



PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF in Olin 103

Notes for Lecture 23: Chap. 7 & App. A-D (F&W)

Generalization of the one dimensional wave equation →
various mathematical problems and techniques including:

-  1. Fourier transforms
-  2. Laplace transforms
- 3. Complex variables
- 4. Contour integrals

PHYSICS COLLOQUIUM

THURSDAY

OCTOBER 20, 2022

More than just noise: Embracing environmental variability and its consequences for ecology

A conspicuous feature of ecological systems is the fact that conditions are rarely the same across space and time. An enduring challenge is to understand how heterogeneous conditions affect individuals, populations, and species, and use that understanding to make predictions. Mathematical models in ecology traditionally represent environmental variability as white noise, that is, independent random perturbations to dynamics that occur under constant conditions. But such a characterization often limits our understanding of potentially relevant biological processes. In this talk, I will give two examples of how environmental noise interacts with population growth processes to yield novel, emergent phenomena. The first example illustrates how changing environments present opportunities for multiple species to coexist despite competition between them. The second example illustrates how spatio-temporal heterogeneity accelerates population growth. I will demonstrate how local SARS-CoV-2 mitigation policies blind to this effect might inadvertently exacerbate viral spread. Throughout, I emphasize how environmental noise interacts with



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Wake Forest University

4:00 pm - Olin 101*

*Link provided for those unable to attend in person.
Note: For additional information on the seminar
or to obtain the video conference link, contact
wfuphys@wfu.edu

Reception at 3:30pm - Olin Entrance



| | | | | | |
|----|-----------------|---------------|--|---------------------|------------|
| 19 | Mon, 10/03/2022 | Chap. 7 | Sturm-Liouville equations | #15 | 10/05/2022 |
| 20 | Wed, 10/05/2022 | Chap. 7 | Sturm-Liouville equations | | |
| 21 | Fri, 10/07/2022 | Chap. 1-4,6-7 | Review | | |
| | Mon, 10/10/2022 | No class | Take home exam | | |
| | Wed, 10/12/2022 | No class | Take home exam | | |
| | Fri, 10/14/2022 | No class | Fall break | | |
| 22 | Mon, 10/17/2022 | Chap. 7 | Green's function methods for one-dimensional Sturm-Liouville equations | #16 | 10/19/2022 |
| 23 | Wed, 10/19/2022 | Chap. 7 | Fourier and other transform methods | #17 | 10/21/2022 |
| 24 | Fri, 10/21/2022 | Chap. 7 | Complex variables and contour integration | #18 | 10/24/2022 |

PHY 711 -- Assignment #17

Oct. 19, 2022

Continue reading Chapter 7 in **Fetter & Walecka**.

1. Consider the function $f(x) = x^2(1-x)$ in the interval $0 \leq x \leq 1$. Find the coefficients A_n of the Fourier series based on the terms $\sin(n\pi x)$. Extra credit: Plot $f(x)$ and the Fourier series including 3 terms.

Your questions:

From Athul -- I've doubts about how A_n is expressed in slides 9 and 10. Are they the same? Also, if $f(X)$ is imaginary can we use the same equations in the theorems for FT, and if we use algorithms for FFT will they treat them as real or imaginary?

From Banasree -- Can you please explain a little bit about how we can get normalized eigenfunctions in different cases?
I didn't understand why we had to do the backward transform.

Review – Sturm-Liouville equations defined over a range of x .

For $x_a \leq x \leq x_b$

$$\text{Homogenous problem: } \left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \varphi_0(x) = 0$$

$$\text{Inhomogenous problem: } \left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \varphi(x) = F(x)$$

Eigenfunctions:

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x)$$

Note that, because Sturm-Liouville operator is Hermitian, the eigenvalues are real and the eigenfunctions are orthogonal. In the last lecture, we argued that the eigenfunctions form a “complete” set over the range of x defined for the particular system.

Formal statement of the completeness of eigenfunctions:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x') \quad \text{where} \quad N_n \equiv \int_{x_a}^{x_b} dx \sigma(x) (f_n(x))^2$$

This means that within the interval $x_a \leq x \leq x_b$,

an arbitrary function $h(x)$ can be expanded: $h(x) = \sum_n A_n f_n(x)$.

Example for $\tau(x) = 1 = \sigma(x)$ and $v(x) = 0$ with

$0 \leq x \leq L$ and $f_n(0) = 0 = f_n(L)$

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x) \quad \Rightarrow \quad -\frac{d^2 f_n(x)}{dx^2} = \lambda_n f_n(x)$$

In this case, the normalized eigenfunctions are

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, \dots$$

Comment about normalized eigenfunctions:

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, \dots$$

$$\text{Check: } \int_0^L dx (f_n(x))^2 = \frac{2}{L} \int_0^L dx \left(\sin\left(\frac{n\pi x}{L}\right) \right)^2 = \frac{2}{L} \int_0^L dx \frac{1}{2} \left(1 - \cos\left(\frac{2n\pi x}{L}\right) \right) = \frac{2}{L} \frac{L}{2} = 1$$

Joseph Fourier



Jean-Baptiste Joseph Fourier

| | |
|-------------|--|
| Born | 21 March 1768 Auxerre, Burgundy, Kingdom of France (now in Yonne, France) |
| Died | 16 May 1830 (aged 62) |

Special case: $\tau(x) = 1 = \sigma(x)$ $v(x) = 0$

$$-\frac{d^2}{dx^2} f_n(x) = \lambda_n f_n(x) \quad \text{for } 0 \leq x \leq a, \quad \text{with } f_n(0) = f_n(a) = 0$$

$$f_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2$$

Fourier series representation of function $h(x)$ in the interval $0 \leq x \leq a$:

$$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$A_n = \sqrt{\frac{2}{a}} \int_0^a dx' h(x') \sin\left(\frac{n\pi x'}{a}\right)$$

