PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 23: Chap. 7 & App. A-D (F&W)

Generalization of the one dimensional wave equation → various mathematical problems and techniques including:

1. Fourier transforms
2. Laplace transforms
3. Complex variables
4. Contour integrals

Physics Colloquium

Thursday

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October 20, 2022

More than just noise: Embracing environmental variability and its consequences for ecology

A conspicuous feature of ecological systems is the fact that conditions are rarely the same across space and time. An enduring challenge is to understand how heterogeneous conditions affect individuals, populations, and species, and use that understanding to make predictions. Mathematical models in ecology traditionally represent environmental variability as white noise, that is, independent random perturbations to dynamics that occur under constant conditions. But such a characterization often limits our understanding of potentially relevant biological processes. In this talk, I will give two examples of how environmental noise interacts with population growth processes to yield novel, emergent phenomena. The first example illustrates how changing environments present opportunities for multiple species to coexist despite competition between them. The second example illustrates how spatio-temporal heterogeneity accelerates population growth. I will demonstrate how local SARS-CoV-2 mitigation policies blind to this effect might inadvertently exacerbate viral spread. Throughout, I emphasize how environmental noise interacts with

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Assistant Professor of Biology

Wake Forest University

4:00 pm - Olin 101*

*Link provided for those unable to attend in person.

Note: For additional information on the seminar or to obtain the video conference link, contact wfuphys@wfu.edu

Reception at 3:30pm - Olin Entrance

19	Mon, 10/03/2022	Chap. 7	Sturm-Liouville equations	#15	10/05/2022
20	Wed, 10/05/2022	Chap. 7	Sturm-Liouville equations		
21	Fri, 10/07/2022	Chap. 1-4,6-7	Review		
	Mon, 10/10/2022	No class	Take home exam		
	Wed, 10/12/2022	No class	Take home exam		
	Fri, 10/14/2022	No class	Fall break		
22	Mon, 10/17/2022	Chap. 7	Green's function methods for one-dimensional Sturm-Liouville equations	#16	10/19/2022
23	Wed, 10/19/2022	Chap. 7	Fourier and other transform methods	#17	10/21/2022
24	Fri, 10/21/2022	Chap. 7	Complex variables and contour integration	#18	10/24/2022

PHY 711 -- Assignment #17

Oct. 19, 2022

Continue reading Chapter 7 in **Fetter & Walecka**.

1. Consider the function $f(x) = x^2 (1-x)$ in the interval $0 \leq x \leq 1$. Find the coefficients A_n of the Fourier series based on the terms sin($n \pi x$). Extra credit: Plot $f(x)$ and the Fourier series including 3 terms.

Review – Sturm-Liouville equations defined over a range of x.

For $x_a \leq x \leq x_b$

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

Note that, because Sturm-Liouville operator is Hermitian, the eigenvalues are real and the eigenfunctions are orthogonal. In the last lecture, we argued that the eigenfunctions form a "complete" set over the range of x defined for the particular system.

Formal statement of the completeness of eigenfunctions:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x') \quad \text{where} \quad N_n \equiv \int_{x_a}^{x_b} dx\sigma(x)\left(f_n(x)\right)^2$$

This means that within the interval $x_a \leq x \leq x_b$,

an arbitrary function $h(x)$ can be expanded: $h(x) = \sum_n A_n f_n(x)$.

Example for $\tau(x) = 1 = \sigma(x)$ and $v(x) = 0$ with

$0 \leq x \leq L$ and $f_n(0) = 0 = f_n(L)$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)\right)f_n(x) = \lambda_n\sigma(x)f_n(x) \quad \Rightarrow -\frac{d^2 f_n(x)}{dx^2} = \lambda_n f_n(x$$

In this case, the normalized eigenfunctions are

$$f_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, ...$$

Joseph Fourier

Jean-Baptiste Joseph Fourier

Born	21 March 1768 Auxerre, Burgundy, Kingdom of France (now in Yonne, France)
Died	16 May 1830 (aged 62)

Special case: $\tau(x) = 1 = \sigma(x) \qquad v(x) = 0$

$$-\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \quad \text{for } 0 \le x \le a, \quad \text{with } f_n(0) = f_n(a) = 0$$

$$f_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \qquad \lambda_n = \left(\frac{n\pi}{a}\right)^2$$

Fourier series representation of function $h(x)$ in the interval $0 \le x \le a$:

$$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$A_n = \sqrt{\frac{2}{a}} \int_0^a dx'\ h(x')\ \sin\left(\frac{n\pi x'}{a}\right)$$

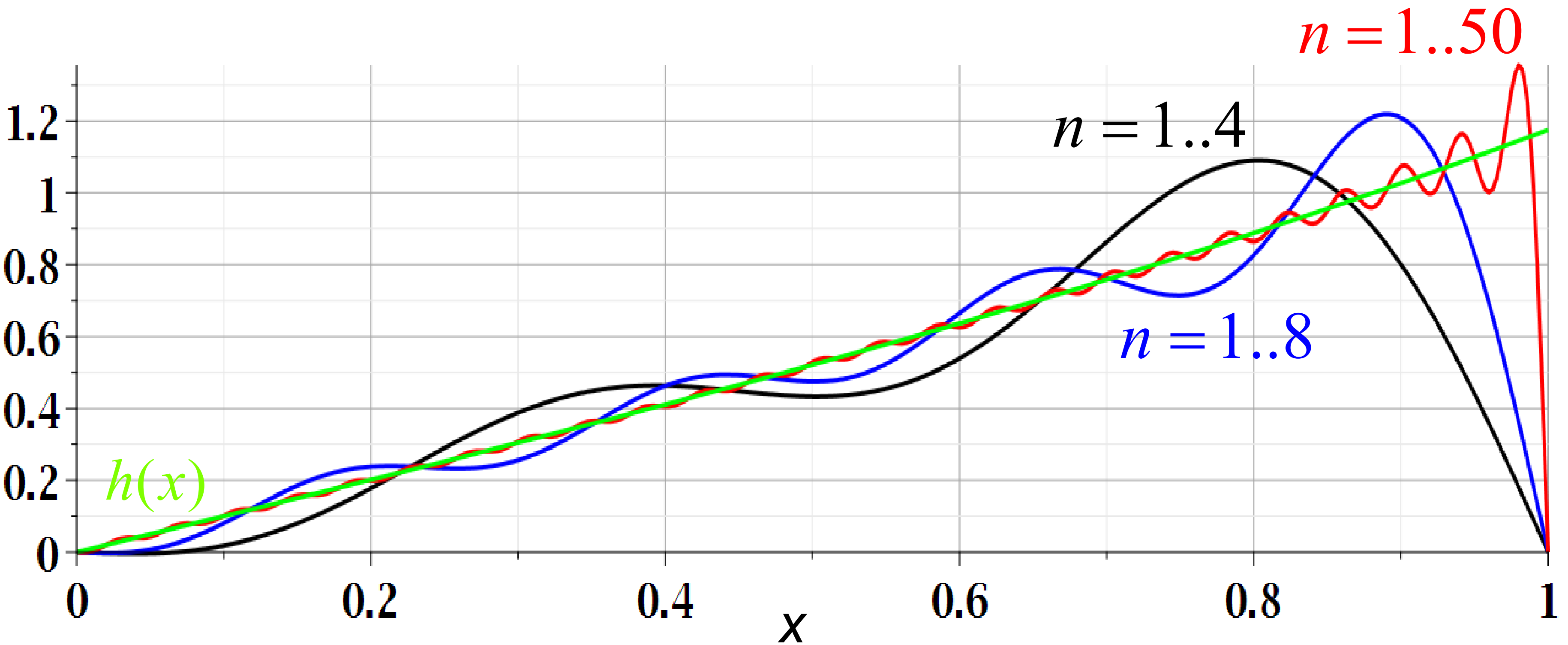
*Note that if $h(x)$ does not vanish at $x = 0$ and $x = a$, the more general

