PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103
Notes for Lecture 27 – Chap. 8 (F & W)

Motions of elastic membranes

1. **Review of standing waves on a string**
2. **Standing waves on a two dimensional membrane.**
3. **Boundary value problems**

22	Mon, 10/17/2022	Chap. 7	Green's function methods for one-dimensional Sturm-Liouville equations	#16	10/19/2022
23	Wed, 10/19/2022	Chap. 7	Fourier and other transform methods	#17	10/21/2022
24	Fri, 10/21/2022	Chap. 7	Complex variables and contour integration	#18	10/24/2022
25	Mon, 10/24/2022	Chap. 5	Rigid body motion	#19	10/26/2022
26	Wed, 10/26/2022	Chap. 5	Rigid body motion	#20	10/28/2022
27	Fri, 10/28/2022	Chap. 8	Elastic two-dimensional membranes		
28	Mon, 10/31/2022	Chap. 9	Mechanics of 3 dimensional fluids		

Elastic media in two or more dimensions --

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

Initial value problem: $\mu(x,0) = \phi(x)$ and $\frac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

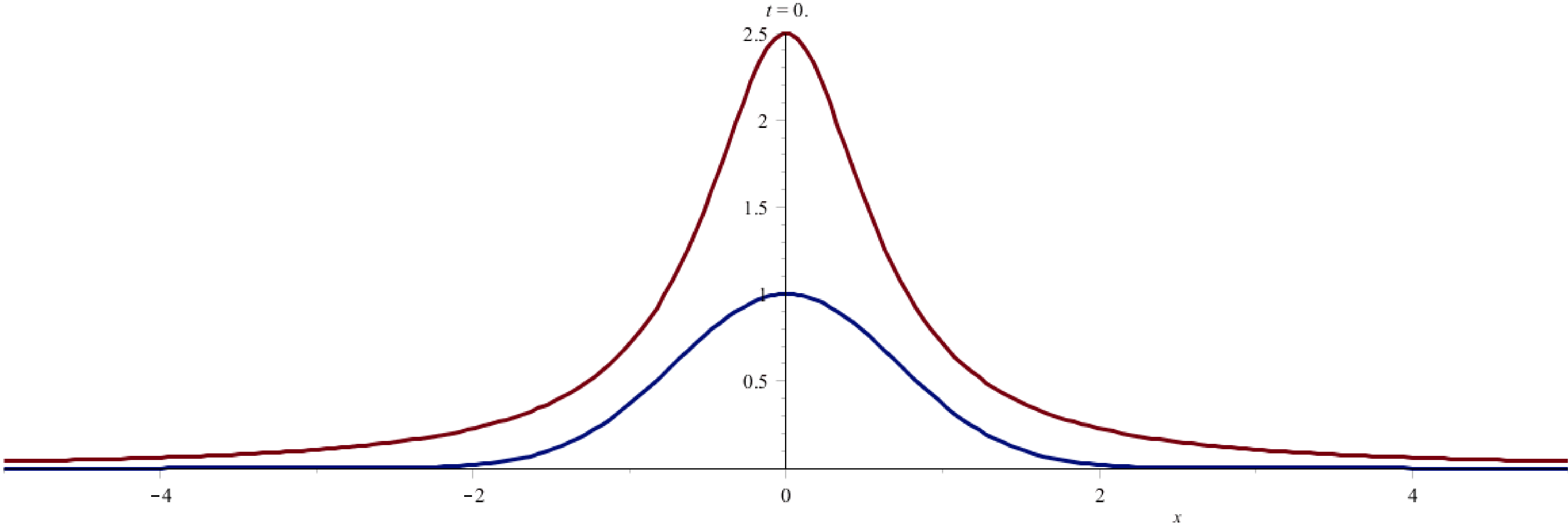
For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\big(\phi(x-ct) + \phi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Example with $\psi(x) = 0$ and $\phi(x) = \dfrac{1}{x^2 + 0.4}$