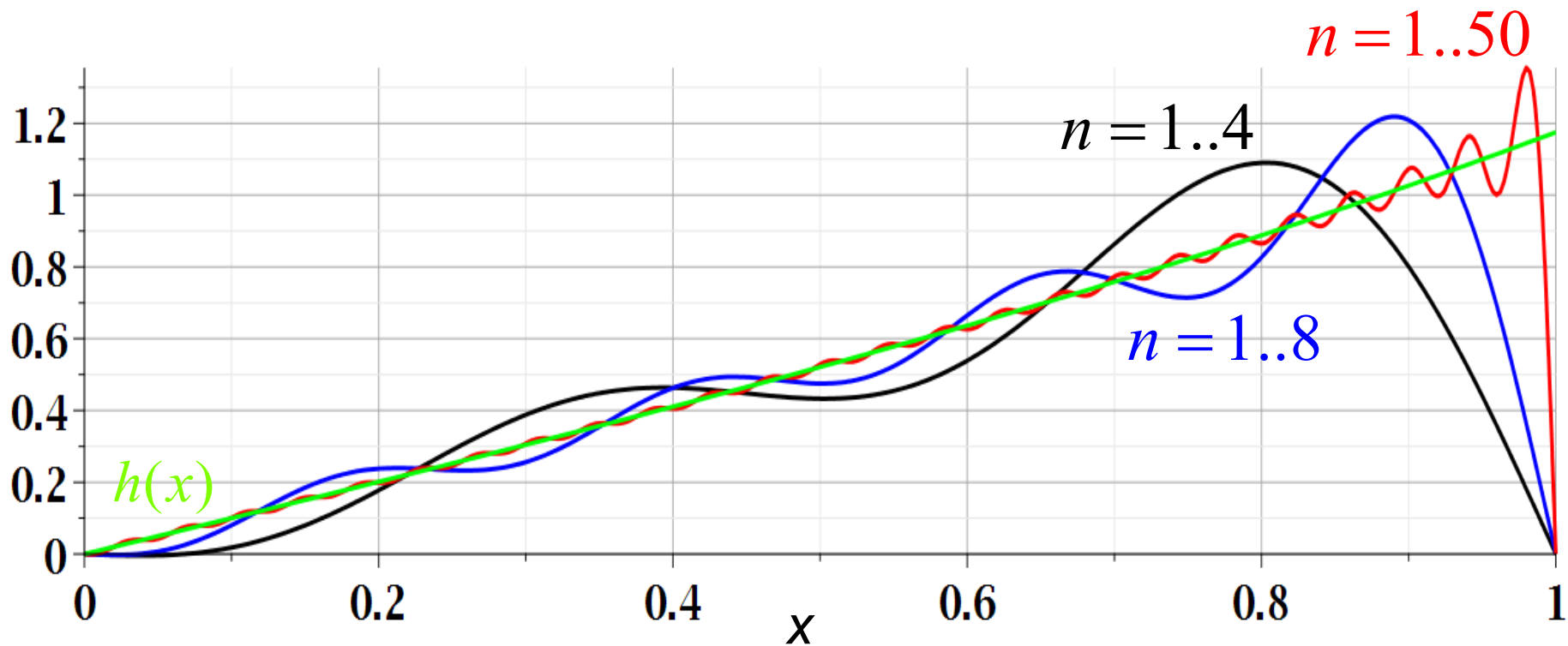
*Note that if $h(x)$ does not vanish at $x = 0$ and $x = a$, the more general

expression applies:
$$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) + \sum_{n=0}^{\infty} B_n \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right)$$

(with some restrictions).

Example

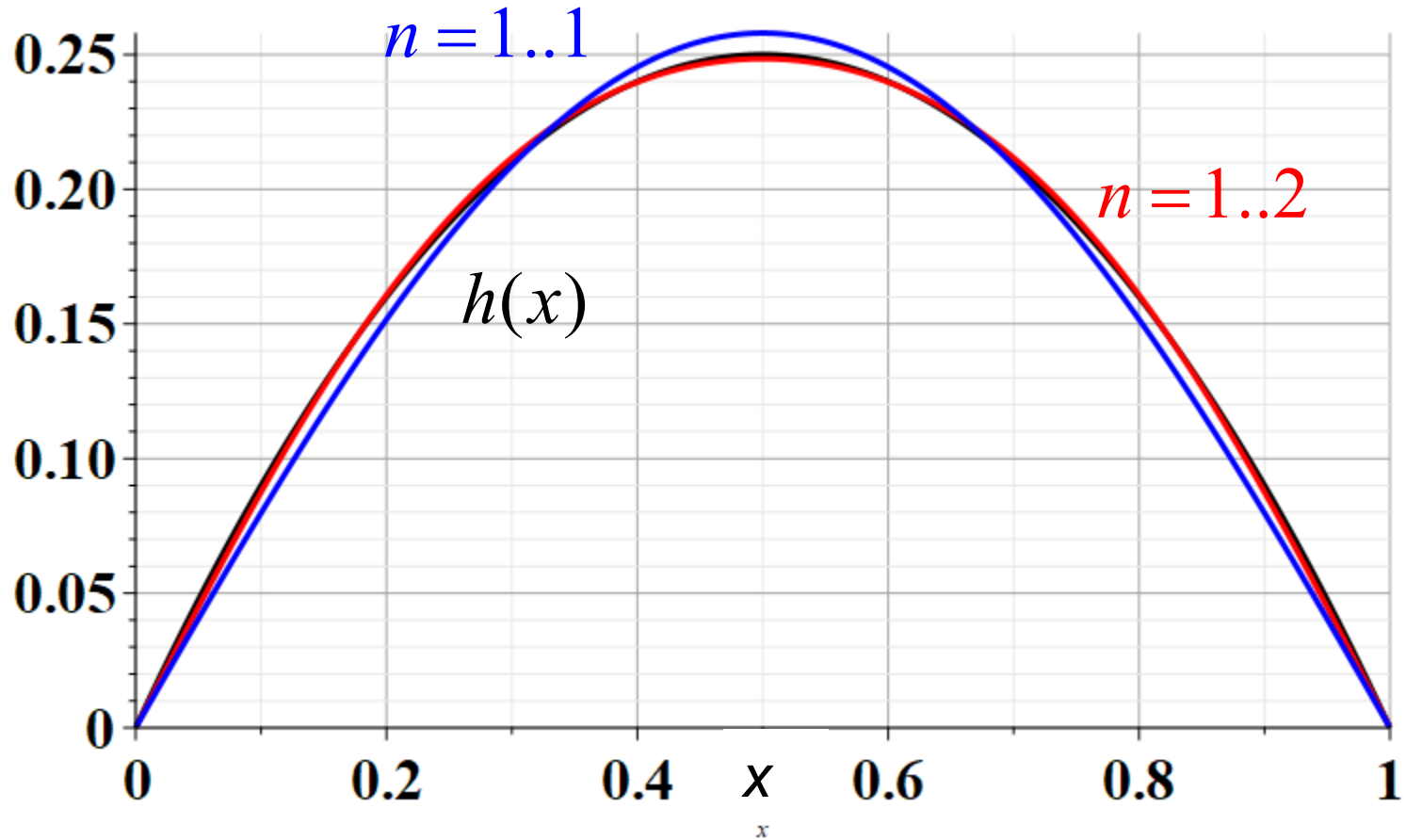
$$h(x) = \sinh(x) = \sqrt{2\pi} \sinh(1) \left(\frac{\sin(\pi x)}{\pi^2 + 1} - \frac{2\sin(2\pi x)}{4\pi^2 + 1} + \dots - (-1)^n n \frac{\sin(n\pi x)}{n^2\pi^2 + 1} + \dots \right)$$



Example

$$h(x) = x(1-x)$$

$$= \sum_{n=1}^{\infty} A_n \sqrt{2} \sin(n\pi x) \quad A_n = \begin{cases} \frac{4\sqrt{2}}{n^3 \pi^3} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$



Fourier series representation of function $h(x)$ in the interval $0 \leq x \leq a$:

$$h(x) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad \text{with} \quad A_n = \sqrt{\frac{2}{a}} \int_0^a dx' h(x') \sin\left(\frac{n\pi x'}{a}\right)$$

Can show that the series converges provided that $h(x)$ is

piecewise continuous.