expression applies: $$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) + \sum_{n=0}^{\infty} B_n \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right)$$

(with some restrictions).

Example

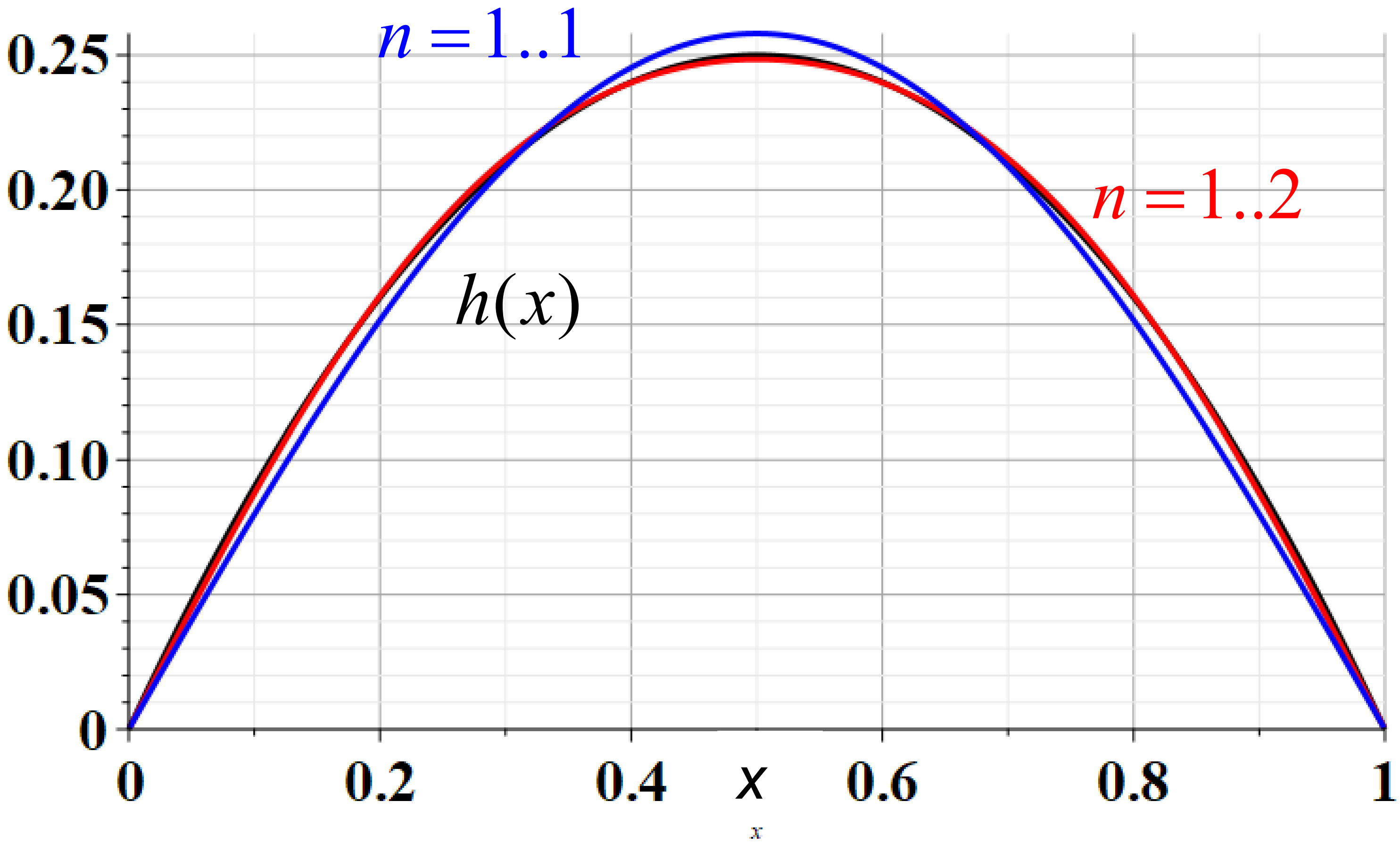
$$h(x)=\sinh(x)=\sqrt{2}\pi\sinh(1)\left(\frac{\sin(\pi x)}{\pi^2+1}-\frac{2\sin(2\pi x)}{4\pi^2+1}+\ldots-(-1)^n n\frac{\sin(n\pi x)}{n^2\pi^2+1}+\ldots\right)$$



Example

$$h(x) = x(1-x)$$

$$= \sum_{n=1}^{\infty} A_n \sin(n\pi x) \qquad A_n = \begin{cases} \dfrac{4\sqrt{2}}{n^3\pi^3} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$



Fourier series representation of function $h(x)$ in the interval $0 \le x \le a$:

$$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad \text{with} \qquad A_n = \sqrt{\frac{2}{a}} \int_0^a dx' \; h(x') \; \sin\left(\frac{n\pi x'}{a}\right)$$

Can show that the series converges provided that $h(x)$ is

piecewise continuous.

Note that this analysis can also apply to time dependent functions. In the remainder of the lecture, we will consider time dependent functions.

$$x \to t \qquad a \to T \qquad 0 \le t \le T \qquad \frac{n\pi}{a} \to \frac{n\pi}{T} \equiv \omega_n$$

$$h(t) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{T}} \sin(\omega_n t) \qquad A_n = \sqrt{\frac{2}{T}} \int_0^T dt' \; h(t') \; \sin(\omega_n t')$$

Note that for this finite time range, Fourier series is discrete in frequency and continuous in time.