Example with $\psi(x) = 0$ and $\phi(x) = e^{-x^2}$

Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

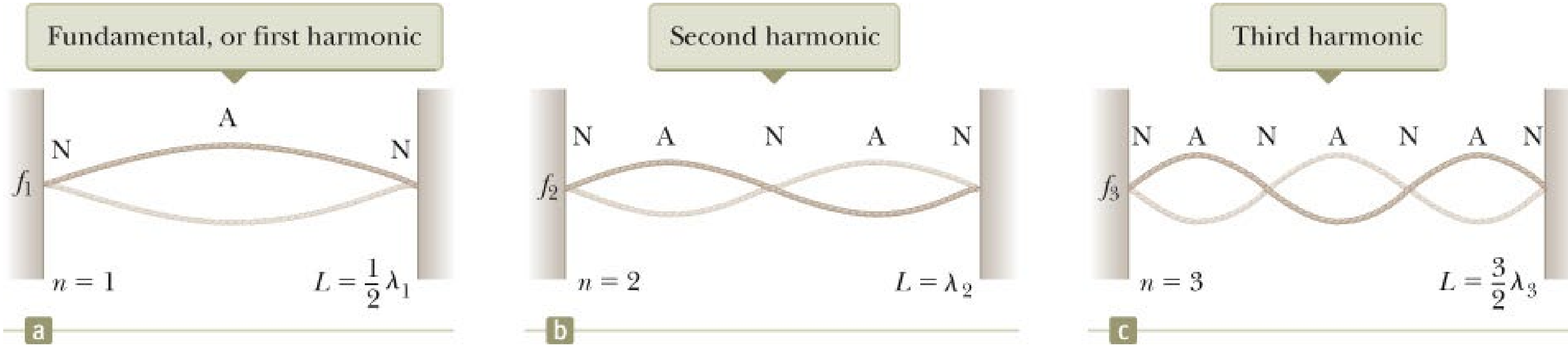
with $\mu(0,t) = \mu(L,t) = 0.$

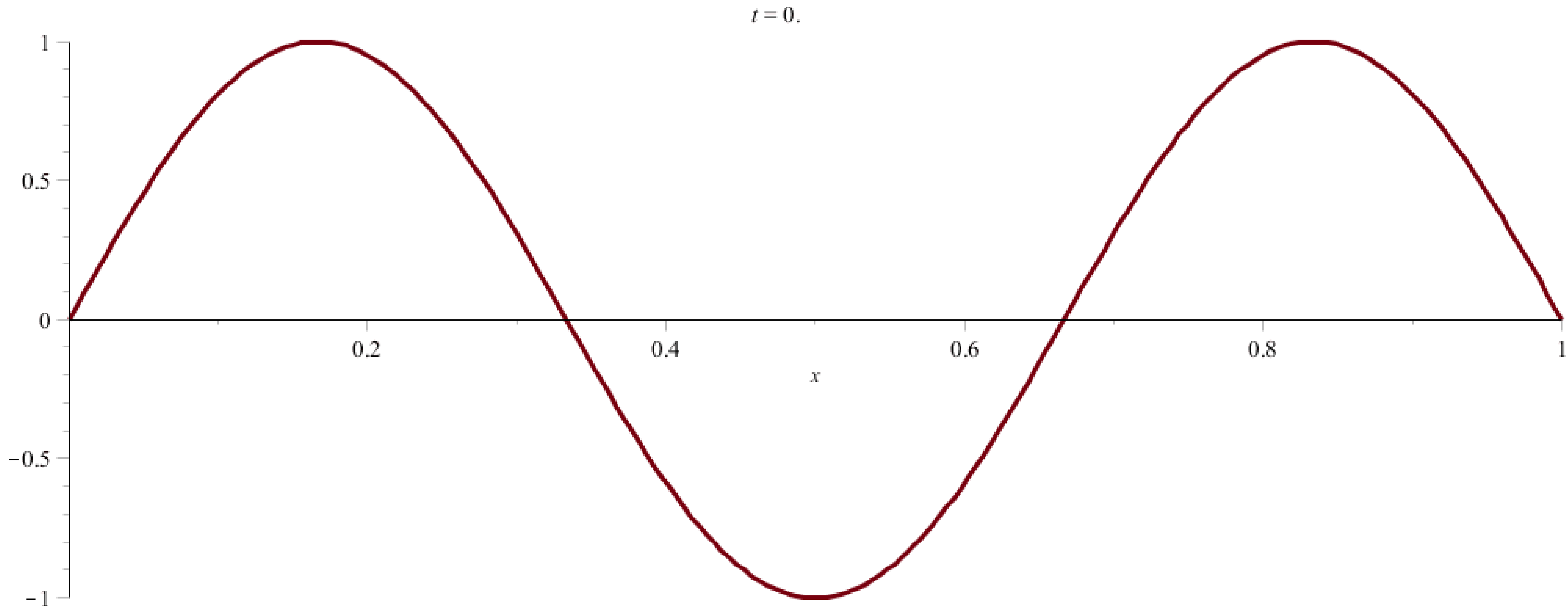
Assume: $\mu(x,t) = \Re\left(e^{-i\omega t}\rho(x)\right)$

where $\frac{d^2\rho(x)}{dx^2} + k^2\rho(x) = 0$ $\qquad k = \frac{\omega}{c}$

$$\rho_\nu(x) = A\sin\left(\frac{\nu\pi x}{L}\right)$$

$$k_\nu = \frac{\nu\pi}{L} \qquad \omega_\nu = ck_\nu$$


Fundamental, or first harmonic
Second harmonic
Third harmonic
A
N
N
f_1
$n = 1$
$L = \frac{1}{2}\lambda_1$
a
N A N A N
f_2
$n = 2$
$L = \lambda_2$
b
N A N A N A N
f_3
$n = 3$
$L = \frac{3}{2}\lambda_3$
c


t = 0.
1
0.5
0
−0.5
−1
0.2
0.4
0.6
0.8
1
x

Wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

Two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

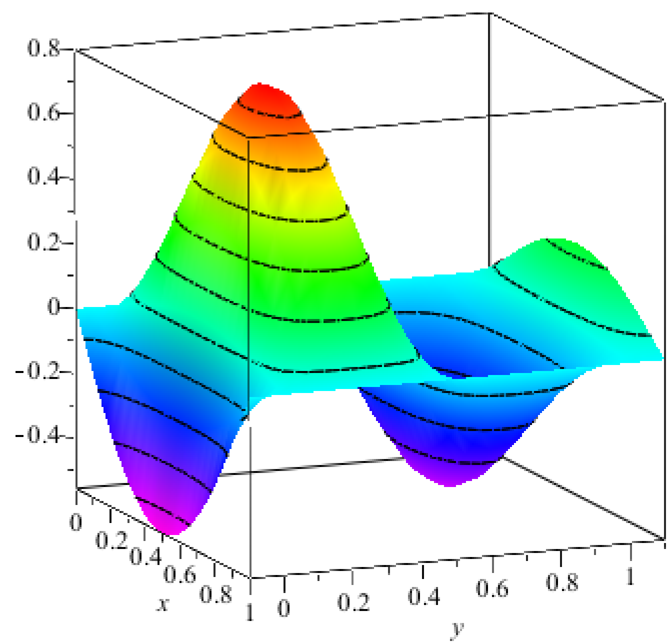
Standing wave solutions:

$$u(x,y,t) = \Re\left(e^{-i\omega t}\rho(x,y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x,y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Note that here we are visualizing transverse waves. Longitudinal waves can also exist.

