PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Lecture 2
Two particle interactions and scattering theory

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 AM	**OPL 103**	**http://www.wfu.edu/~natalie/f22phy711/**

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/22/2022		Introduction	#1	8/26/2022
2	Wed, 8/24/2022	Chap. 1	Scattering theory		
3	Fri, 8/26/2022	Chap. 1	Scattering theory	#2	8/29/2022
4	Mon, 8/29/2022	Chap. 1	Scattering theory		

Comment on quiz questions

1. $$g(t)=\int_0^t\left(x^2+t\right)dx$$

$$\frac{dg}{dt}=\int_0^t\frac{d\left(x^2+t\right)}{dt}dx+\left(x^2+t\right)\Big|_{x=t}$$

$$=\int_0^t dx+\left(t^2+t\right)=t^2+2t$$

2. Evaluate the integral $\oint\frac{dz}{z}$ for a closed contour about the origin.

Suppose that $z=e^{i\theta}$ $\quad dz=e^{i\theta}id\theta$ $\quad \oint\frac{dz}{z}=\int_0^{2\pi}\frac{e^{i\theta}id\theta}{e^{i\theta}}=2\pi i$

3. $$\frac{df}{dx}=f \quad \Rightarrow\frac{df}{f}=dx \quad \Rightarrow d\left(\ln f\right)=dx \quad \Rightarrow f\left(x\right)=Ae^x$$

$$f\left(x=0\right)=A=1 \quad \Rightarrow A=1 \quad f\left(x\right)=e^x$$

4. $$\sum_{n=1}^{N}a^n=\frac{a-a^{N+1}}{1-a}$$

Let $S\equiv\sum_{n=1}^{N}a^n$ Note that $aS-S=a^{N+1}-a$

$$\Rightarrow S=\frac{a^{N+1}-a}{a-1}$$

Some more details on Question #1:

Suppose $$G(t) = \int_{A(t)}^{B(t)} f(x,t)\,dx$$

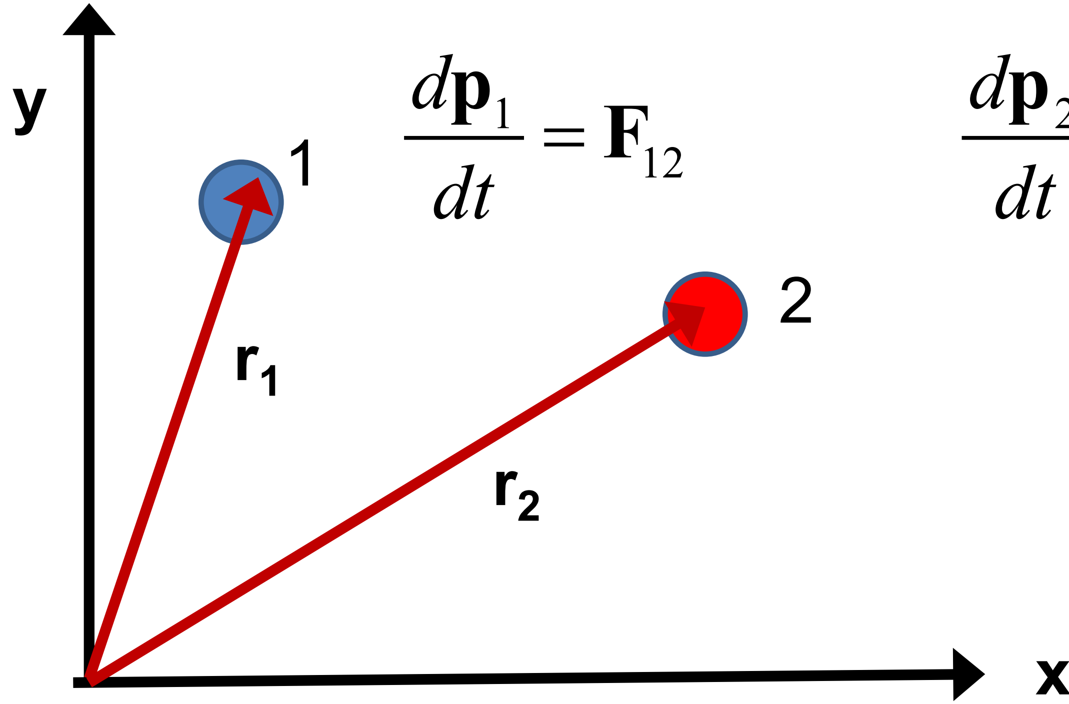
Then $$\frac{dG}{dt} = \int_{A(t)}^{B(t)} \frac{\partial f(x,t)}{\partial t} dx + \frac{dB}{dt} f(x = B(t), t) - \frac{dA}{dt} f(x = A(t), t)$$

A more systematic discussion of these and other mathematical details will be discussed throughout the course.

Introduction to the analysis of the energy and forces between two particles --

First consider fundamental picture of particle interactions

Classical mechanics of a conservative 2-particle system.