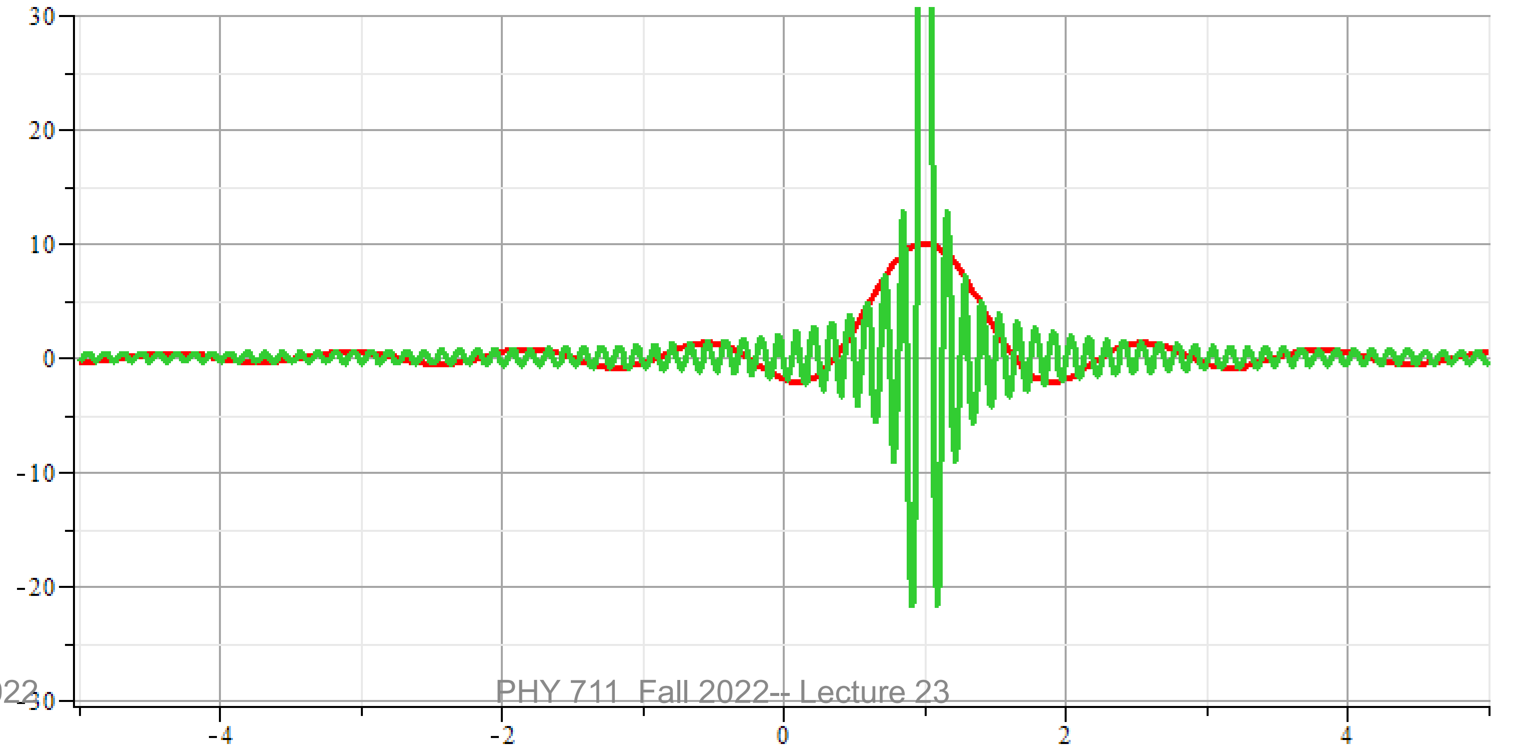
Generalization to infinite range -- Fourier transforms

A useful identity

$$\int_{-\infty}^{\infty} dt\ e^{-i(\omega-\omega_0)t} = 2\pi\delta(\omega-\omega_0)$$

Note that

$$\int_{-T}^{T} dt\ e^{-i(\omega-\omega_0)t} = \frac{2\sin\left[(\omega-\omega_0)T\right]}{\omega-\omega_0} \underset{T\to\infty}{\approx} 2\pi\delta(\omega-\omega_0)$$



Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega \, F(\omega) \, e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, f(t) \, e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \, f(t') \, e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt' \, f(t') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt' \, f(t') \, \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt\,\left(f(t)\right)^* f(t) = 2\pi \int_{-\infty}^{\infty} d\omega\,\left(F(\omega)\right)^* F(\omega)$$

Check:

$$\int_{-\infty}^{\infty} dt\,\left(f(t)\right)^* f(t) = \int_{-\infty}^{\infty} dt \left(\left(\int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t} \right)^* \int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t} \right)$$

$$= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega') \int_{-\infty}^{\infty} dt\; e^{i(\omega'-\omega)t}$$

$$= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega')\, 2\pi\delta(\omega'-\omega)$$

$$=2\pi \int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega)$$

Note that for an infinite time range, the Fourier transform is continuous in both time and frequency.

Use of Fourier transforms to solve wave equation

Wave equation: $$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Suppose $u(x,t) = e^{-i\omega t}\tilde{F}(x,\omega)$ where $\tilde{F}(x,\omega)$ satisfies the equation:

$$\frac{\partial^2 \tilde{F}(x,\omega)}{\partial x^2} = -\frac{\omega^2}{c^2}\tilde{F}(x,\omega) \equiv -k^2\tilde{F}(x,\omega)$$

More generally:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F(x,\omega) e^{-i\omega t}$$

Further assume that fixed boundary conditions apply: $0 \le x \le L$

with $\tilde{F}(0,\omega) = 0$ and $\tilde{F}(L,\omega) = 0$

For $n = 1,2,3\cdots$

$$\tilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right) \qquad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$$

$$u(x,t) = e^{-i\omega_n t}\sin(k_n x) = e^{-i\omega_n t}\frac{\left(e^{ik_n x} - e^{-ik_n x}\right)}{2i} = \frac{\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right)}{2i}$$

Use of Fourier transforms to solve wave equation -- continued

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Using superposition: Suppose $u(x,t) = \sum_n A_n e^{-i\omega_n t} \tilde{F}_n(x,\omega_n)$

$$\frac{\partial^2 \tilde{F}_n(x,\omega_n)}{\partial x^2} = -\frac{\omega_n^2}{c^2} \tilde{F}(x,\omega_n) \equiv -k_n^2 \, \tilde{F}(x,\omega_n)$$

For $\tilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right)$ $\qquad k \rightarrow k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$

$$\Rightarrow u(x,t) = \sum_n A_n e^{-i\omega_n t} \sin\left(k_n x\right) = \sum_n \frac{A_n}{2i} e^{-i\omega_n t} \left(e^{ik_n x} - e^{-ik_n x}\right)$$

$$= \sum_n \frac{A_n}{2i} \left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right) \equiv f(x-ct) + g(x+ct)$$

Note that at this point, we do not know the coefficients A_n; however, it clear that the solutions are consistent with D'Alembert's analysis of the wave equation.