$\rho(x,y)$



In this case, we have mapped the one dimensional elastic string to a two dimensional elastic membrane

$$\frac{\partial^2}{\partial x^2} \rightarrow \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{(in Cartesian coordinates)}$$

Lagrangian density: $\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right)$

$$L = \int \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) dxdy$$

Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u / \partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u / \partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u / \partial y)} = 0$$

Lagrangian density for elastic membrane with constant σ and τ :

$$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau(\nabla u)^2$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u / \partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u / \partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u / \partial y)} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Two-dimensional wave equation:

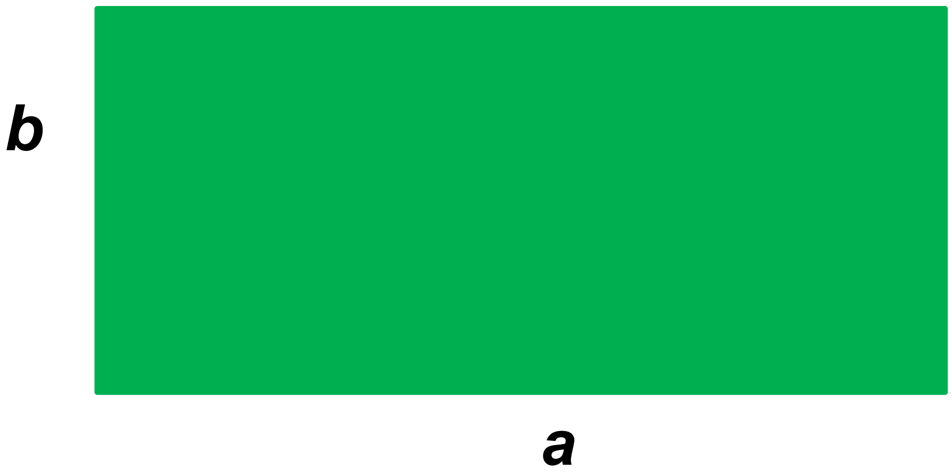
$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x, y, t) = \Re\left(e^{-i\omega t}\rho(x, y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x, y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Consider a rectangular boundary:

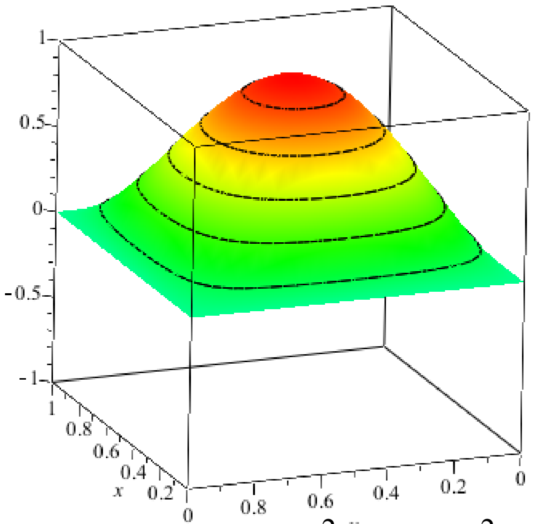


$$\text{Clamped boundary conditions}: \qquad \left(\nabla^2 + k^2\right)\rho(x,y) = 0$$

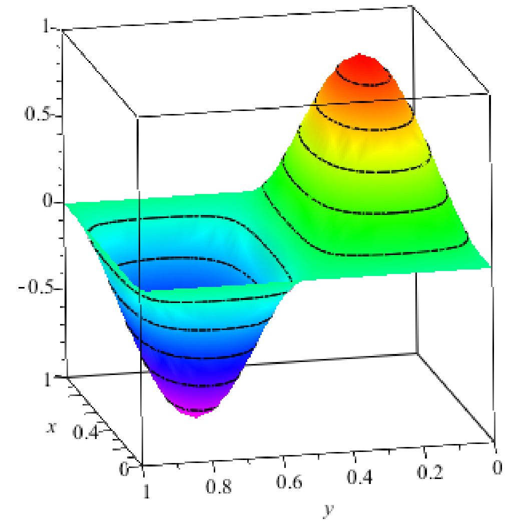
$$\rho(0,y) = \rho(a,y) = \rho(x,0) = \rho(x,b) = 0$$

$$\Rightarrow \rho_{mn}(x,y) = A \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad \text{where } k = \frac{\omega}{c}$$

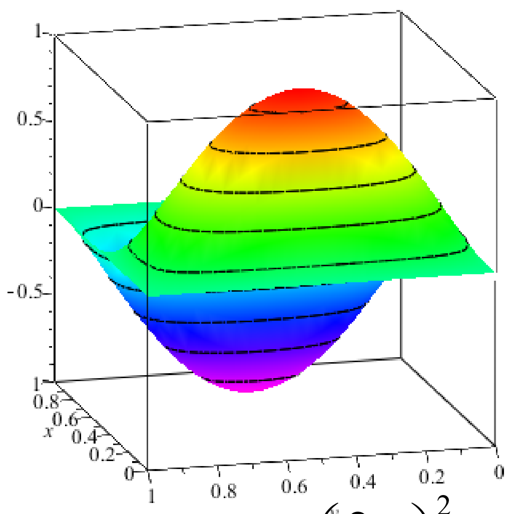
$$k_{mn}^{\ 2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$



