PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 21 –
Review of Chap. 1-4,6-7 in F&W

- Comments on Take-home exam
- Review of specific topics

15	Fri, 9/23/2022	Chap. 4	Small oscillations about equilibrium	#12	9/26/2022
16	Mon, 9/26/2022	Chap. 4	Normal modes of vibration	#13	9/28/2022
17	Wed, 9/28/2022	Chap. 4	Normal modes of more complicated systems	#14	10/03/2022
18	Fri, 9/30/2022	Chap. 7	Motion of strings		
19	Mon, 10/03/2022	Chap. 7	Sturm-Liouville equations	#15	10/05/2022
20	Wed, 10/05/2022	Chap. 7	Sturm-Liouville equations		
21	Fri, 10/07/2022	Chap. 1-4,6-7	Review		
	Mon, 10/10/2022	No class	Take home exam		
	Wed, 10/12/2022	No class	Take home exam		
	Fri, 10/14/2022	No class	Fall break		
22	Mon, 10/17/2022	Chap. 7	Class resumes		

Exam due

Comments about the exam

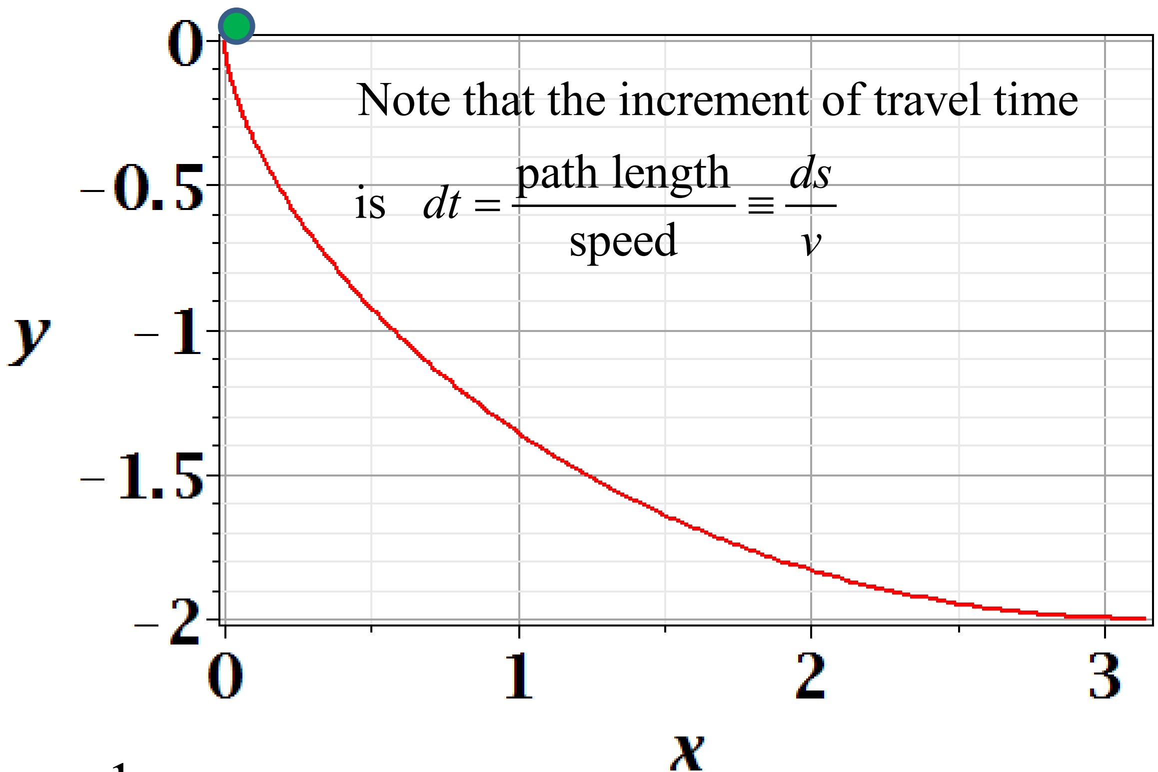
- It must be your own work, under the honor code
- Please make sure that the grader can read your answers.
- Grading is based on the correct answer AND the correct reasoning to arrive at the correct answer. Full credit is obtained only with both. Partial credit also benefits from clear reasoning and results.
- Please meet with me **only** (in person today or by email after today) if you have questions about the exam.