Now consider the Fourier transform for a time periodic function:

Suppose $f(t+nT)=f(t)$ for any integer n

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\ f(t)e^{i\omega t}\ =\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{0}^{T}dt\ f(t)e^{i\omega(t+nT)}\ \right)$$

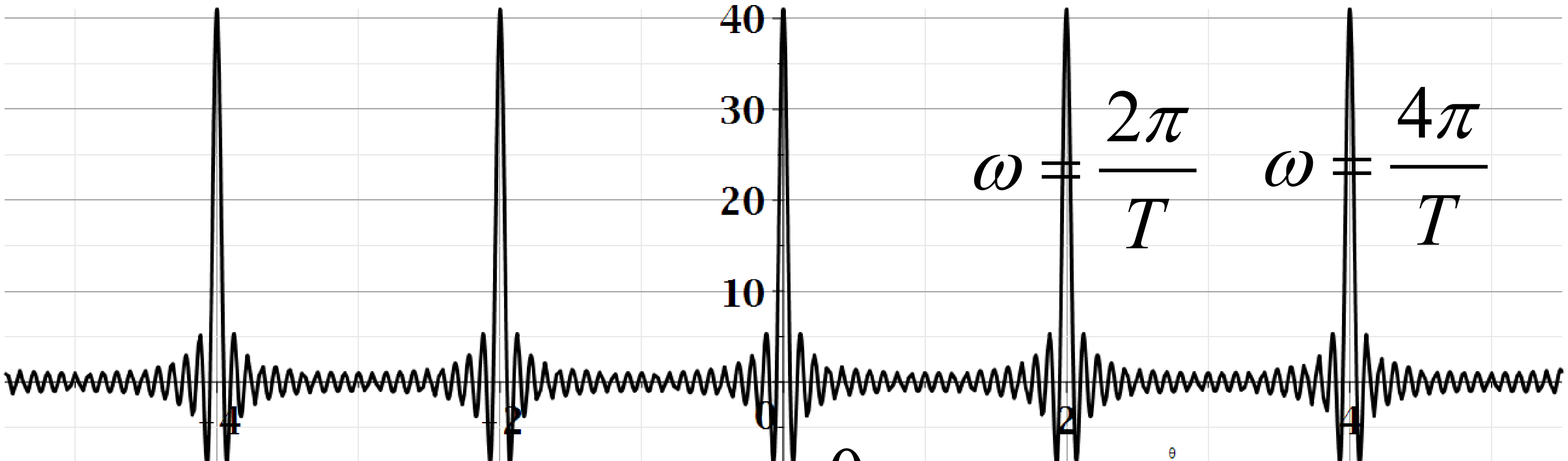
Note that:

$$\sum_{n=-\infty}^{\infty}e^{in\omega T}=\Omega\sum_{\nu=-\infty}^{\infty}\delta\left(\omega-\nu\Omega\right),\quad \text{where}\quad \Omega\equiv\frac{2\pi}{T}$$

Details:

$$\sum_{n=-\infty}^{\infty}e^{in\omega T}=\lim_{N\to\infty}\sum_{n=-N}^{N}e^{in\omega T}=\lim_{N\to\infty}\frac{\sin\left(\left(N+\frac{1}{2}\right)\omega \mathrm{T}\right)}{\sin\left(\frac{1}{2}\omega \mathrm{T}\right)}$$

$$\frac{\sin\left(\left(N+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$



Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where} \quad \Omega \equiv \frac{2\pi}{T}$$

Geometric summation: $$\sum_{n=-N}^{N} e^{in\omega T} = \frac{\sin\left(\left(N+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\lim_{N\to\infty}\left(\frac{\sin\left(\left(N+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}\right) = 2\pi\sum_{\nu}\delta\left(\omega T-\nu\Omega T\right) = \frac{2\pi}{T}\sum_{\nu}\delta\left(\omega-\nu\Omega\right)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega\sum_{\nu=-\infty}^{\infty}\delta(\omega-\nu\Omega), \quad \text{where} \quad \Omega \equiv \frac{2\pi}{T}$$

$$\Rightarrow F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ f(t)e^{i\omega t} = \frac{1}{2\pi}\sum_{\nu=-\infty}^{\infty}\Omega\delta\left(\omega-\nu\Omega\right)\left(\int_{0}^{T} dt\ f(t)e^{i\omega t}\right)$$

Thus, for a time periodic function

$$f(t) = \int_{-\infty}^{\infty} d\omega\ F(\omega)\ e^{-i\omega t} = \sum_{\nu=-\infty}^{\infty}\bar{F}\left(\nu\Omega\right)e^{-i\nu\Omega t},$$

