that for any reasonable function $h(x)$, defined within the interval $a < x < b$, we can expand that function as a linear combination of the eigenfunctions $f_n(x)$

$$h(x) \approx \sum_n C_n f_n(x),$$

where
$$C_n = \frac{1}{N_n} \int_a^b \sigma(x') h(x') f_n(x') dx'.$$

These ideas lead to the notion that the set of eigenfunctions $f_n(x)$ form a "complete" set in the sense of "spanning" the space of all functions in the interval $a < x < b$, as summarized by the statement:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x').$$



Comment on “completeness” -- continued

$$h(x) \approx \sum_n C_n f_n(x),$$

$$\text{where } C_n = \frac{1}{N_n} \int_a^b \sigma(x') h(x') f_n(x') dx'.$$

Consider the squared error of the expansion:

$$\epsilon^2 = \int_a^b dx \sigma(x) \left(h(x) - \sum_n C_n f_n(x) \right)^2$$

ϵ^2 can be minimized:

$$\frac{\partial \epsilon^2}{\partial C_m} = 0 = -2 \int_a^b dx \sigma(x) \left(h(x) - \sum_n C_n f_n(x) \right) f_m(x)$$

$$\Rightarrow C_m = \frac{1}{N_m} \int_a^b dx \sigma(x) h(x) f_m(x)$$