Comments on exam

- The purpose of the exam is to help with your understanding of the material
- In accordance with the honor code, the solutions you hand in must be your own work. That is, if you have any questions, please consult with me, **but no one else.**
- You will get credit for the reasoning and derivations as well as for the right answer, including Mathematica, Maple, etc work sheets.
- This is an open "book" exam which means that you can consult your textbook and lecture notes as long as you cite them. (Of course, if you find a source that works out the same problem, hopefully you will refrain from looking at that…).
- It is often helpful approach problems in more than one way – recalling that undergraduate physics is still true.

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html



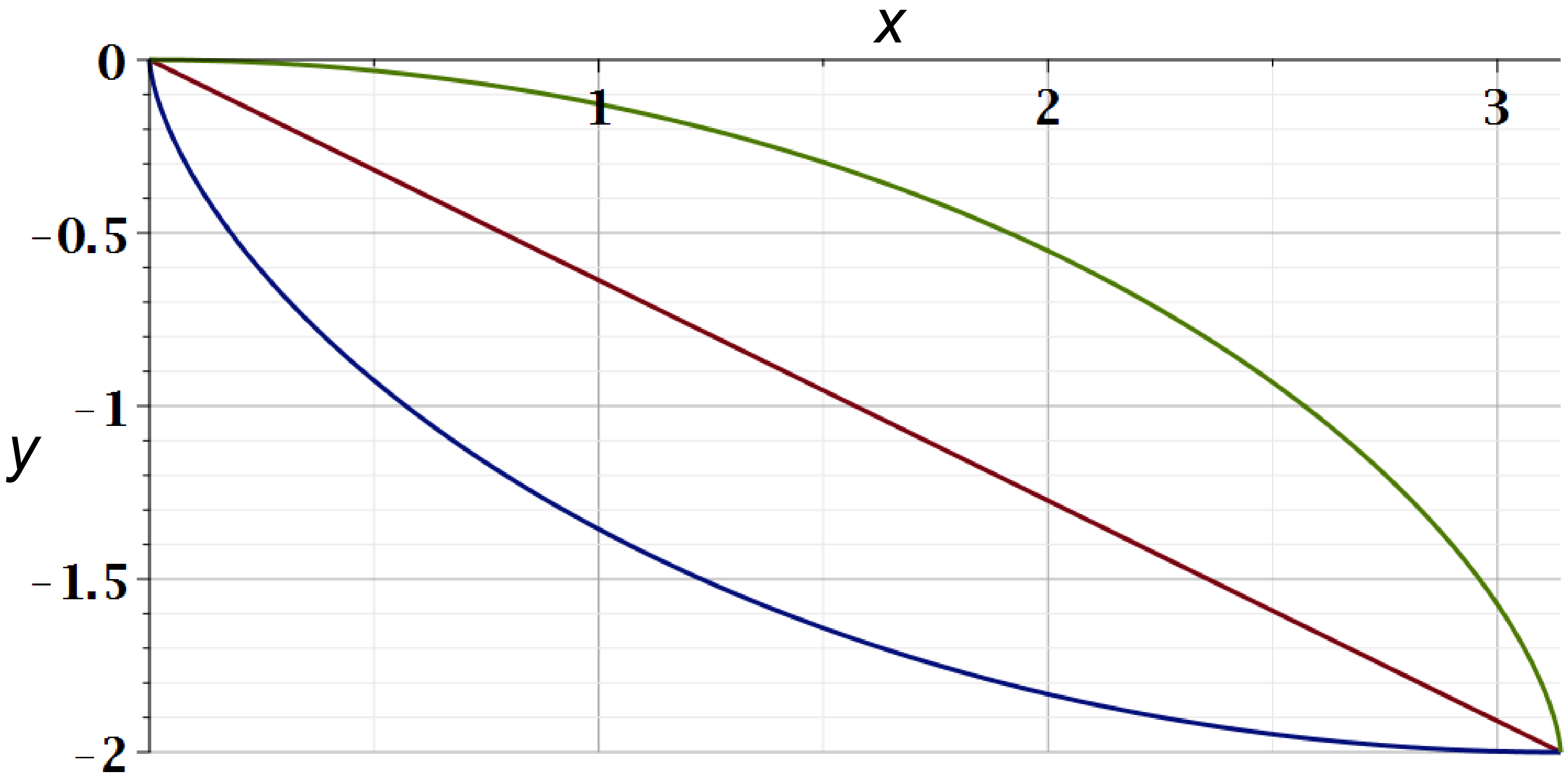
Note that the increment of travel time is $dt = \frac{\text{path length}}{\text{speed}} \equiv \frac{ds}{v}$