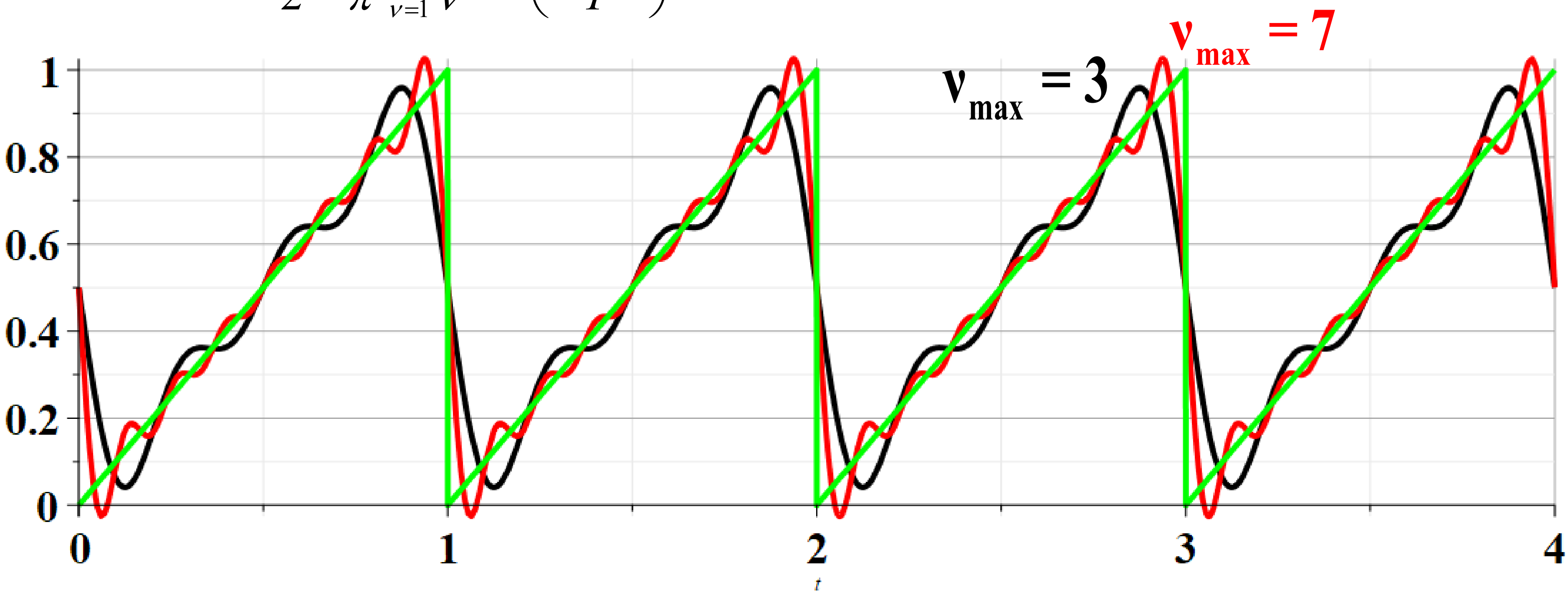
$$\text{where} \quad \bar{F}\left(\nu\Omega\right) = \frac{1}{T}\int_{0}^{T} dt\ f(t)e^{i\nu\Omega t}$$

Example:

Suppose: $f(t) = \frac{t - nT}{T}$ for $nT \le t \le (n+1)T$; $n = \ldots -3, -2, -1, 0, 1, 2, 3 \ldots$

$$\bar{F}(\nu\Omega) = \frac{1}{T}\int_0^T \frac{t}{T}\, e^{i\frac{\nu 2\pi t}{T}}\, dt = \bar{F}^*(-\nu\Omega) = \frac{-i}{2\pi\nu} \text{ for } \nu = 1, 2, 3 \ldots \quad \bar{F}(0) = \frac{1}{2}$$

$$f(t) = \frac{1}{2} - \frac{2}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{\nu}\sin\left(\frac{2\pi\nu t}{T}\right)$$



Summary –

Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega \; F(\omega) \; e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \;\; f(t) \, e^{i\omega t}$$

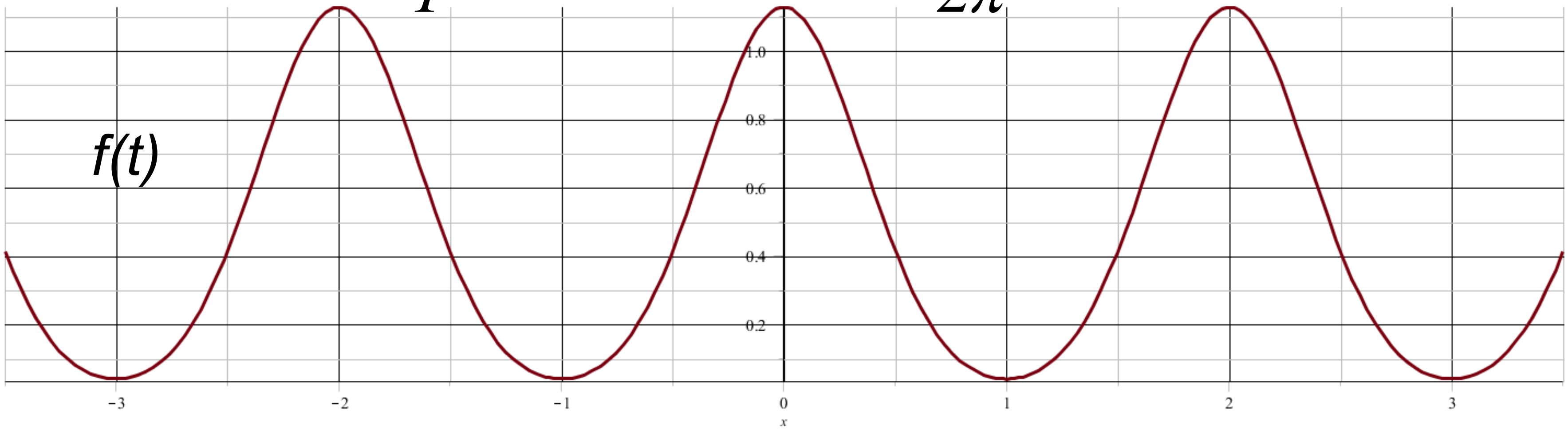
Find discrete frequencies ω for functions *f(t)* over finite time domain of for functions *f(t)* which are periodic*: f(t)=f(t+nT)*

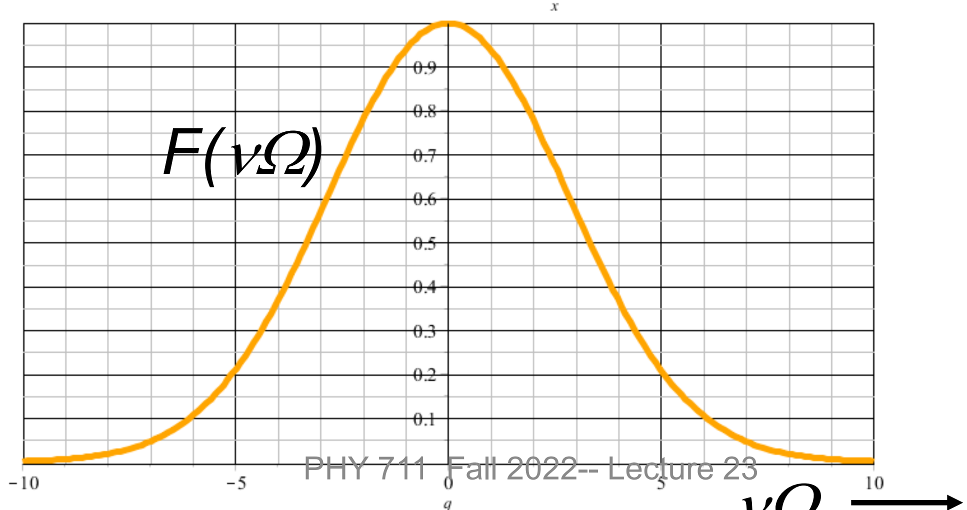
➔Numerically, there is an advantage of tabulating double discrete Fourier transforms (discrete in ω and in *t*).