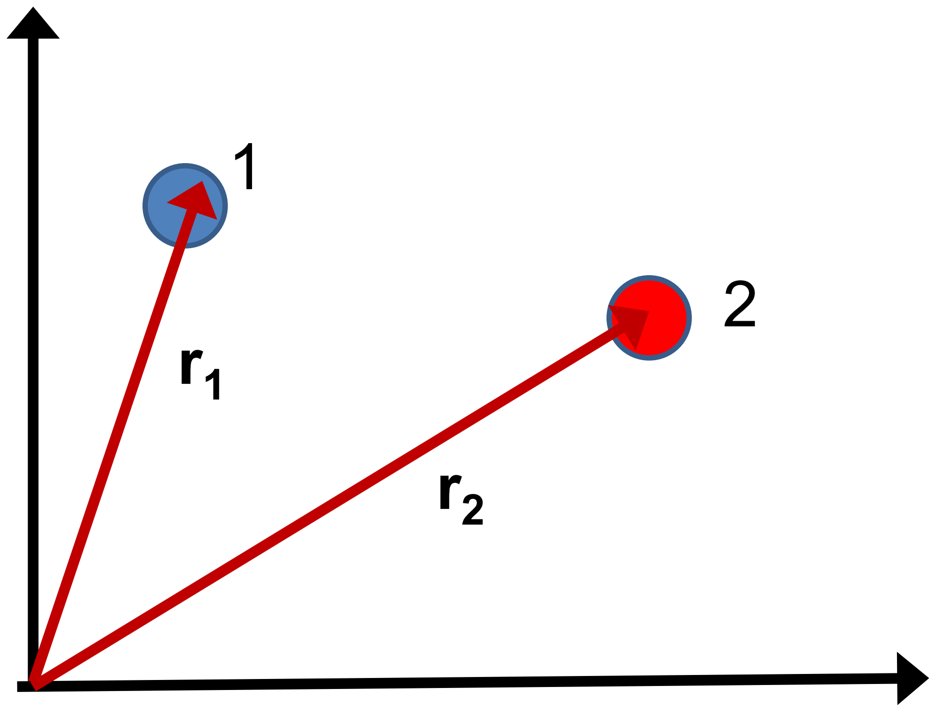


$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21}$$

$$\mathbf{F}_{12} = -\nabla_1 V\left(\mathbf{r}_1 - \mathbf{r}_2\right) \qquad \Rightarrow E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

For this discussion, we will assume that $V(\mathbf{r})=V(r)$ (a central potential).

Energy is conserved: $E=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2+V(\mathbf{r}_1-\mathbf{r}_2)$



For a central potential $V(\mathbf{r})=V(r)$, angular momentum is conserved. For the moment we also make the simplifying assumption that $m_2 >> m_1$ so that particle 1 dominates the motion.

Typical two-particle interactions –

Central potential: $$V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$$

Hard sphere: $$V(r) = \begin{cases} \infty & r \leq a \\ 0 & r > a \end{cases}$$

Coulomb or gravitational: $$V(r) = \frac{K}{r}$$

Lennard-Jones: $$V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$$

More details of two particle interaction potentials

Central potential: $$V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$$

This means that the interaction only depends on the distance between the particles and not on the angle between them. This would typically be true of the particles are infinitesimal points without any internal structure such as two infinitesimal charged particles or two infinitesimal masses separated by a distance *r*:

$$V(r) = \frac{K}{r}$$

Example – Interaction between a proton and an electron. Note we are treating the interactions with classical mechanics; in some cases, quantum effects are non-trivial.