A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from *y(0)=0* to *y(π)=-2*?

$$E = \frac{1}{2}mv^2 + mgy$$

With the choice of initial conditions, $E = 0$

Vote for your favorite path



Which gives the shortest time?

a. Green
b. Red
c. Blue

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}} dx \quad \text{because} \quad \tfrac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Note that for the original form of Euler-Lagrange equation:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0,$$

differential equation is more complicated:

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y^3}} - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f-\frac{\partial f}{\partial\left(dy/dx\right)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow\frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right)=0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

Question – why this choice? Answer – because the answer will be more beautiful. (Be sure that was not my cleverness.)

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right)=K\equiv 2a$$

$$\frac{dy}{dx}=-\sqrt{\frac{2a}{-y}-1}$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=dx$$

Let $y=-2a\sin^2\frac{\theta}{2}=a(\cos\theta-1)$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}}=dx$$

$$x=\int_0^\theta a(1-\cos\theta')d\theta'=a(\theta-\sin\theta)$$

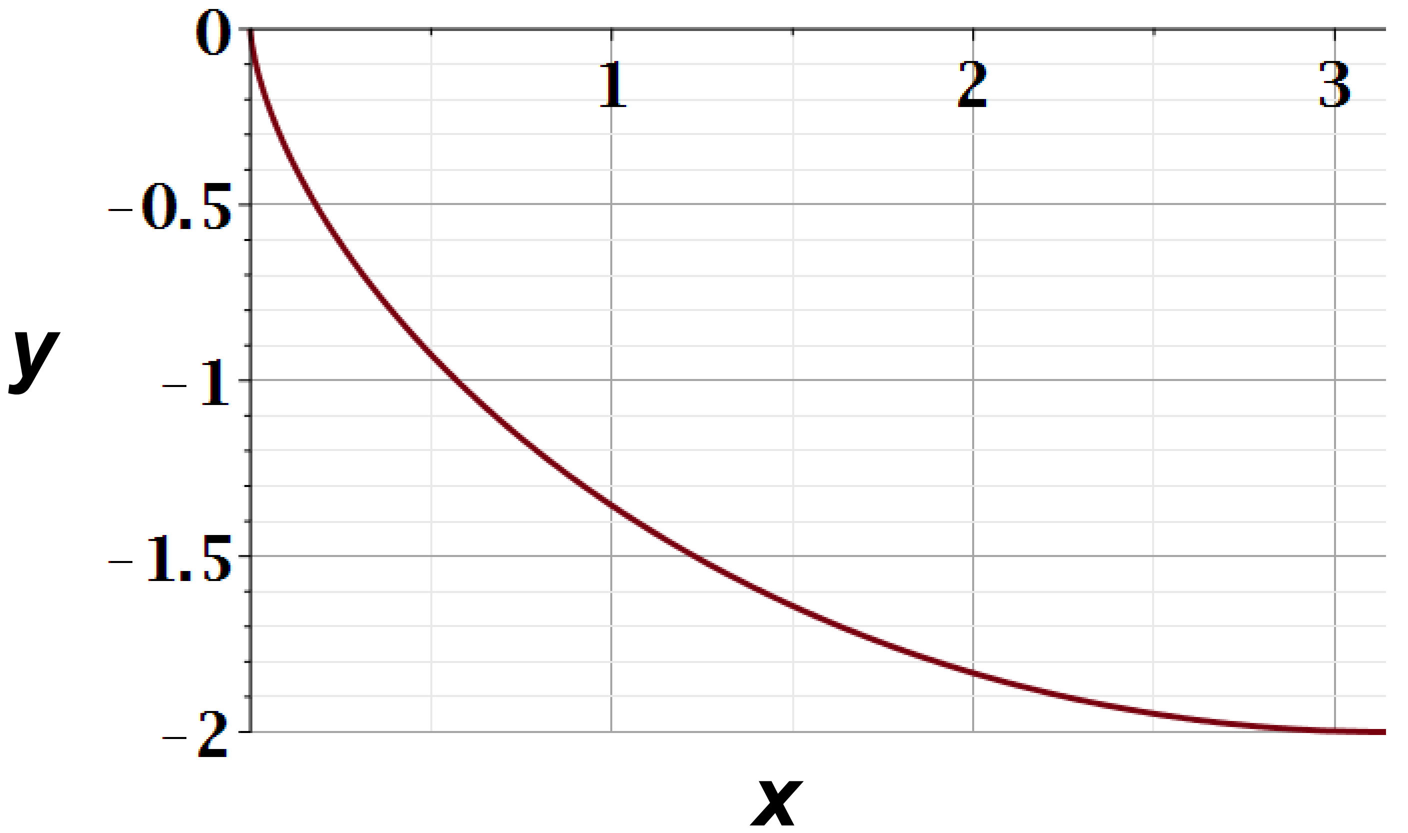
Parametric equations for Brachistochrone:

$$x=a(\theta-\sin\theta)$$

$$y=a(\cos\theta-1)$$

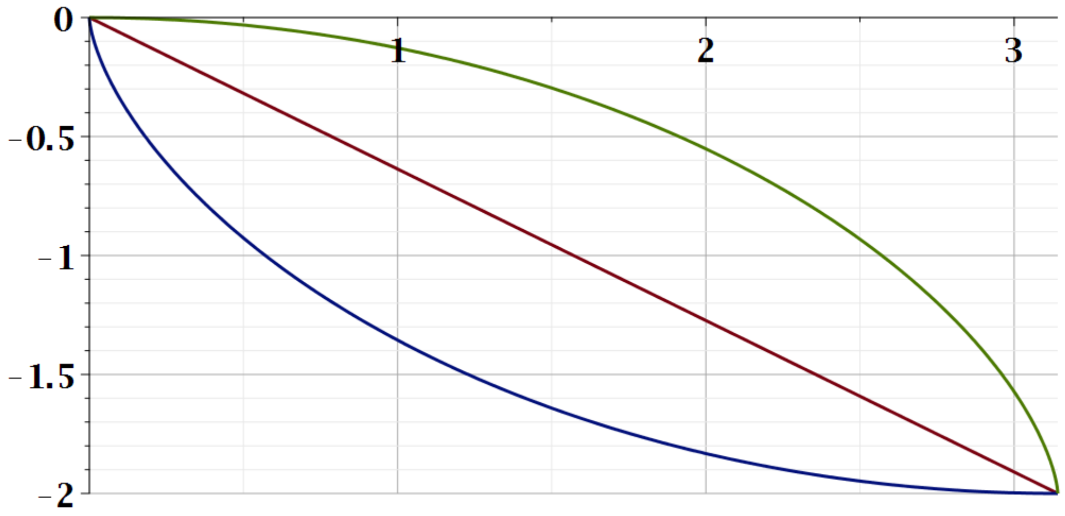
Parametric plot --

plot([theta-sin(theta), cos(theta)-1, theta = 0 .. Pi])



Checking the results

$$T=\int_{x_i y_i}^{x_f y_f}\frac{ds}{v}=\int_{x_i}^{x_f}\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}dx$$



T=infinite
T=5.2668
T=4.4429

$$\text{(units of } \frac{1}{\sqrt{(2g)}}\text{)}$$

Some comments on perturbation theory –

Perturbation theory methods are useful for solving differential equations when some of the contributions are well known but other "smaller" contributions are also present. The concept is used in many contexts although the details vary.

An example --

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t) + \epsilon\left(y(t)\right)^2 \qquad \text{where } \epsilon \text{ is small in some measure}$$

Perturbation theory expansion:

Suppose: $y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t)....$

$$\frac{d^2\left(y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t)...\right)}{dt^2} = -\omega^2\left(y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t)...\right) + \epsilon\left(\left(y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t)...\right)\right)^2$$

Collecting terms by power of ϵ :

$$\epsilon^0: \quad \frac{d^2y_0(t)}{dt^2} = -\omega^2 y_0(t)$$

➔solve for $y_0(t)$

$$\epsilon^1: \quad \frac{d^2y_1(t)}{dt^2} = -\omega^2 y_1(t) + \left(y_0(t)\right)^2$$

➔solve for $y_1(t)$

PHY 711 – Assignment #8

September 12, 2022

The material for this exercise is covered in the lecture notes and in Chapters 3 and 6 of Fetter and Walecka.