Example:

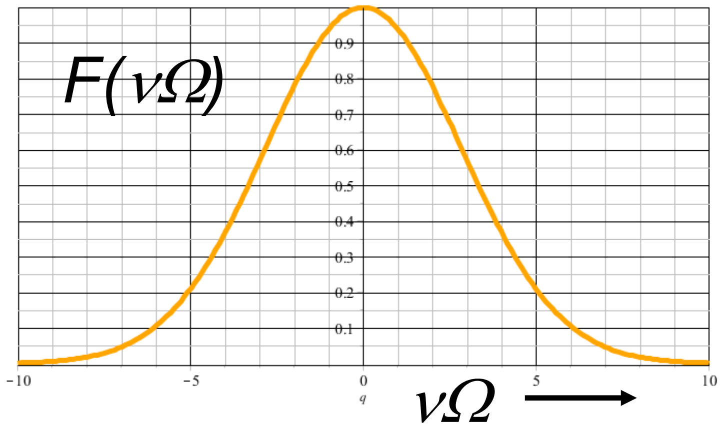
Suppose: $f(t)=\frac{1}{a\sqrt{\pi}}\sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2} = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$

where $\Omega \equiv \frac{2\pi}{T}$ and $F(\nu\Omega)=\frac{1}{2\pi} e^{-a^2\nu^2\Omega^2/4}$





Continued: $f(t)=\frac{1}{a\sqrt{\pi}}\sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2}=\sum_{\nu=-\infty}^{\infty} F(\nu\Omega)e^{-i\nu\Omega t}$



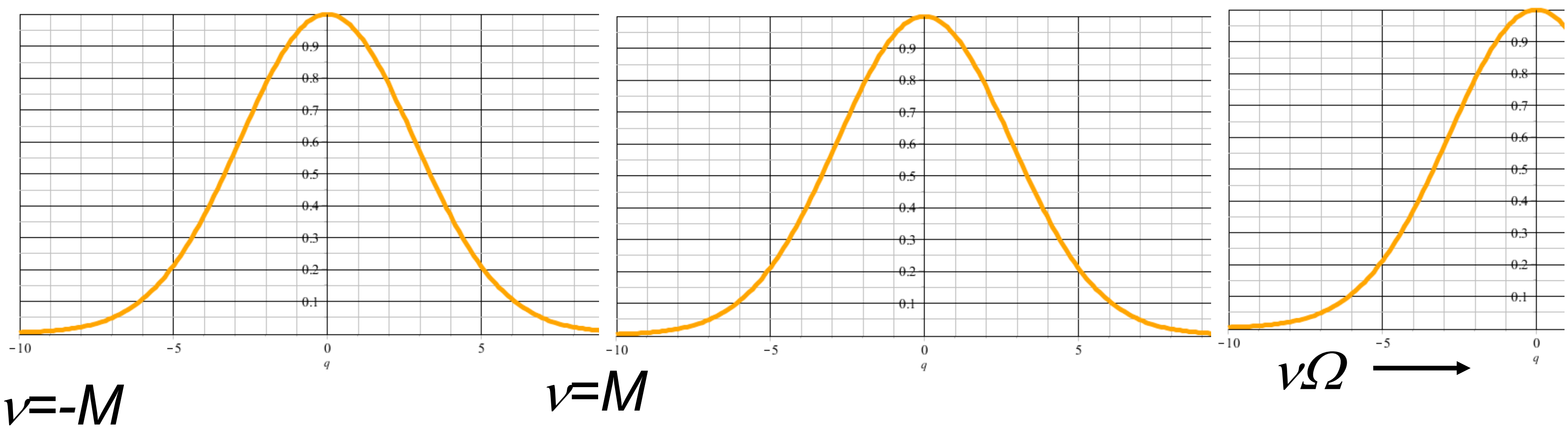
$$\Omega=\frac{2\pi}{T}$$

Note:

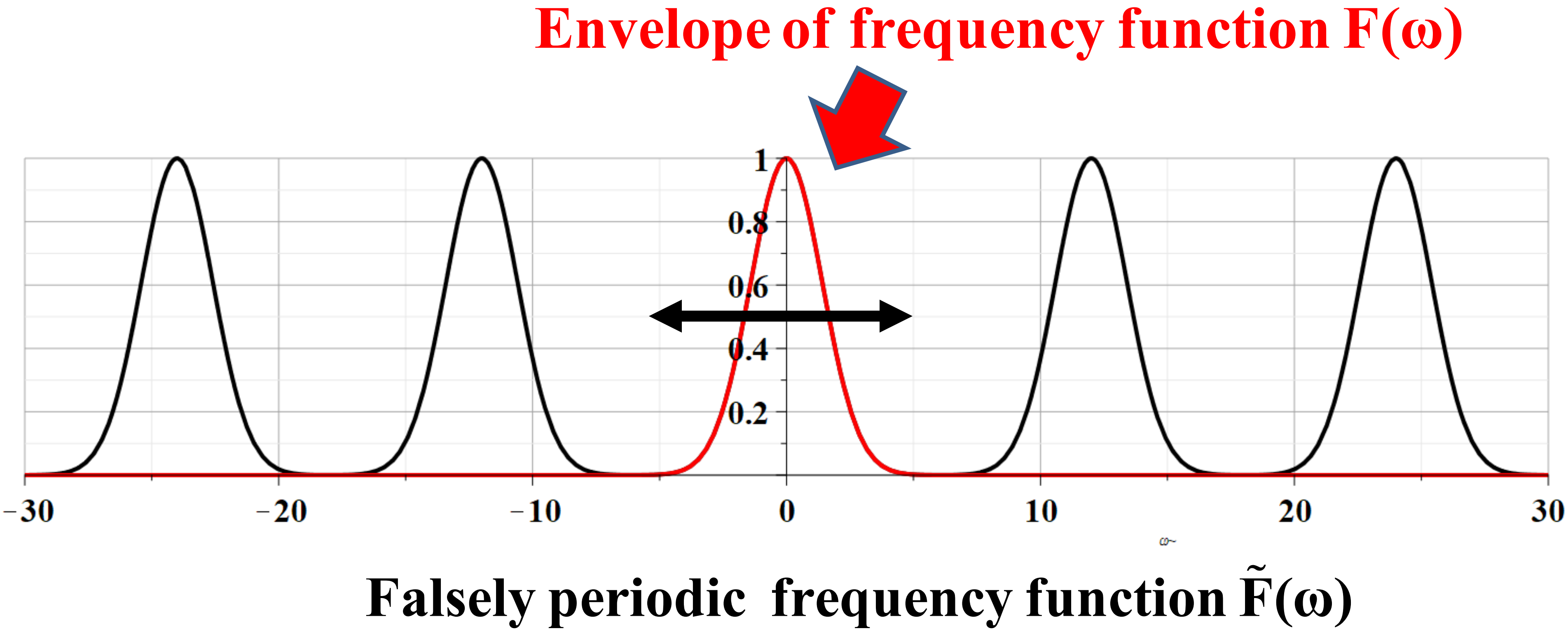
$$f(t)\approx\sum_{\nu=-M}^{M} F(\nu\Omega)e^{-i\nu\Omega t}$$