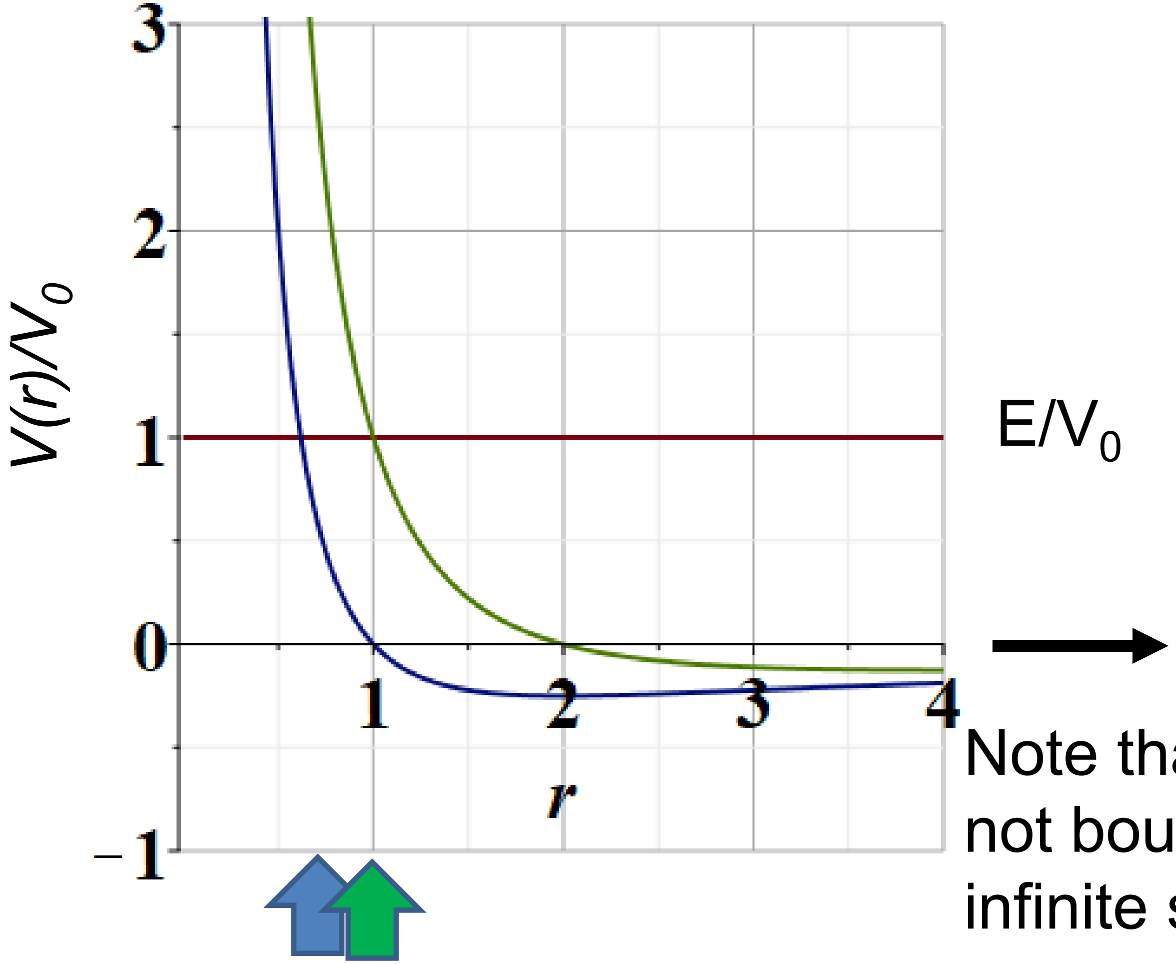
Other examples of central potentials --

		Example
Hard sphere:	$V(r)=\begin{cases}\infty & r\leq a\\ 0 & r>a\end{cases}$	Two marbles
Lennard-Jones:	$V(r)=\frac{A}{r^{12}}-\frac{B}{r^{6}}$	Two Ar atoms

Note – not all systems are described by this form. Some counter examples:

1. Molecules (internal degrees of freedom)
2. Systems with more than two particles such as crystals

Representative plot of *V(r)*



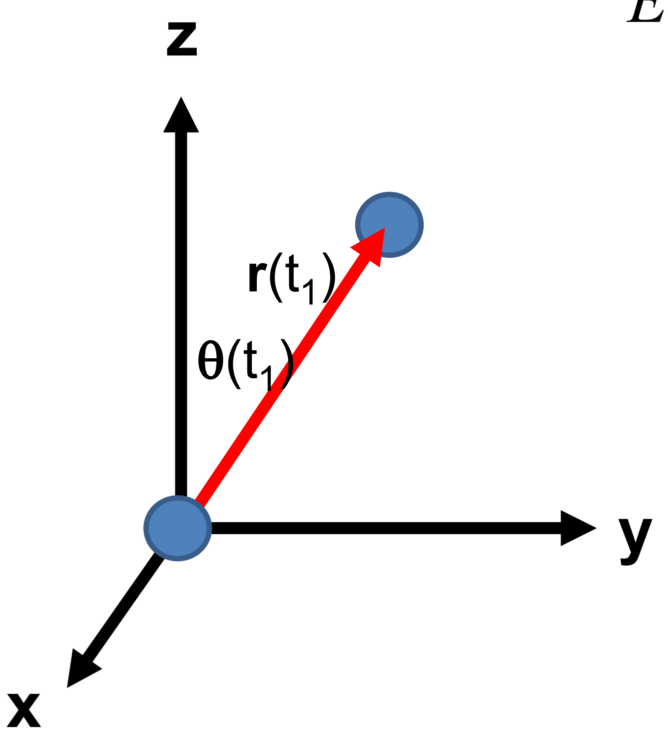
Note that particles are not bound; can reach infinite separation

Distances of closest approach

Some more details --

Here we are assuming that the target particle is stationary and $m_1 \equiv m$.

The origin of our coordinate system is taken at the position of the target particle.



Conservation of energy:

$$E = \frac{1}{2} m \left(\frac{d\mathbf{r}}{dt} \right)^2 + V(r)$$

$$= \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \right) + V(r)$$

Conservation of angular momentum:

$$L = m r^2 \frac{d\theta}{dt}$$

Comments continued --

Conservation of energy:

$$E = \frac{1}{2} m \left(\frac{d\mathbf{r}}{dt} \right)^2 + V(r)$$

$$= \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \right) + V(r)$$

$$= \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \boxed{\frac{L^2}{2mr^2} + V(r)}$$

⇦ $V_{eff}(r)$

Conservation of angular momentum:

$$L = mr^2 \frac{d\theta}{dt}$$

Also note that when $r \rightarrow \infty, \quad V(r) \rightarrow 0$

$$\mathbf{L} \equiv \mathbf{r} \times m \frac{d\mathbf{r}}{dt} \qquad L = b\sqrt{2mE}$$

For $r \rightarrow \infty, \quad \frac{d\mathbf{r}}{dt} \rightarrow v_\infty = \sqrt{\frac{2E}{m}}$