Note that this analysis can also apply to time dependent functions. In the remainder of the lecture, we will consider time dependent functions.

$$x \rightarrow t \quad a \rightarrow T \quad 0 \leq t \leq T \quad \frac{n\pi}{a} \rightarrow \frac{n\pi}{T} \equiv \omega_n$$

$$h(t) = \sum_{n=1}^{\infty} A_n \sqrt{\frac{2}{T}} \sin(\omega_n t) \quad A_n = \sqrt{\frac{2}{T}} \int_0^T dt' h(t') \sin(\omega_n t')$$

Note that for this finite time range, Fourier series is discrete in frequency and continuous in time.

Comment on complex functions –

We can also treat complex functions, typically separately considering the real and imaginary parts.

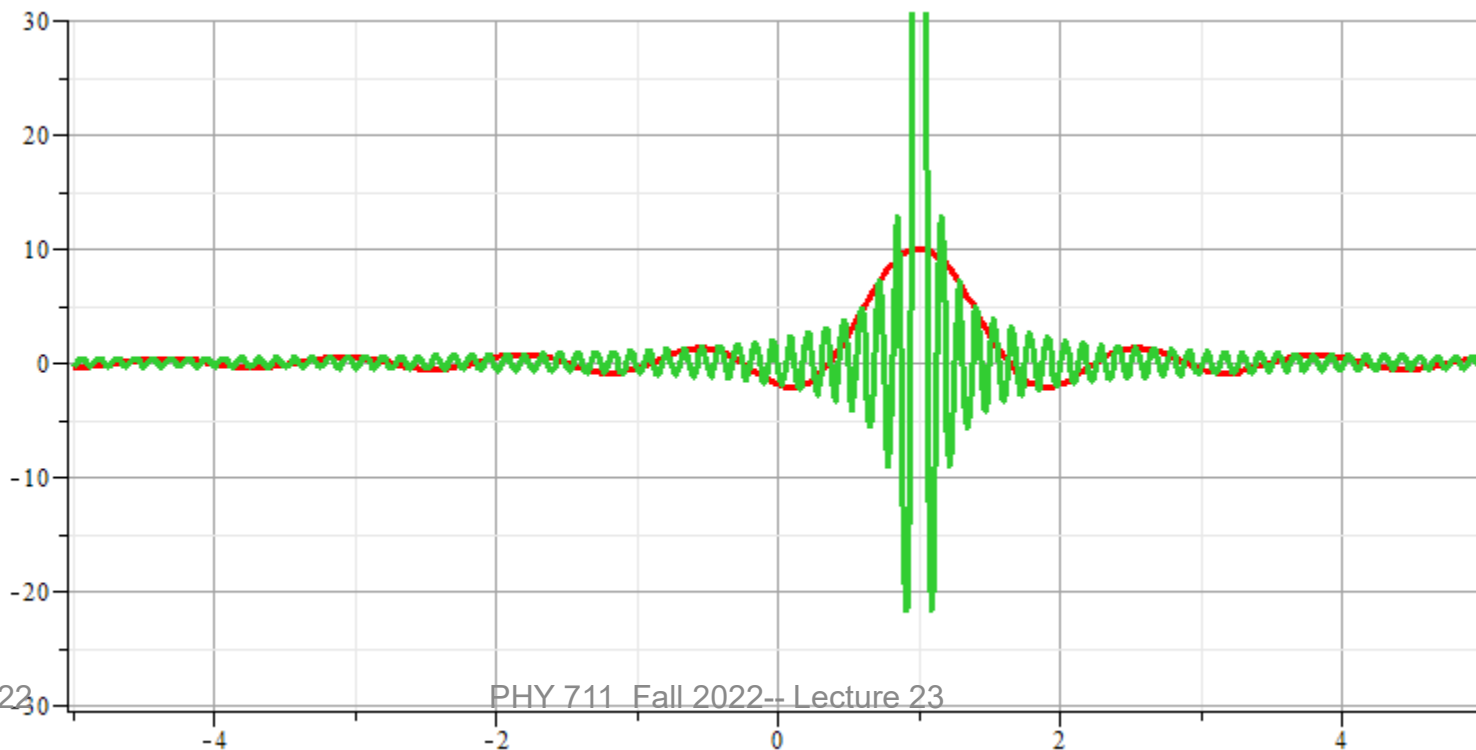
Generalization to infinite range -- Fourier transforms

A useful identity

$$\int_{-\infty}^{\infty} dt e^{-i(\omega - \omega_0)t} = 2\pi\delta(\omega - \omega_0)$$

Note that

$$\int_{-T}^T dt e^{-i(\omega - \omega_0)t} = \frac{2 \sin[(\omega - \omega_0)T]}{\omega - \omega_0} \underset{T \rightarrow \infty}{\approx} 2\pi\delta(\omega - \omega_0)$$



Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega F(\omega) e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dt' f(t') e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt' f(t') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt' f(t') \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

Question about forward and backward transforms –

Comment – Nomenclature for forward and backward can be confusing, but the point is that if we know the time dependence, we can determine the frequency dependence and if we know the frequency dependence, we can determine the time dependence.

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt (f(t))^* f(t) = 2\pi \int_{-\infty}^{\infty} d\omega (F(\omega))^* F(\omega)$$

Check:

$$\begin{aligned} \int_{-\infty}^{\infty} dt (f(t))^* f(t) &= \int_{-\infty}^{\infty} dt \left(\left(\int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t} \right)^* \int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t} \right) \\ &= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega') \int_{-\infty}^{\infty} dt e^{i(\omega' - \omega)t} \\ &= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega') 2\pi \delta(\omega' - \omega) \\ &= 2\pi \int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega) \end{aligned}$$

Note that for an infinite time range, the Fourier transform is continuous in both time and frequency.