because $F(\nu'\Omega)\approx 0$

for $|\nu'|>M$



Constructed frequency periodic function --



Thus, for any periodic function: $f(t)=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$

Now suppose that the transformed function is bounded;

$\left|F(\nu\Omega)\right|\leq\varepsilon$ for $|\nu|\geq N$

Define a periodic transform function

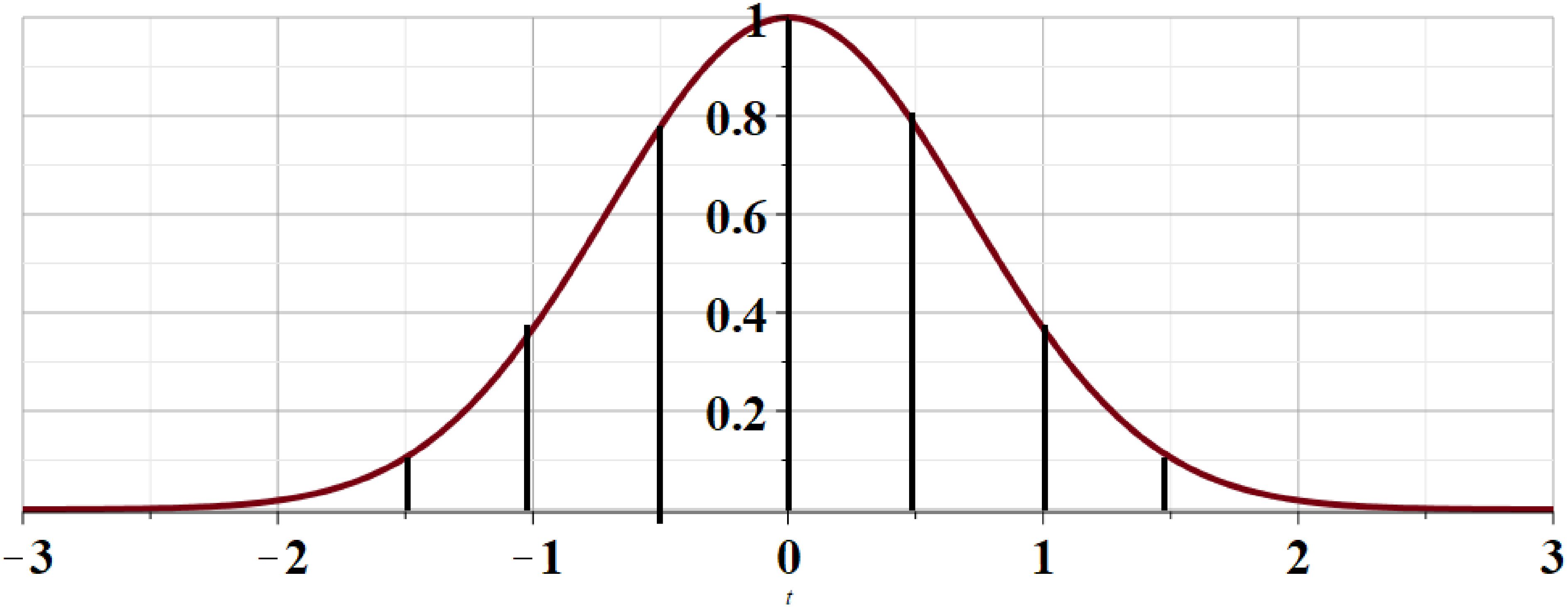
$\tilde{F}(\nu\Omega+\sigma W)\equiv\tilde{F}(\nu\Omega)$ for $\sigma=...-3,-2,-1,0,1,2,3...$ where $W\equiv\left((2N+1)\Omega\right)$

Recall that: $\sum_{n=-\infty}^{\infty}e^{in\omega T}=\Omega\sum_{\nu=-\infty}^{\infty}\delta(\omega-\nu\Omega),$ where $\Omega\equiv\frac{2\pi}{T}$

$$f(t)=\sum_{\nu=-\infty}^{\infty}\tilde{F}(\nu\Omega)e^{-i\nu\Omega t}=\frac{2\pi}{(2N+1)\Omega}\sum_{\nu=-N}^{N}\tilde{F}(\nu\Omega)e^{-i\nu\Omega t}\sum_{\mu}\delta\left(t-\frac{\mu T}{2N+1}\right)$$

For $t=\frac{m}{2N+1}T$: $\Rightarrow f\left(\frac{mT}{2N+1}\right)=\sum_{\nu=-M}^{M}F(\nu\Omega)e^{-i2\pi\nu m/(2N+1)}$

Falsely discretized time function $\bar{f}(t)$



$t \rightarrow$

$$t = \frac{m}{2N+1}T:$$

Doubly periodic functions

$$t \to \frac{\mu T}{2N+1}$$

$$\tilde{f}_\mu = \frac{1}{2N+1} \sum_{\nu=-N}^{N} \tilde{F}_\nu e^{-i2\pi\nu\mu/(2N+1)}$$

$$\tilde{F}_\nu = \sum_{\mu=-N}^{N} \tilde{f}_\mu e^{i2\pi\nu\mu/(2N+1)}$$

More convenient notation

$$2N+1 \rightarrow M$$

$$\tilde{f}_\mu = \frac{1}{M}\sum_{\nu=0}^{M-1} \tilde{F}_\nu e^{-i2\pi\nu\mu/M}$$

$$\tilde{F}_\nu = \sum_{\mu=0}^{M} \tilde{f}_\mu e^{i2\pi\nu\mu/M}$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

However, $W^M = \left(e^{i2\pi/M}\right)^M = 1$

and $W^{M/2} = \left(e^{i2\pi/M}\right)^{M/2} = -1$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, "An algorithm for machine calculation of complex Fourier series" Math. Computation 19, 297-301 (1965) http://www.fftw.org/

FFTW

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Introduction

FFTW is a C subroutine library for computing the discrete Fourier transform (DFT) in one or more dimensions, of arbitrary input size, and of both real and complex data (as well as of even/odd data, i.e. the discrete cosine/sine transforms or DCT/DST). We believe that FFTW, which is free software, should become the FFT library of choice for most applications.