1. A particle of mass m and charge q is subjected to a vector potential $\mathbf{A}(\mathbf{r},t) = -(E_0 ct + B_0 x)\hat{\mathbf{z}}$. (Note that we are using the cgs Gaussian units of your text book.) Here E_0 denotes a constant electric field amplitude and B_0 denotes a constant magnetic field amplitude. The initial particle position is $\mathbf{r}(0) = 0$ and the initial particle velocity is $\dot{\mathbf{r}}(0) = 0$.

   (a) Determine the Lagrangian $L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$ which describes the particle's motion.

   (b) Write the Euler-Lagrange equations for this system.

   (c) Find and evaluate the constants of motion for this system.

   (d) Find the particle trajectories $x(t)$, $y(t)$, $z(t)$ by solving the equations and imposing the given initial conditions.

   (e) Determine the Hamiltonian for this system and evaluate dH/dt.

Steps for tackling a problem –

1. What are the basic concepts the apply to this problem.
2. Write down the fundamental equations
3. Solve
4. Check.

In this case, we expect that we should use the Lagrangian formalism and thus we need to know how to represent electric and magnetic fields in the Lagrangian.

$$L(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) = L_{mech}(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) - q\left(\Phi(\mathbf{r},t) - \frac{1}{c}\mathbf{A}(\mathbf{r},t)\right)$$

In this example: $\Phi(\mathbf{r},t) = 0$

$$\mathbf{A}(\mathbf{r},t) = -\hat{\mathbf{z}}\left(E_0 ct + B_0 x\right)$$

$$L(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) = L_{mech}(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) - q\left(\Phi(\mathbf{r},t) - \frac{1}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t) \right)$$

In this example: $\Phi(\mathbf{r},t) = 0$

$$\mathbf{A}(\mathbf{r},t) = -\hat{\mathbf{z}}\left(E_0 ct + B_0 x\right)$$

Note that this corresponds to an electric field:

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = E_0\hat{\mathbf{z}}$$

and a magnetic field:

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t) = B_0\hat{\mathbf{y}}$$

$$L(\mathbf{r}(t),\dot{\mathbf{r}}(t),t) = \frac{1}{2}m\left|\dot{\mathbf{r}}(t)\right|^2 - \frac{q\dot{z}}{c}\left(E_0 ct + B_0 x\right)$$

$$L(\mathbf{r}(t),\dot{\mathbf{r}}(t),t)=\frac{1}{2}m\left|\dot{\mathbf{r}}(t)\right|^2-\frac{q\dot{z}}{c}\left(E_0ct+B_0x\right)$$

Digression on forming the Hamiltonian for this case:

$$p_x=\frac{\partial L}{\partial\dot{x}}=m\dot{x}$$

$$p_y=\frac{\partial L}{\partial\dot{y}}=m\dot{y}$$

$$p_z=\frac{\partial L}{\partial\dot{z}}=m\dot{z}-\frac{q}{c}\left(E_0ct+B_0x\right)$$

$$H=p_x\dot{x}+p_y\dot{y}+p_z\dot{z}-L=\frac{1}{2}m\left|\dot{\mathbf{r}}(t)\right|^2$$

Is this correct?

$$H(\mathbf{r}(t),\mathbf{p}(t),t)=\frac{1}{2m}p_x^2+\frac{1}{2m}p_y^2+\frac{1}{2m}\left(p_z+qE_0t+\frac{q}{c}B_0x\right)^2$$

Note that $\frac{dH}{dt}=\frac{\partial H}{\partial t}=\frac{qE}{m}\left(p_z+qE_0t+\frac{q}{c}B_0x\right)$ why?

Comment on solving equations of motion

$$L(\mathbf{r}(t),\dot{\mathbf{r}}(t),t)=\frac{1}{2}m\left|\dot{\mathbf{r}}(t)\right|^2-\frac{q\dot{z}}{c}\left(E_0ct+B_0x\right)$$

Initial values: $\mathbf{r}(0)=0 \qquad \dot{\mathbf{r}}(0)=0$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x}=\frac{d}{dt}m\dot{x}+\frac{qB_0\dot{z}}{c} \qquad \Rightarrow \dot{x}+\frac{qB_0z}{mc}=K=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}}=0 \quad \Rightarrow y(t)=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}}=0=\frac{d}{dt}\left(m\dot{z}-\frac{q}{c}\left(E_0ct+B_0x\right)\right) \qquad \Rightarrow m\dot{z}-\frac{q}{c}\left(E_0ct+B_0x\right)=K'=0$$

Use these two equations to decouple x(t) and z(t)