Use of Fourier transforms to solve wave equation

Wave equation:
$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Suppose $u(x,t) = e^{-i\omega t} \tilde{F}(x,\omega)$ where $\tilde{F}(x,\omega)$ satisfies the equation:

$$\frac{\partial^2 \tilde{F}(x,\omega)}{\partial x^2} = -\frac{\omega^2}{c^2} \tilde{F}(x,\omega) \equiv -k^2 \tilde{F}(x,\omega)$$

More generally:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega F(x,\omega) e^{-i\omega t}$$

Further assume that fixed boundary conditions apply: $0 \leq x \leq L$

with $\tilde{F}(0,\omega) = 0$ and $\tilde{F}(L,\omega) = 0$

For $n = 1, 2, 3 \dots$

$$\tilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right) \quad k \rightarrow k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$$

$$u(x,t) = e^{-i\omega_n t} \sin(k_n x) = e^{-i\omega_n t} \frac{(e^{ik_n x} - e^{-ik_n x})}{2i} = \frac{(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)})}{2i}$$

Use of Fourier transforms to solve wave equation -- continued

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Using superposition: Suppose $u(x,t) = \sum_n A_n e^{-i\omega_n t} \tilde{F}_n(x, \omega_n)$

$$\frac{\partial^2 \tilde{F}_n(x, \omega_n)}{\partial x^2} = -\frac{\omega_n^2}{c^2} \tilde{F}_n(x, \omega_n) \equiv -k_n^2 \tilde{F}_n(x, \omega_n)$$

For $\tilde{F}_n(x, \omega) = \sin\left(\frac{n\pi x}{L}\right)$ $k \rightarrow k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$

$$\begin{aligned} \Rightarrow u(x,t) &= \sum_n A_n e^{-i\omega_n t} \sin(k_n x) = \sum_n \frac{A_n}{2i} e^{-i\omega_n t} (e^{ik_n x} - e^{-ik_n x}) \\ &= \sum_n \frac{A_n}{2i} (e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}) \equiv f(x-ct) + g(x+ct) \end{aligned}$$

Note that at this point, we do not know the coefficients A_n ; however, it clear that the solutions are consistent with D'Alembert's analysis of the wave equation.

Now consider the Fourier transform for a time periodic function:

Suppose $f(t + nT) = f(t)$ for any integer n

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_0^T dt f(t) e^{i\omega(t+nT)} \right)$$

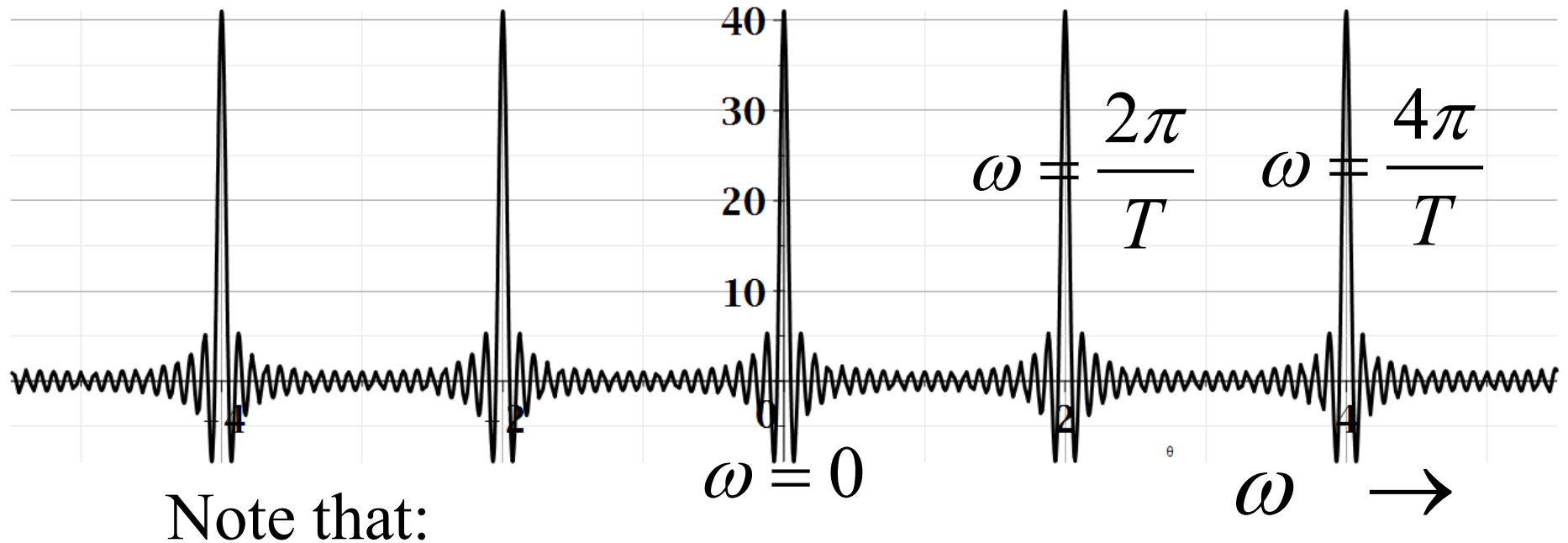
Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where } \Omega \equiv \frac{2\pi}{T}$$

Details:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \lim_{N \rightarrow \infty} \sum_{n=-N}^N e^{in\omega T} = \lim_{N \rightarrow \infty} \frac{\sin\left(\left(N + \frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\frac{\sin\left(\left(N + \frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$



$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where } \Omega \equiv \frac{2\pi}{T}$$

Geometric summation:
$$\sum_{n=-N}^N e^{in\omega T} = \frac{\sin\left(\left(N + \frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\lim_{N \rightarrow \infty} \left(\frac{\sin\left(\left(N + \frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)} \right) = 2\pi \sum_{\nu} \delta(\omega T - \nu\Omega T) = \frac{2\pi}{T} \sum_{\nu} \delta(\omega - \nu\Omega)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where } \Omega \equiv \frac{2\pi}{T}$$

$$\Rightarrow F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t} = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} \Omega \delta(\omega - \nu\Omega) \left(\int_0^T dt f(t) e^{i\omega t} \right)$$

Thus, for a time periodic function

$$f(t) = \int_{-\infty}^{\infty} d\omega F(\omega) e^{-i\omega t} = \sum_{\nu=-\infty}^{\infty} \bar{F}(\nu\Omega) e^{-i\nu\Omega t},$$