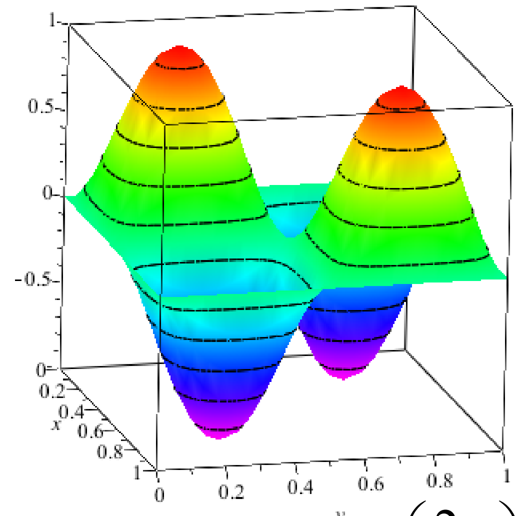
$$k_{11}{}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



$$k_{12}{}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$



$$k_{21}{}^2 = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



$$k_{22}{}^2 = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2$$

More general boundary conditions:

$$\tau \nabla u\big|_b = \kappa u\big|_b$$ represents bounded side constrained with spring

$$\tau \nabla u\big|_b = 0$$ represents "free" side

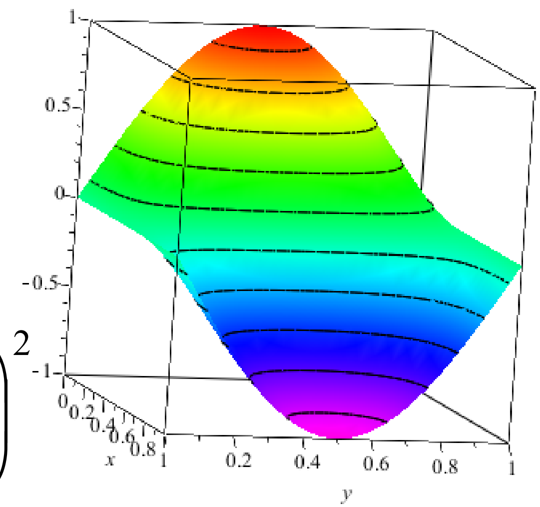
Mixed boundary conditions :

$$\rho(x,0) = \rho(x,b) = \frac{\partial \rho(0,y)}{\partial x} = \frac{\partial \rho(a,y)}{\partial x} = 0$$

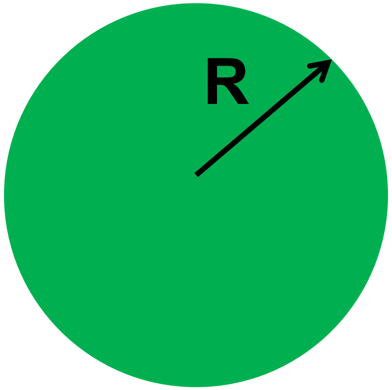
$$\Rightarrow \rho_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^{\ 2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn} = ck_{mn}$$

$$k_{11}^{\ 2} = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



Consider a circular boundary:



Clamped boundary conditions for $\rho(r,\varphi)$:

$\rho(R,\varphi)=0$

$$\left(\nabla^2+k^2\right)\rho(r,\phi)=0 \qquad \text{where } k=\frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

Assume: $\quad \rho(r,\varphi)=f(r)\Phi(\varphi)$

Let: $\quad \Phi(\varphi)=e^{im\varphi}$

Note: $\quad \Phi(\varphi)=\Phi(\varphi+2\pi)$

$$\Rightarrow m=\text{integer}$$

Consider circular boundary -- continued

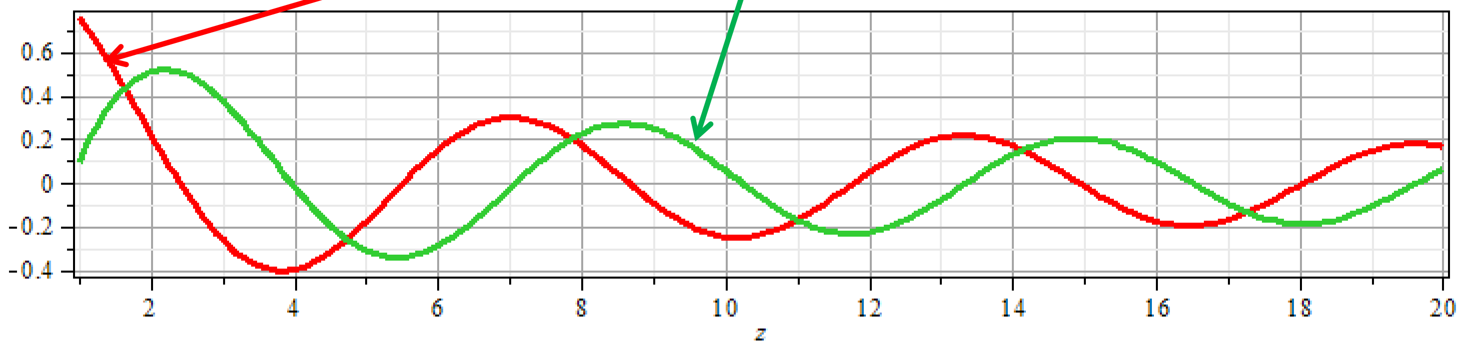
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$ also called $Y_m(z)$



Some properties of Bessel functions

$$\text{Asending series:}\quad J_m(z)=\left(\frac{z}{2}\right)^m\sum_{j=0}^{\infty}\frac{(-1)^j}{j!(j+m)!}\left(\frac{z}{2}\right)^{2j}$$

$$\text{Recursion relations:}\quad J_{m-1}(z)+J_{m+1}(z)=\frac{2m}{z}J_m(z)$$

$$J_{m-1}(z)-J_{m+1}(z)=2\frac{dJ_m(z)}{dz}$$

$$\text{Asymptotic form:}\quad J_m(z)\xrightarrow[z>>1]{}\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{m\pi}{2}-\frac{\pi}{4}\right)$$