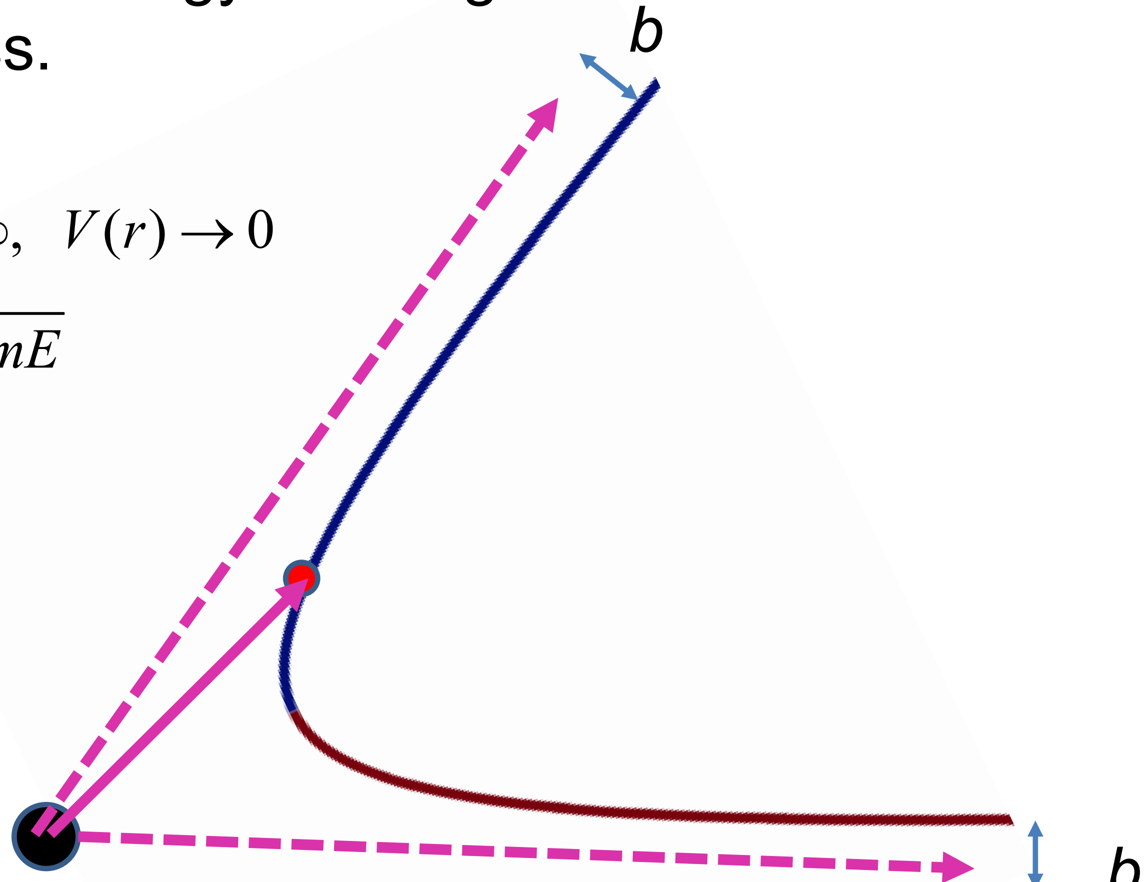
$$E = \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{b^2 E}{r^2} + V(r)$$

What is the impact parameter?

Briefly, a convenient distance that depends on the conserved energy and angular momentum of the process.

Also note that when $r \to \infty, \quad V(r) \to 0$

$$\mathbf{L} \equiv \mathbf{r} \times m \frac{d\mathbf{r}}{dt} \qquad L = b\sqrt{2mE}$$



Which of the following are true:

a. The particle moves in a plane.
b. For any interparticle potential the trajectory can be determined/calculated.
c. Only for a few special interparticle potential forms can the trajectory be determined.

Why should we care about this?

a. We shouldn’t really care.
b. It is only of academic interest
c. It is of academic interest but can be measured.
d. Many experiments can be analyzed in terms of the particle trajectory.

Scattering theory:

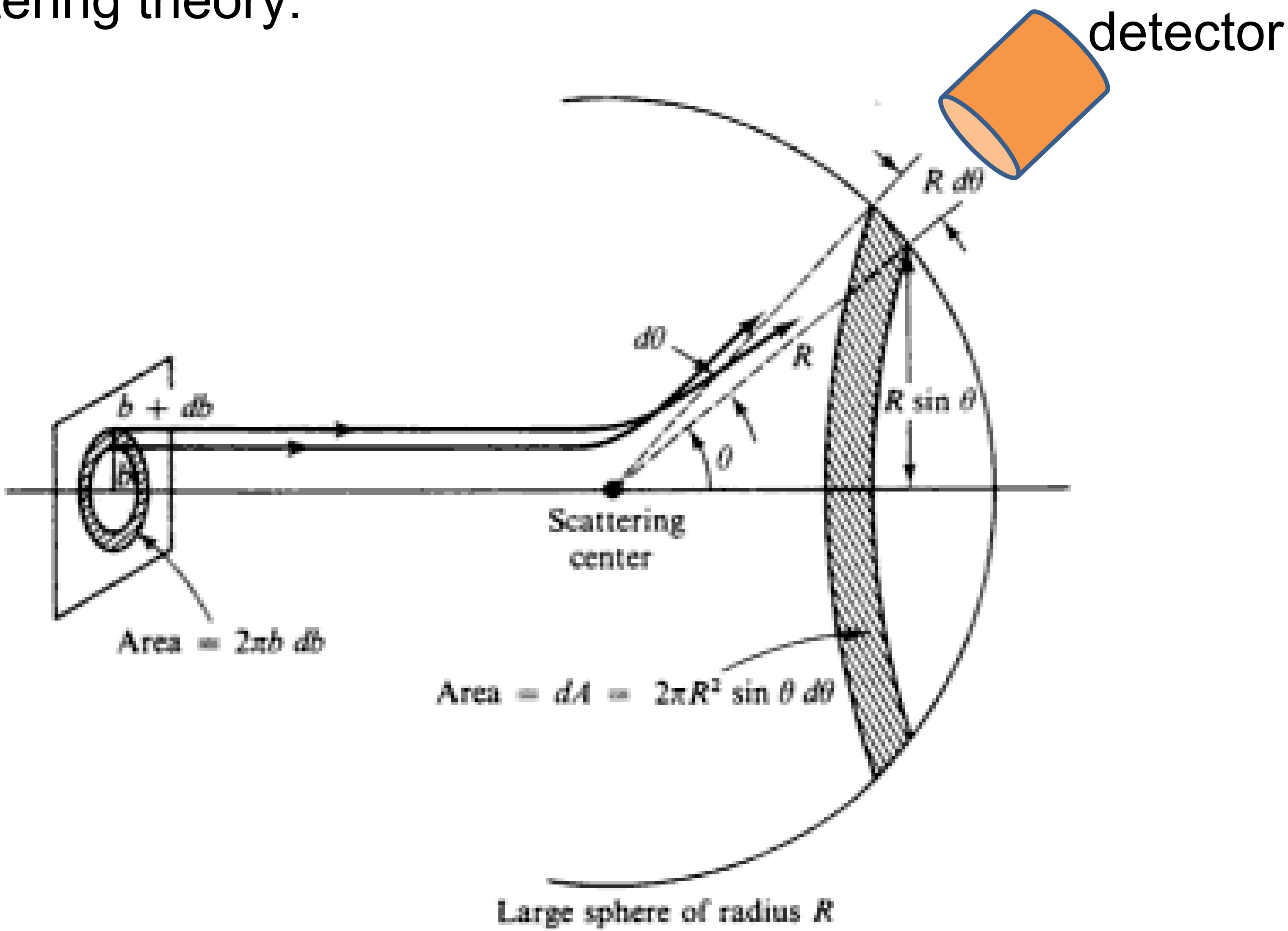


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Scattering theory:

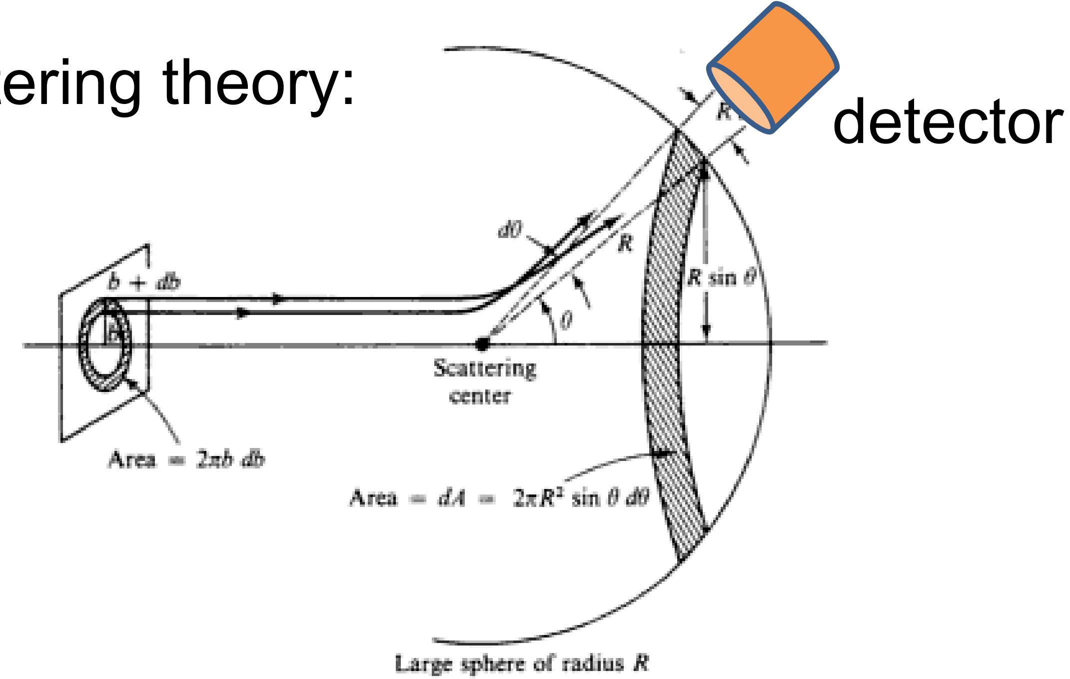
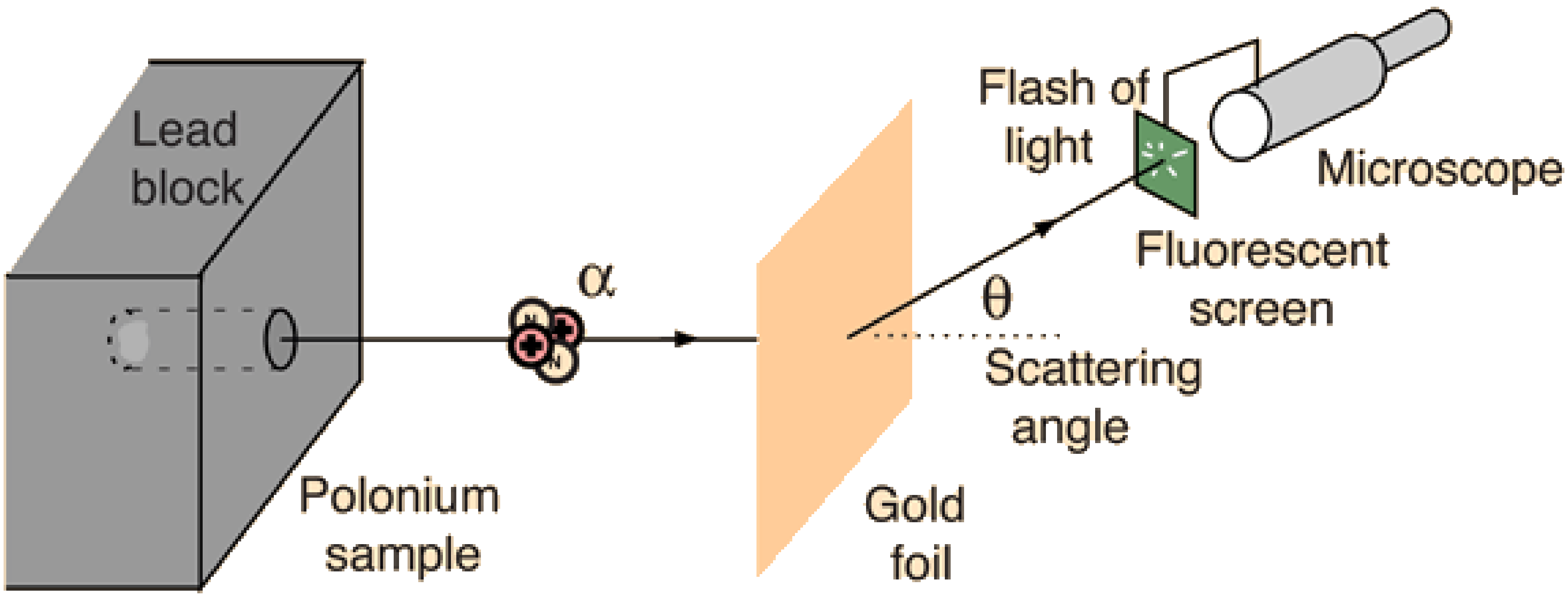