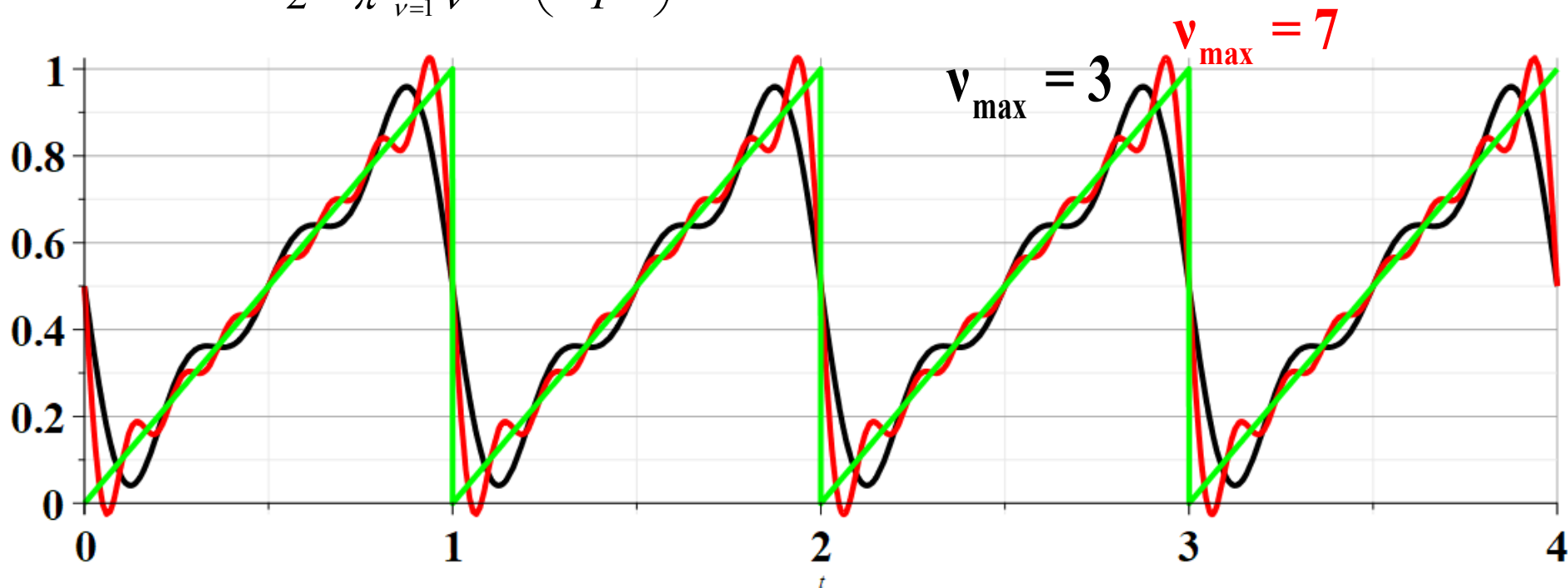
$$\text{where } \bar{F}(\nu\Omega) = \frac{1}{T} \int_0^T dt f(t) e^{i\nu\Omega t}$$

Example:

Suppose: $f(t) = \frac{t - nT}{T}$ for $nT \leq t \leq (n+1)T$; $n = \dots -3, -2, -1, 0, 1, 2, 3, \dots$

$$\bar{F}(v\Omega) = \frac{1}{T} \int_0^T \frac{t}{T} e^{i\frac{v2\pi t}{T}} dt = \bar{F}^*(-v\Omega) = \frac{-i}{2\pi v} \text{ for } v = 1, 2, 3, \dots \quad \bar{F}(0) = \frac{1}{2}$$

$$f(t) = \frac{1}{2} - \frac{2}{\pi} \sum_{v=1}^{\infty} \frac{1}{v} \sin\left(\frac{2\pi vt}{T}\right)$$



Summary –

Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega F(\omega) e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$$

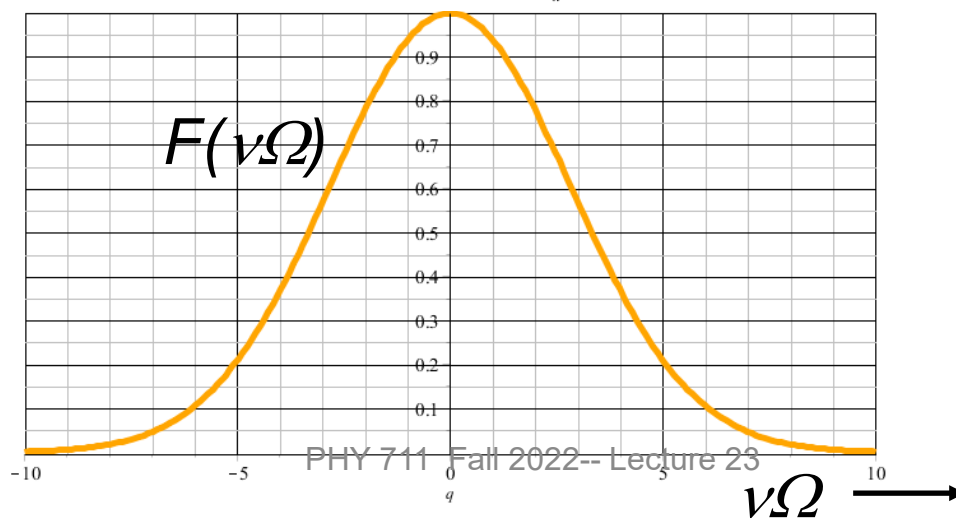
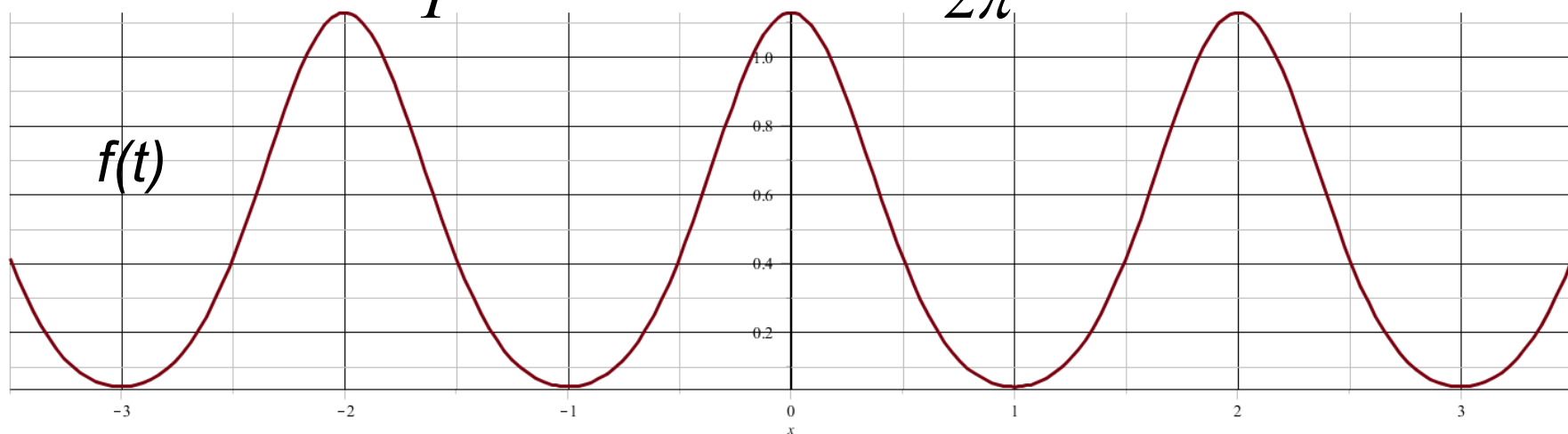
Find discrete frequencies ω for functions $f(t)$ over finite time domain of for functions $f(t)$ which are periodic: $f(t) = f(t + nT)$

➔ Numerically, there is an advantage of tabulating double discrete Fourier transforms (discrete in ω and in t).