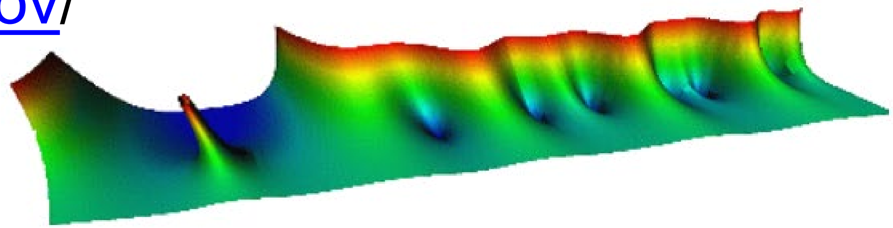
$$\text{Zeros of Bessel functions}\quad J_m(z_{mn})=0$$

$$m=0:\qquad z_{0n}=2.406,\ 5.520,\ 8.654,\ldots$$

$$m=1:\qquad z_{1n}=3.832,\ 7.016,\ 10.173,\ldots$$

$$m=2:\qquad z_{2n}=5.136,\ 8.417,\ 11.620,\ldots$$

http://dlmf.nist.gov/



NIST Digital Library of Mathematical Functions

Project News

2014-08-29 *DLMF Update; Version 1.0.9*
2014-04-25 *DLMF Update; Version 1.0.8; errata & improved MathML*
2014-03-21 *DLMF Update; Version 1.0.7; New Features improve Math & 3D Graphics*
2013-08-16 *Bille C. Carlson, DLMF Author, dies at age 89*
More news

Series expansions of Bessel and Neumann functions

$$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\tfrac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu+k+1)}.$$

$$Y_n(z) = -\frac{\left(\tfrac{1}{2}z\right)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\tfrac{1}{4}z^2\right)^k + \frac{2}{\pi}\ln\left(\tfrac{1}{2}z\right) J_n(z)$$

$$- \frac{\left(\tfrac{1}{2}z\right)^{n}}{\pi} \sum_{k=0}^{\infty} \left(\psi(k+1) + \psi(n+k+1)\right) \frac{\left(-\tfrac{1}{4}z^2\right)^k}{k!\,(n+k)!},$$

Some properties of Bessel functions -- continued

Note: It is possible to prove the following identity for the functions $J_m\left(\frac{z_{mn}}{R}r\right)$:

$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right)J_m\left(\frac{z_{mn'}}{R}r\right)rdr = \frac{R^2}{2}\left(J_{m+1}\left(z_{mn}\right)\right)^2\delta_{nn'}$$

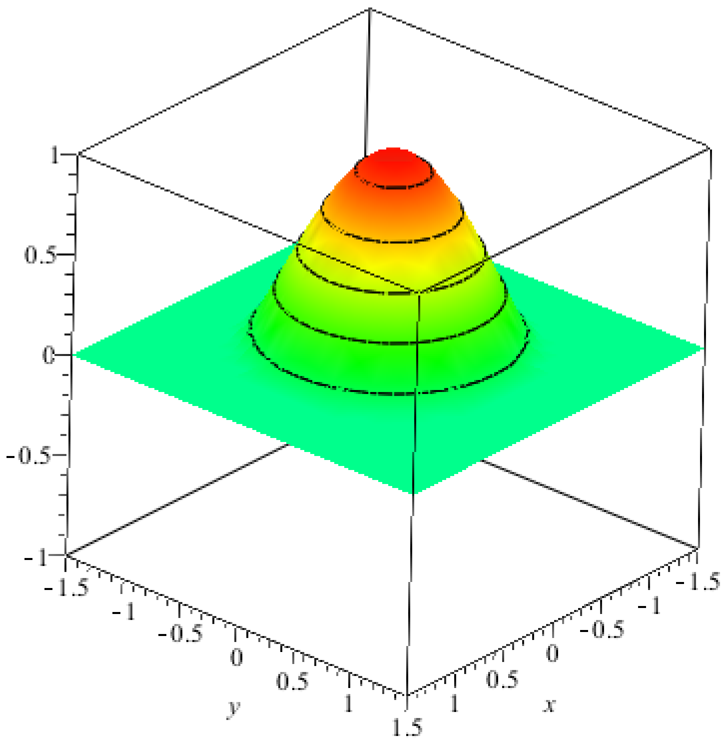
Returning to differential equation for radial function:

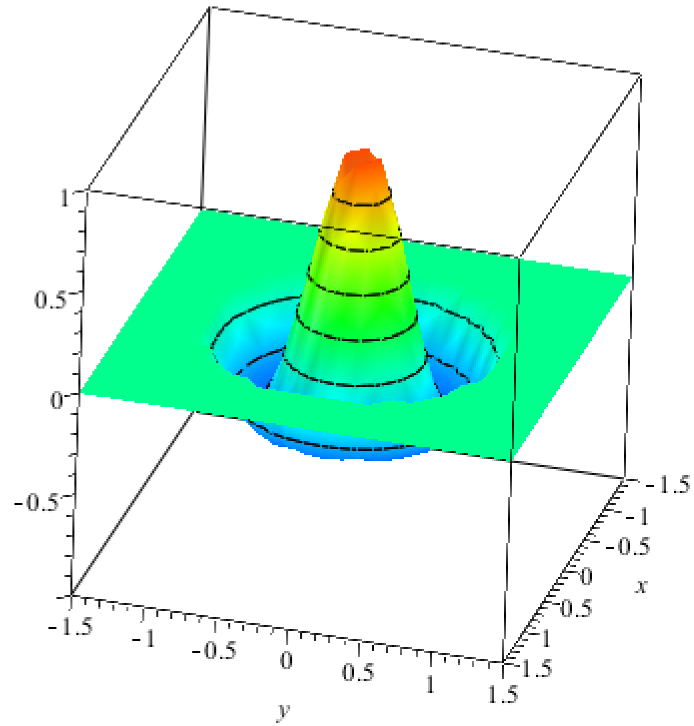
$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$$\Rightarrow \;\; f_{mn}(r) = AJ_m\left(\frac{z_{mn}}{R}r\right); \qquad k_{mn} = \frac{z_{mn}}{R}$$