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

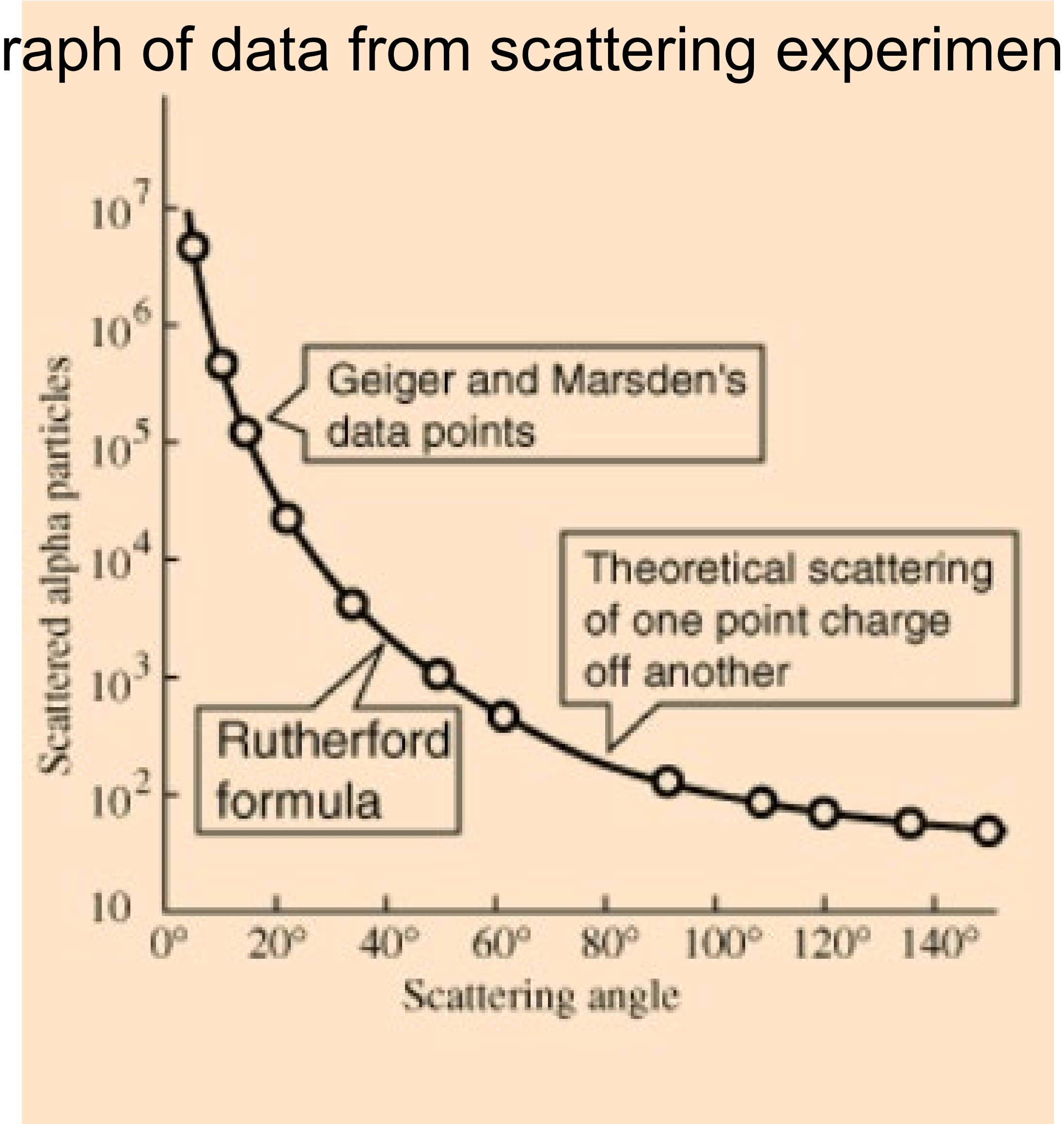
Some reasons that scattering theory is useful:

1. It allows comparison between measurement and theory
2. The analysis depends on knowledge of the scattering particles when they are far apart
3. The scattering results depend on the interparticle interactions

Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Graph of data from scattering experiment



From website: http://hyperphysics.phy-astr.gsu.edu/hbase/Nuclear/rutsca2.html

Standardization of scattering experiments --

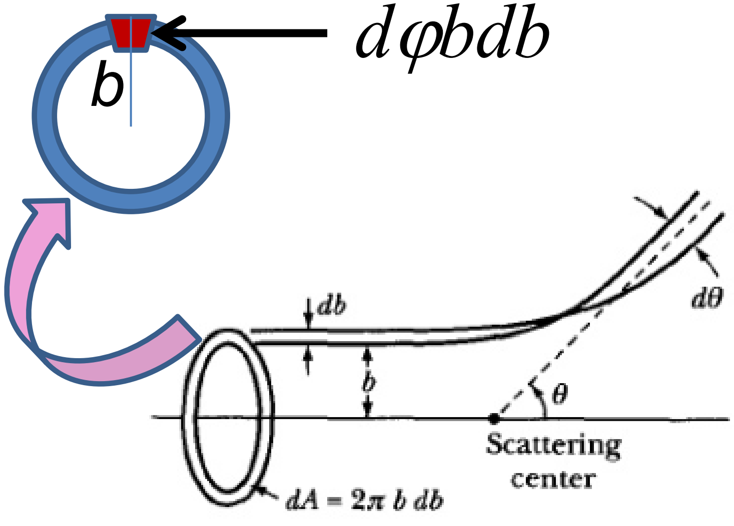
Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector at angle } \theta$$

Impact parameter: *b*

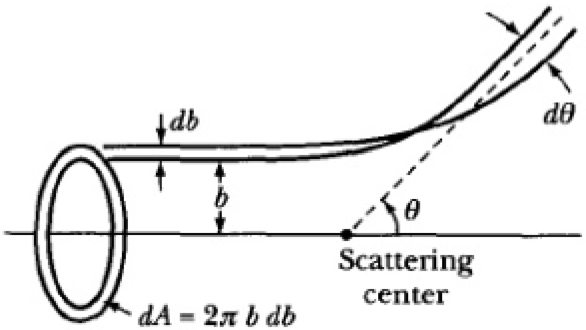
$$d\varphi b db$$



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



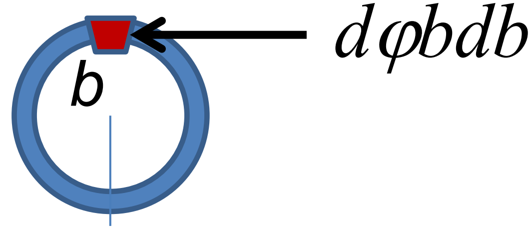
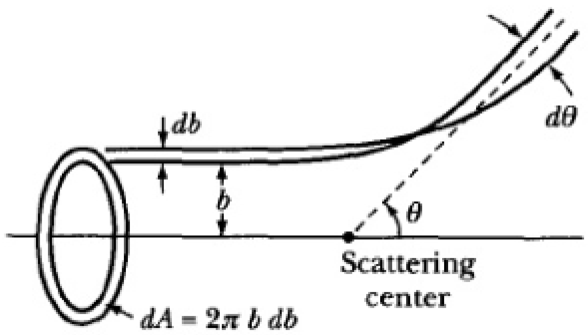


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Note: We are assuming that the process is isotropic in φ

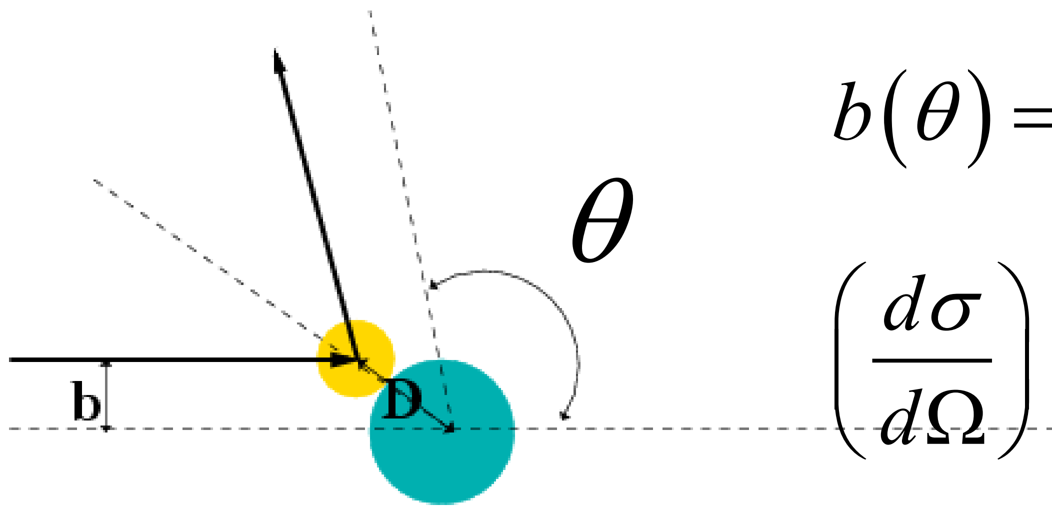
Simple example – collision of hard spheres having mutual radius D; very large target mass