Example:

$$\text{Suppose: } f(t) = \frac{1}{a\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2} = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$$

$$\text{where } \Omega \equiv \frac{2\pi}{T} \text{ and } F(\nu\Omega) = \frac{1}{2\pi} e^{-a^2\nu^2\Omega^2/4}$$



Continued: $f(t) = \frac{1}{a\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2} = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$

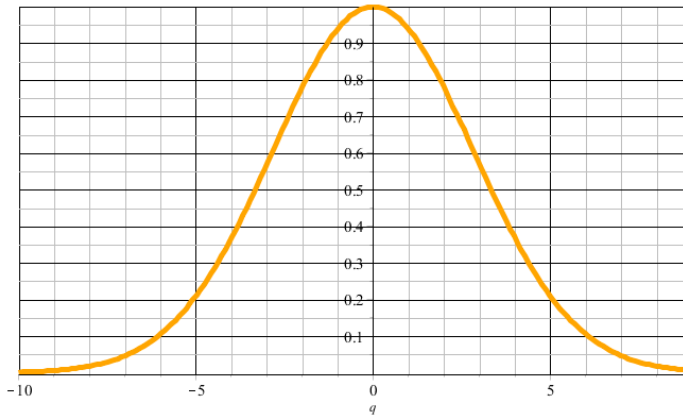
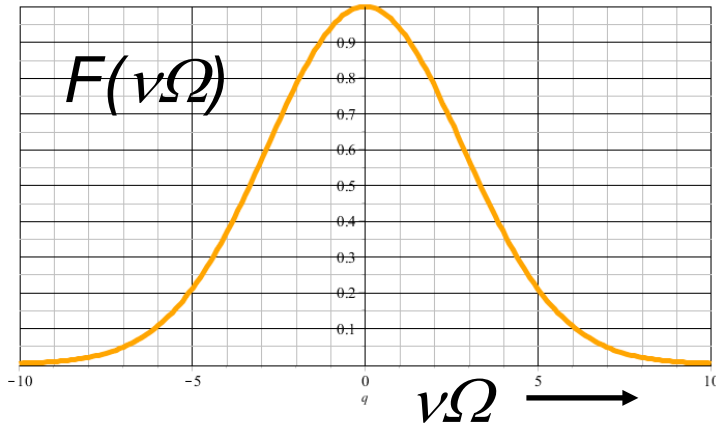
Note:

$$\Omega = \frac{2\pi}{T}$$

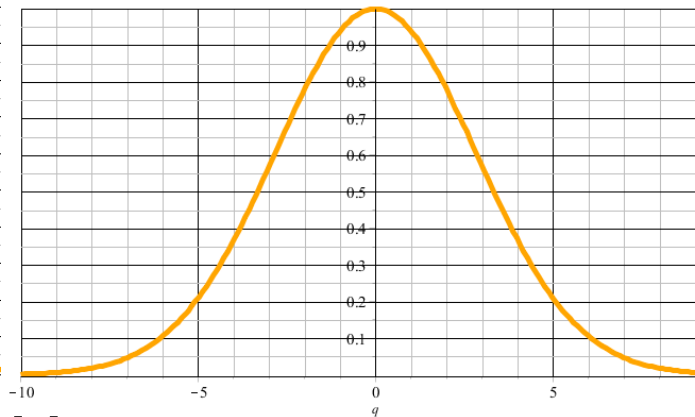
$$f(t) \approx \sum_{\nu=-M}^M F(\nu\Omega) e^{-i\nu\Omega t}$$

because $F(\nu'\Omega) \approx 0$

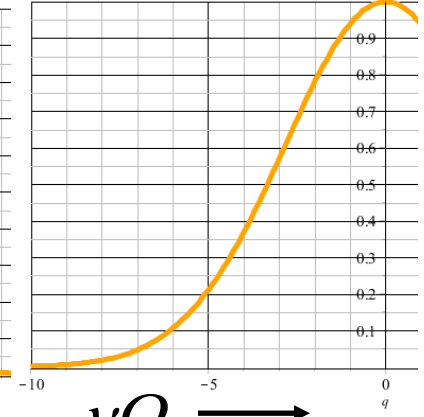
for $|\nu'| > M$



$\nu = -M$

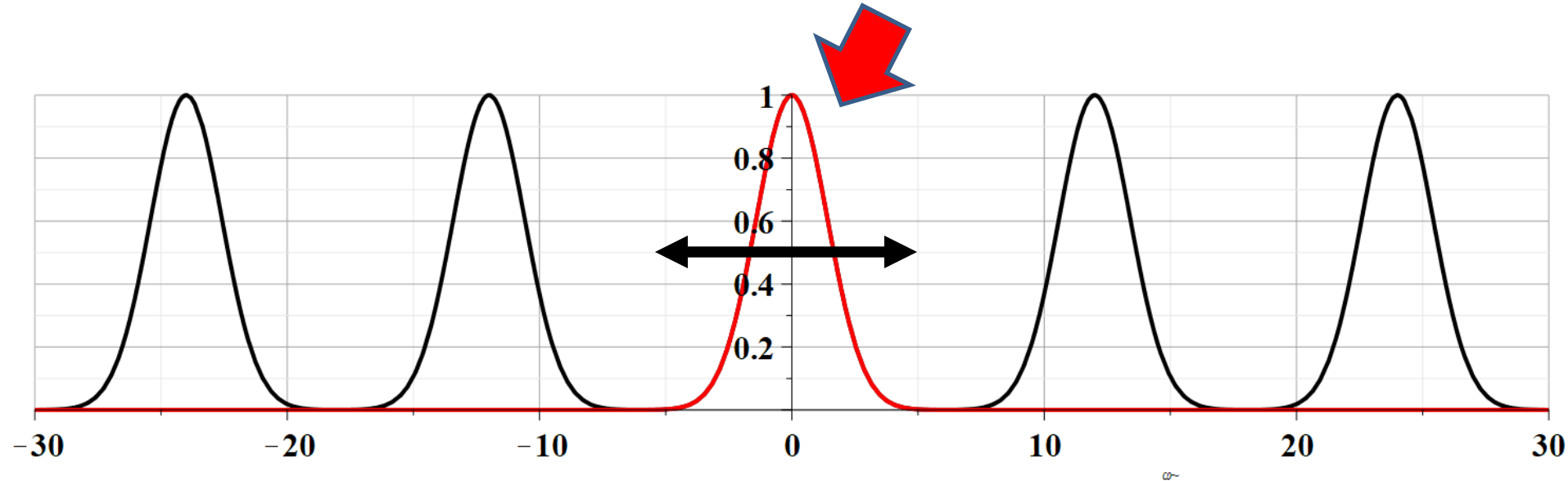


$\nu = M$



Constructed frequency periodic function --

Envelope of frequency function $F(\omega)$



Falsely periodic frequency function $\tilde{F}(\omega)$

Thus, for any periodic function: $f(t) = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega)e^{-i\nu\Omega t}$