$$\rho_{01}(r,\varphi) = f_{01}(r) = AJ_0\left(\frac{z_{01}}{R}r\right)$$

$$\rho_{02}(r,\varphi) = f_{02}(r) = AJ_0\left(\frac{z_{02}}{R}r\right)$$



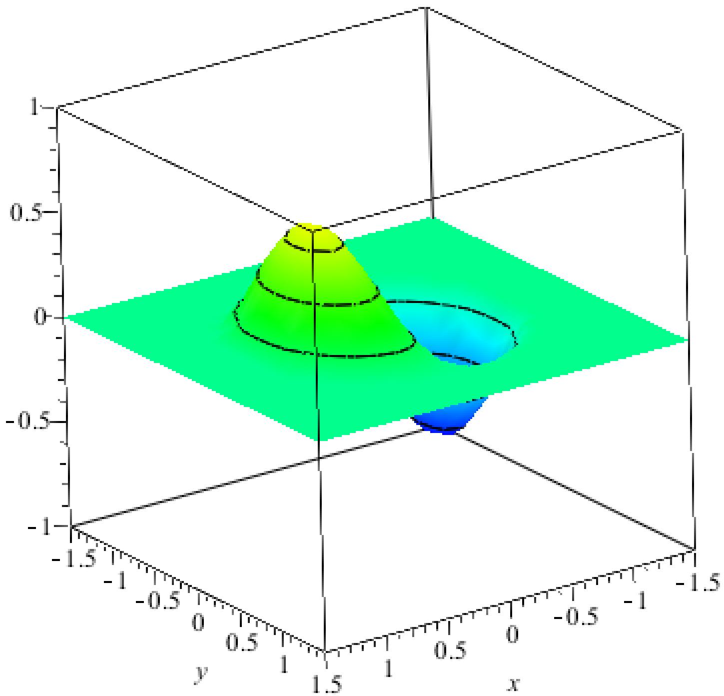


$$k_{01} = \frac{2.406}{R}$$

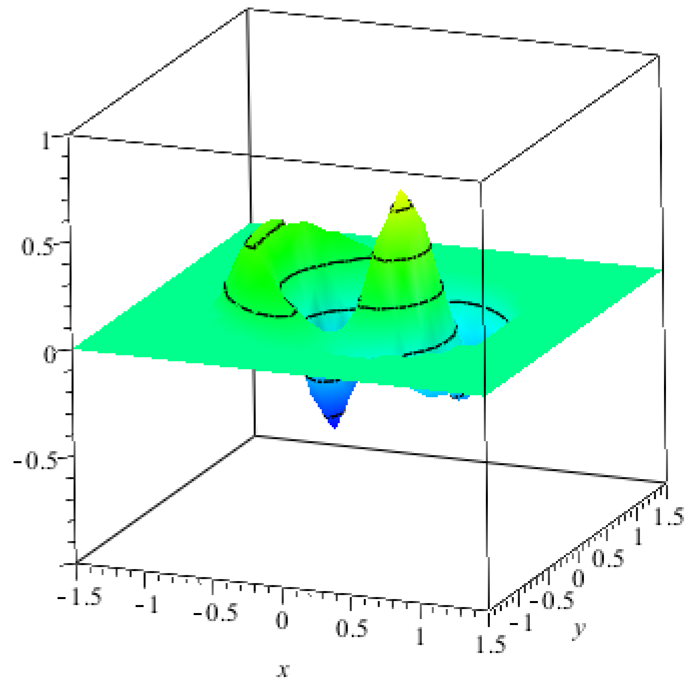
$$k_{02} = \frac{5.520}{R}$$

$$\rho_{11}(r,\varphi) = f_{11}(r)\cos(\varphi)$$
$$= AJ_1\left(\frac{z_{11}}{R}r\right)\cos(\varphi)$$

$$\rho_{12}(r,\varphi) = f_{12}(r)\cos(\varphi)$$
$$= AJ_1\left(\frac{z_{12}}{R}r\right)\cos(\varphi)$$