$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Microscopic view:

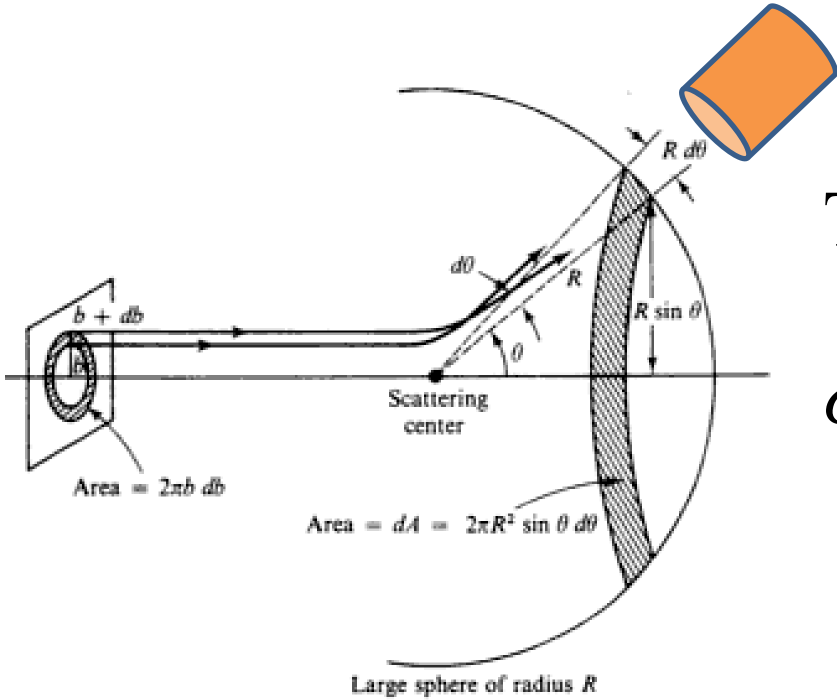
$$b(\theta)=?$$



$$b(\theta)=D\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{D^2}{4}$$

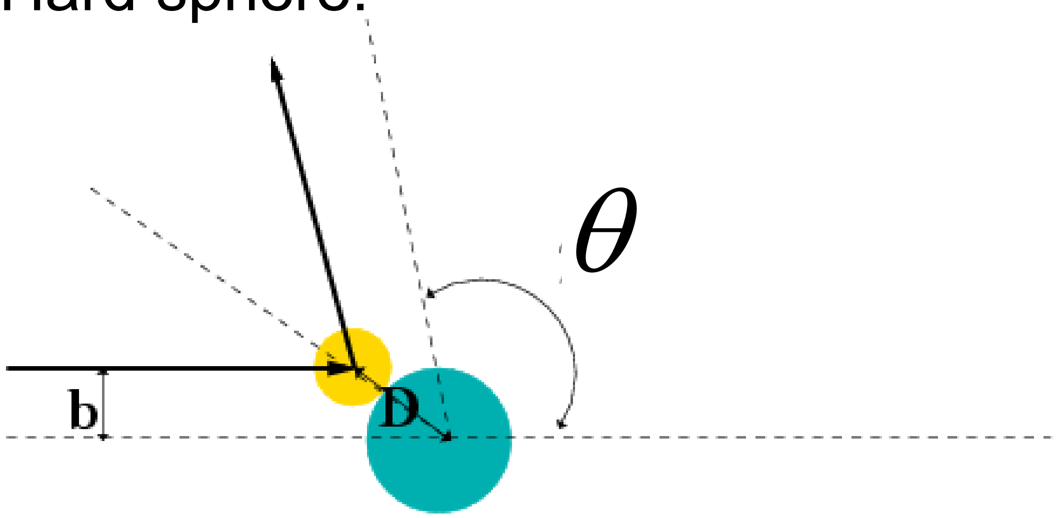
Simple example – collision of hard spheres -- continued



Total scattering cross section:

$$\sigma = \int \left(\frac{d\sigma}{d\Omega} \right) d\Omega$$

Hard sphere:



$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{D^2}{4}$$

$$\sigma = \pi D^2$$

More details of hard sphere scattering –

Hidden in the analysis are assumptions about the scattering process such as:

- No external forces ➔ linear momentum is conserved
- No dissipative phenomena ➔ energy is conserved
- No torque on the system ➔ angular momentum is conserved
- Target particle is much more massive than scattering particle
- Other assumptions??

Note that for quantum mechanical hard spheres at low energy the total cross section is 4 times as large.