Now suppose that the transformed function is bounded;

$$|F(\nu\Omega)| \leq \varepsilon \quad \text{for} \quad |\nu| \geq N$$

Define a periodic transform function

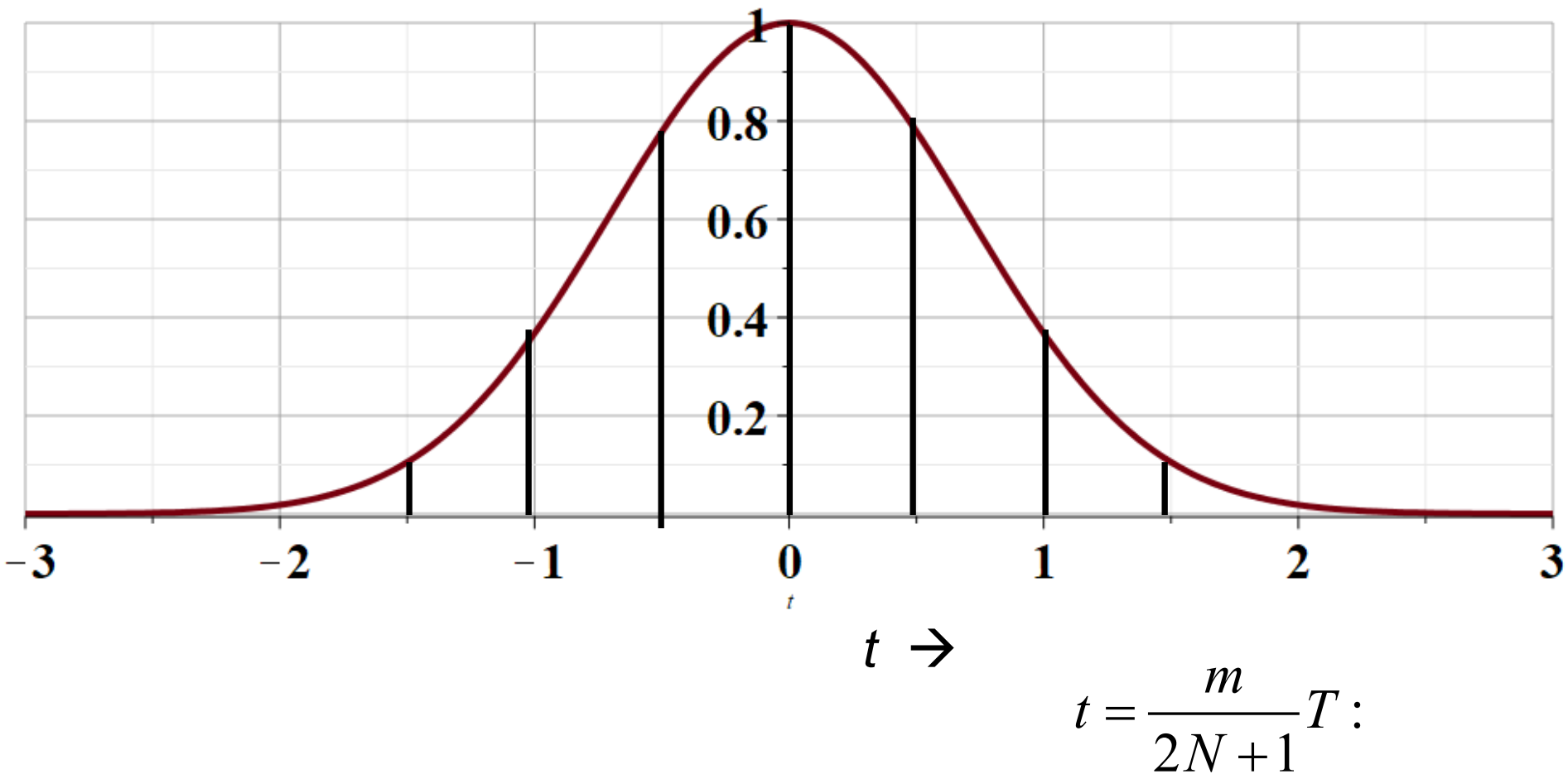
$$\tilde{F}(\nu\Omega + \sigma W) \equiv \tilde{F}(\nu\Omega) \quad \text{for} \quad \sigma = \dots -3, -2, -1, 0, 1, 2, 3 \dots \quad \text{where} \quad W \equiv ((2N+1)\Omega)$$

Recall that: $\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where} \quad \Omega \equiv \frac{2\pi}{T}$

$$f(t) = \sum_{\nu=-\infty}^{\infty} \tilde{F}(\nu\Omega)e^{-i\nu\Omega t} = \frac{2\pi}{(2N+1)\Omega} \sum_{\nu=-N}^N \tilde{F}(\nu\Omega)e^{-i\nu\Omega t} \sum_{\mu} \delta\left(t - \frac{\mu T}{2N+1}\right)$$

$$\text{For } t = \frac{m}{2N+1}T : \Rightarrow f\left(\frac{mT}{2N+1}\right) = \sum_{\nu=-M}^M F(\nu\Omega)e^{-i2\pi\nu m/(2N+1)}$$

Falsely discretized time function $\bar{f}(t)$



Doubly periodic functions

$$t \rightarrow \frac{\mu T}{2N+1}$$

$$\tilde{f}_\mu = \frac{1}{2N+1} \sum_{\nu=-N}^N \tilde{F}_\nu e^{-i2\pi\nu\mu/(2N+1)}$$

$$\tilde{F}_\nu = \sum_{\mu=-N}^N \tilde{f}_\mu e^{i2\pi\nu\mu/(2N+1)}$$

More convenient notation

$$2N + 1 \rightarrow M$$

$$\tilde{f}_\mu = \frac{1}{M} \sum_{\nu=0}^{M-1} \tilde{F}_\nu e^{-i2\pi\nu\mu/M}$$

$$\tilde{F}_\nu = \sum_{\mu=0}^M \tilde{f}_\mu e^{i2\pi\nu\mu/M}$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \dots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \dots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \dots$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \dots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \dots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \dots$$

$$\text{However, } W^M = \left(e^{i2\pi/M} \right)^M = 1$$

$$\text{and } W^{M/2} = \left(e^{i2\pi/M} \right)^{M/2} = -1$$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, “An algorithm for machine calculation of complex Fourier series” Math. Computation 19, 297-301 (1965)



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Introduction

FFTW is a C subroutine library for computing the discrete Fourier transform (DFT) in one or more dimensions, of arbitrary input size, and of both real and complex data (as well as of even/odd data, i.e. the discrete cosine/sine transforms or DCT/DST). We believe that FFTW, which is [free software](#), should become the [FFT](#) library of choice for most applications.