$$k_{11} = \frac{3.832}{R}$$



$$k_{12} = \frac{7.016}{R}$$

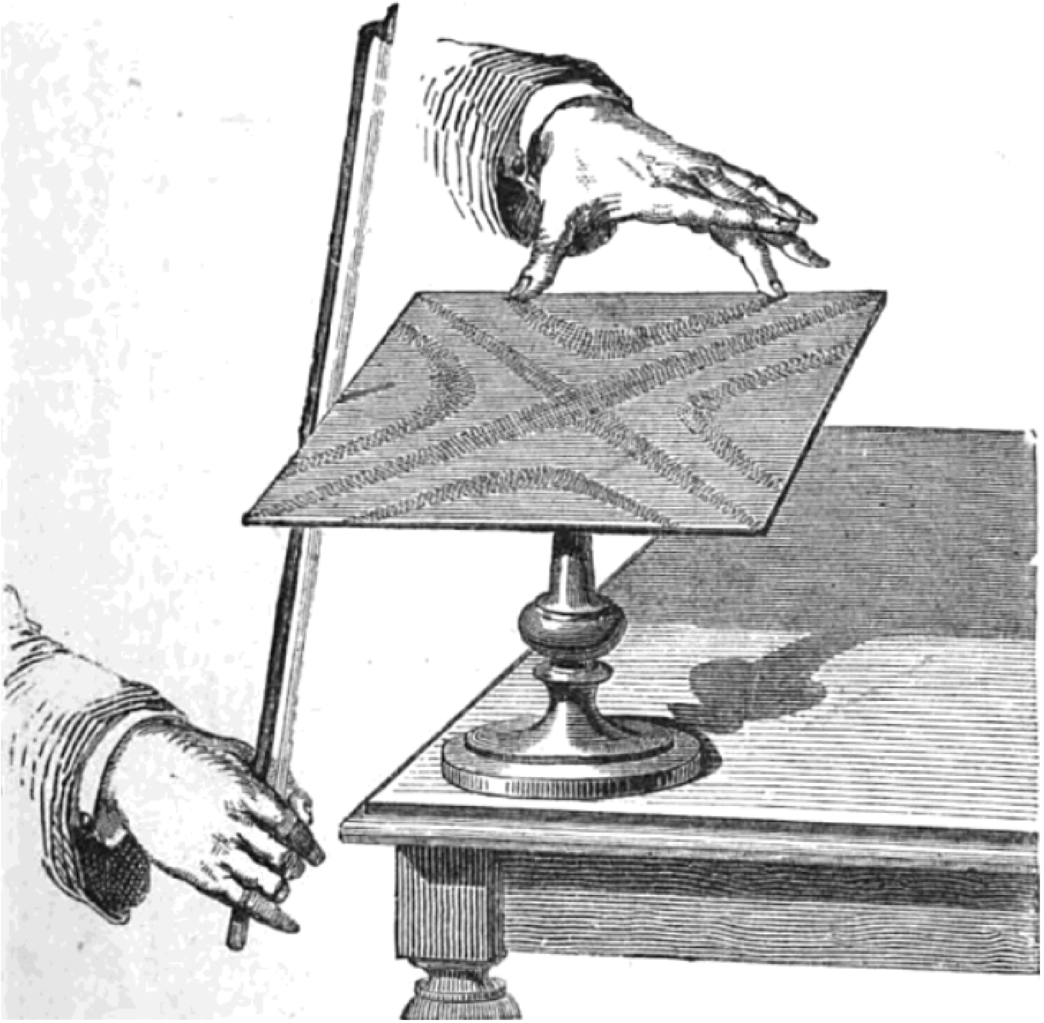
Ernst Chladni

Ernst Chladni

Born	30 November 1756 Wittenberg, Electorate of Saxony in the Holy Roman Empire
Died	3 April 1827 (aged 70) Breslau, Province of Silesia in the Kingdom of Prussia, a part of the German Confederation
Nationality	German
Known for	Study of acoustics Chladni plates and figures Estimating the speed of sound Chladni's law Theory of meteorites' origins