The latest official release of FFTW is version **3.3.10**, available from our download page. Version 3.3 introduced support for the AVX x86 extensions, a distributed-memory implementation on top of MPI, and a Fortran 2003 API. Version 3.3.1 introduced support for the ARM Neon extensions. See the release notes for more information.

The FFTW package was developed at MIT by Matteo Frigo and Steven G. Johnson.

Our benchmarks, performed on on a variety of platforms, show that FFTW's performance is typically superior to that of other publicly available FFT software, and is even competitive with vendor-tuned codes. In contrast to vendor-tuned codes, however, FFTW's performance is *portable*: the same program will perform well on most architectures without modification. Hence the name, "FFTW," which stands for the somewhat whimsical title of "**Fastest Fourier Transform in the West**."

Subscribe to the fftw-announce mailing list to receive release announcements (or use the web feed).

Fourier series and Fourier transforms are useful for solving and analyzing a wide variety of functions, also beyond the Sturm-Liouville context.

In the next several slides we will consider a related concept – the Laplace transform.

Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty e^{-px}\phi(x)dx. \tag{24}$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x \leq \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \tag{25}$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \tag{26}$$

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.
For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \tag{27}$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by e^{-px} and integrating $0 \le x \le \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \tag{28}$$

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty \mathrm{e}^{-px} dx = \frac{1}{p}. \tag{29}$$

$$\mathcal{L}_x = \int_0^\infty x\mathrm{e}^{-px} dx = \frac{1}{p^2}. \tag{30}$$

$$\mathcal{L}_{\mathrm{e}^{-\gamma x}} = \int_0^\infty \mathrm{e}^{-\gamma x}\mathrm{e}^{-px} dx = \frac{1}{p+\gamma}. \tag{31}$$

Noting that

$$-\frac{F_0}{p^2(\gamma+p)} = -\frac{F_0}{\gamma^2}\left(\frac{1}{\gamma+p} - \frac{1}{p} + \frac{\gamma}{p^2}\right), \tag{32}$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2}\left(1 - \mathrm{e}^{-\gamma x} - \gamma x\right). \tag{33}$$

We can check that this a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \qquad \text{for} \quad \phi(0)=0 \qquad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$

Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right) \quad \text{with} \quad \phi(0)=0,\ \frac{d\phi(0)}{dx}=0$$

$$\mathcal{L}_\phi(p)=-\left(\frac{\pi}{L}\right)\frac{F_0}{\left(p^2+1\right)\left(p^2+\left(\frac{\pi}{L}\right)^2\right)}$$

$$=-F_0\left(\frac{\pi/L}{(\pi/L)^2-1}\right)\left(\frac{1}{p^2+1}-\frac{1}{p^2+\left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\displaystyle\int_0^\infty \sin(at)e^{-pt}dt=\frac{a}{a^2+p^2}$

$$\Rightarrow \phi(x)=\frac{F_0}{(\pi/L)^2-1}\left(\sin\left(\frac{\pi x}{L}\right)-\frac{\pi}{L}\sin(x)\right)$$

Does this result look familiar?

a. Yes
b. No

Table of Laplace transforms

Laplace Transform Table

Largely modeled on a table in D'Azzo and Houpis, *Linear Control Systems Analysis and Design*, 1988

$F(s)$	$f(t) \quad 0 \le t$	
1. 1	$\delta(t)$	unit impulse at $t=0$
2. $\frac{1}{s}$	1 or $u(t)$	unit step starting at $t=0$
3. $\frac{1}{s^2}$	$t \cdot u(t)$ or t	ramp function
4. $\frac{1}{s^n}$	$\frac{1}{(n-1)!} t^{n-1}$	n = positive integer
5. $\frac{1}{s} e^{-as}$	$u(t-a)$	unit step starting at $t=a$
6. $\frac{1}{s}(1-e^{-as})$	$u(t)-u(t-a)$	rectangular pulse
7. $\frac{1}{s+a}$	e^{-at}	exponential decay
8. $\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!} t^{n-1} e^{-at}$	n = positive integer
9. $\frac{1}{s(s+a)}$	$\frac{1}{a}(1-e^{-at})$	
10. $\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}(1-\frac{b}{b-a}e^{-at}+\frac{a}{b-a}e^{-bt})$	
11. $\frac{s+\alpha}{s(s+a)(s+b)}$	$\frac{1}{ab}[\alpha-\frac{b(\alpha-a)}{b-a}e^{-at}+\frac{a(\alpha-b)}{b-a}e^{-bt}]$	
12. $\frac{1}{(s+a)(s+b)}$	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	
13. $\frac{s}{(s+a)(s+b)}$	$\frac{1}{a-b}(ae^{-at}-be^{-bt})$	

https://www.dartmouth.edu/~sullivan/22files/New%20Laplace%20Transform%20Table.pdf

Inverse Laplace transform :

In order to evaluate these integrals, we need to use complex analysis.

$$\mathcal{L}_\phi(p) = \int_0^\infty e^{-pt}\phi(t)dt$$

$$\phi(t) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\phi(p)dp$$

Check: $$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\varphi(p)dp = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}dp\int_0^\infty e^{-pu}\varphi(u)du$$

$$\frac{1}{2\pi i}\int_0^\infty \varphi(u)du\int_{\lambda-i\infty}^{\lambda+i\infty} e^{p(t-u)}dp = \frac{1}{2\pi i}\int_0^\infty \varphi(u)du\int_{-\infty}^{\infty} e^{\lambda(t-u)}e^{is(t-u)}ids$$

$$= \frac{1}{2\pi i}\int_0^\infty \varphi(u)du\left(e^{\lambda(t-u)}2\pi i\ \delta(t-u)\right)$$

$$= \begin{cases} \varphi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In general – to calculate inverse Laplace transforms, we need to introduce concepts of complex numbers and contour integration

Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

$$\text{Define} \quad z = x + iy$$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

$$\text{Polar representation}$$

$$z = \rho(\cos\phi + i\sin\phi) = \rho\, e^{i\phi}$$