The latest official release of FFTW is version **3.3.10**, available from [our download page](#). Version 3.3 introduced support for the AVX x86 extensions, a distributed-memory implementation on top of MPI, and a Fortran 2003 API. Version 3.3.1 introduced support for the ARM Neon extensions. See the [release notes](#) for more information.

The FFTW package was developed at [MIT](#) by [Matteo Frigo](#) and [Steven G. Johnson](#).

Our [benchmarks](#), performed on a variety of platforms, show that FFTW's performance is typically superior to that of other publicly available FFT software, and is even competitive with vendor-tuned codes. In contrast to vendor-tuned codes, however, FFTW's performance is *portable*: the same program will perform well on most architectures without modification. Hence the name, "FFTW," which stands for the somewhat whimsical title of "**Fastest Fourier Transform in the West**."

Subscribe to the [fftw-announce mailing list](#) to receive release announcements (or use the web feed ).



Only got to this slide --

Fourier series and Fourier transforms are useful for solving and analyzing a wide variety of functions, also beyond the Sturm-Liouville context.

In the next several slides we will consider a related concept – the Laplace transform.



Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty e^{-px} \phi(x) dx. \quad (24)$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x \leq \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \quad (25)$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \quad (26)$$



These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.

For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \quad (27)$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by e^{-px} and integrating $0 \leq x \leq \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \quad (28)$$

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty e^{-px} dx = \frac{1}{p}. \quad (29)$$

$$\mathcal{L}_x = \int_0^\infty xe^{-px} dx = \frac{1}{p^2}. \quad (30)$$

$$\mathcal{L}_{e^{-\gamma x}} = \int_0^\infty e^{-\gamma x} e^{-px} dx = \frac{1}{p + \gamma}. \quad (31)$$

Noting that

$$-\frac{F_0}{p^2(\gamma + p)} = -\frac{F_0}{\gamma^2} \left(\frac{1}{\gamma + p} - \frac{1}{p} + \frac{\gamma}{p^2} \right), \quad (32)$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2} (1 - e^{-\gamma x} - \gamma x). \quad (33)$$

We can check that this a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \quad \text{for} \quad \phi(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$

Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right) \quad \text{with } \phi(0) = 0, \frac{d\phi(0)}{dx} = 0$$

$$\mathcal{L}_\phi(p) = -\left(\frac{\pi}{L}\right) \frac{F_0}{\left(p^2 + 1\right)\left(p^2 + \left(\frac{\pi}{L}\right)^2\right)}$$

$$= -F_0 \left(\frac{\pi/L}{(\pi/L)^2 - 1}\right) \left(\frac{1}{p^2 + 1} - \frac{1}{p^2 + \left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\int_0^\infty \sin(at)e^{-pt} dt = \frac{a}{a^2 + p^2}$

$$\Rightarrow \phi(x) = \frac{F_0}{(\pi/L)^2 - 1} \left(\sin\left(\frac{\pi x}{L}\right) - \frac{\pi}{L} \sin(x)\right)$$

Does this result look familiar?

- a. Yes
- b. No



Table of Laplace transforms

Laplace Transform Table

Largely modeled on a table in D'Azzo and Houpis, *Linear Control Systems Analysis and Design*, 1988

| $F(s)$ | $f(t) \quad 0 \leq t$ |
|------------------------------------|--|
| 1. 1 | $\delta(t)$ unit impulse at $t = 0$ |
| 2. $\frac{1}{s}$ | 1 or $u(t)$ unit step starting at $t = 0$ |
| 3. $\frac{1}{s^2}$ | $t \cdot u(t)$ or t ramp function |
| 4. $\frac{1}{s^n}$ | $\frac{1}{(n-1)!} t^{n-1}$ $n = \text{positive integer}$ |
| 5. $\frac{1}{s} e^{-as}$ | $u(t-a)$ unit step starting at $t = a$ |
| 6. $\frac{1}{s}(1 - e^{-as})$ | $u(t) - u(t-a)$ rectangular pulse |
| 7. $\frac{1}{s+a}$ | e^{-at} exponential decay |
| 8. $\frac{1}{(s+a)^n}$ | $\frac{1}{(n-1)!} t^{n-1} e^{-at}$ $n = \text{positive integer}$ |
| 9. $\frac{1}{s(s+a)}$ | $\frac{1}{a}(1 - e^{-at})$ |
| 10. $\frac{1}{s(s+a)(s+b)}$ | $\frac{1}{ab} \left(1 - \frac{b}{b-a} e^{-at} + \frac{a}{b-a} e^{-bt} \right)$ |
| 11. $\frac{s+\alpha}{s(s+a)(s+b)}$ | $\frac{1}{ab} \left[\alpha - \frac{b(\alpha-a)}{b-a} e^{-at} + \frac{a(\alpha-b)}{b-a} e^{-bt} \right]$ |
| 12. $\frac{1}{(s+a)(s+b)}$ | $\frac{1}{b-a} (e^{-at} - e^{-bt})$ |
| 13. $\frac{s}{(s+a)(s+b)}$ | $\frac{1}{a-b} (ae^{-at} - be^{-bt})$ |

<https://www.dartmouth.edu/~sullivan/22files/New%20Laplace%20Transform%20Table.pdf>

Inverse Laplace transform :

$$\mathcal{L}_\phi(p) = \int_0^{\infty} e^{-pt} \phi(t) dt$$

$$\phi(t) = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt} \mathcal{L}_\phi(p) dp$$

Check:
$$\frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt} \mathcal{L}_\phi(p) dp = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt} dp \int_0^{\infty} e^{-pu} \phi(u) du$$

$$\begin{aligned} \frac{1}{2\pi i} \int_0^{\infty} \phi(u) du \int_{\lambda-i\infty}^{\lambda+i\infty} e^{p(t-u)} dp &= \frac{1}{2\pi i} \int_0^{\infty} \phi(u) du \int_{-\infty}^{\infty} e^{\lambda(t-u)} e^{is(t-u)} i ds \\ &= \frac{1}{2\pi i} \int_0^{\infty} \phi(u) du \left(e^{\lambda(t-u)} 2\pi i \delta(t-u) \right) \\ &= \begin{cases} \phi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

In order to evaluate these integrals, we need to use complex analysis.



In general – to calculate inverse Laplace transforms, we need to introduce concepts of complex numbers and contour integration

Complex numbers

$$i \equiv \sqrt{-1} \quad i^2 = -1$$

$$\text{Define } z = x + iy$$

$$|z|^2 = zz^* = (x + iy)(x - iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos \phi + i \sin \phi) = \rho e^{i\phi}$$