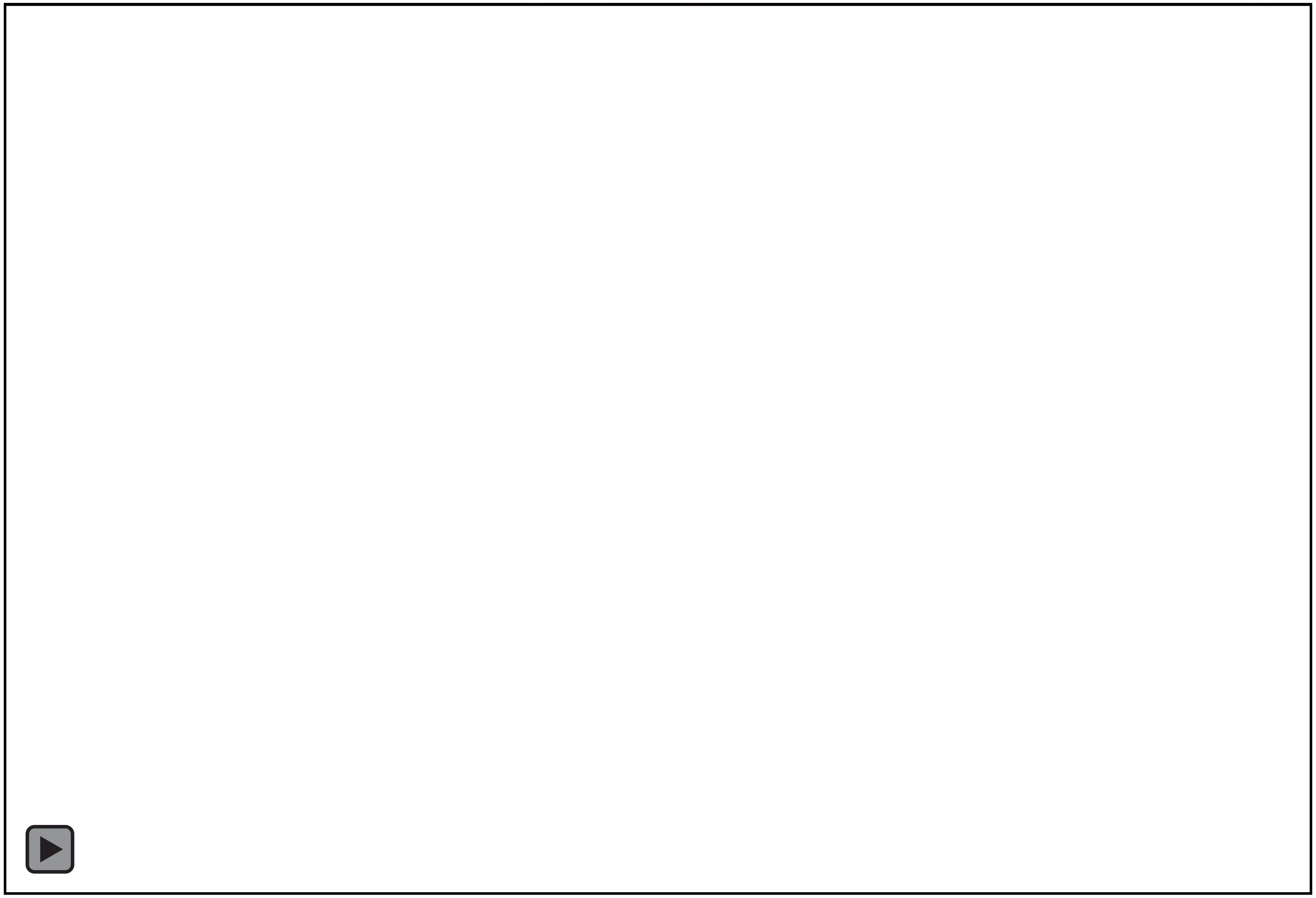
Scientific career

Fields	Physics

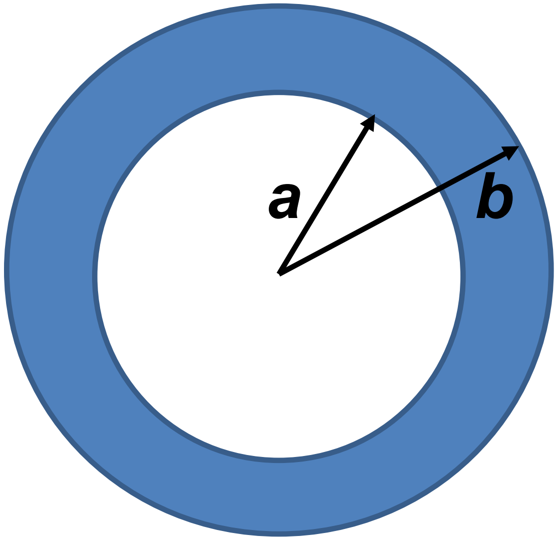


Demonstration with motor in the middle – (PASCO)

http://www.physics.wfu.edu/resources/education-resources/demo-videos/waves/



More complicated geometry – annular membrane



In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

Assume: $\rho(r,\varphi) = f(r)\Phi(\varphi)$

Let: $\Phi(\varphi) = e^{im\varphi}$

Note: $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$\Rightarrow m = \text{integer}$

Consider circular boundary -- continued

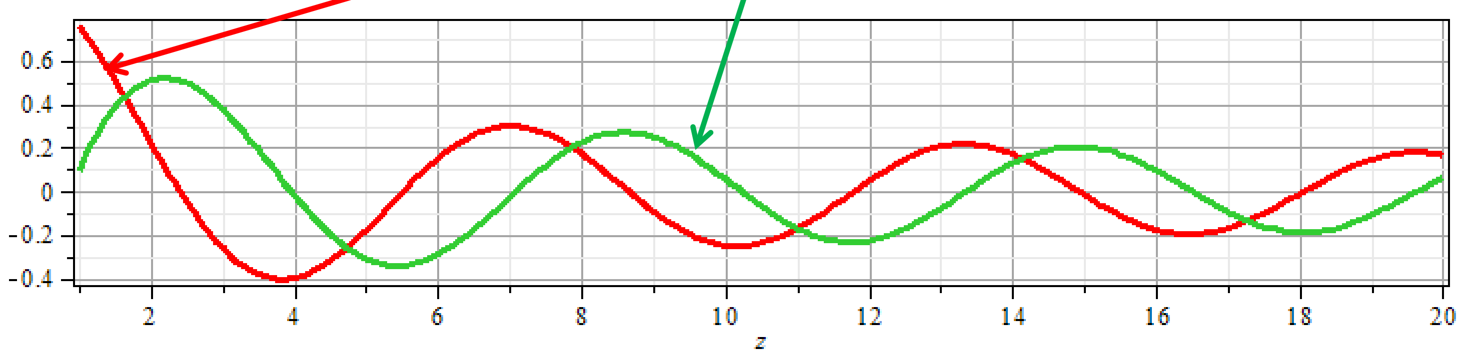
Differential equation for radial function :

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

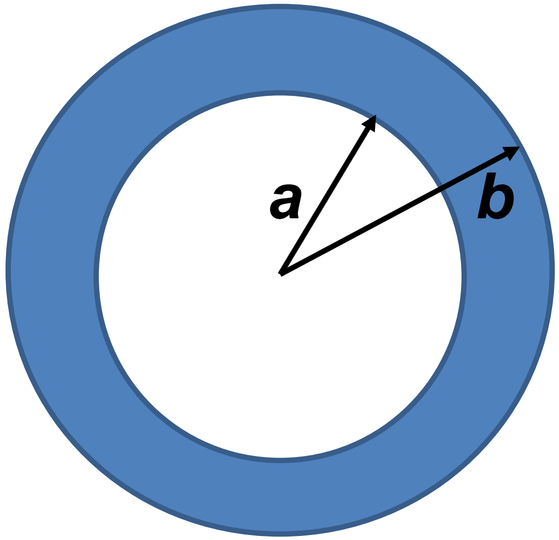
$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$



Normal modes of an annular membrane -- continued

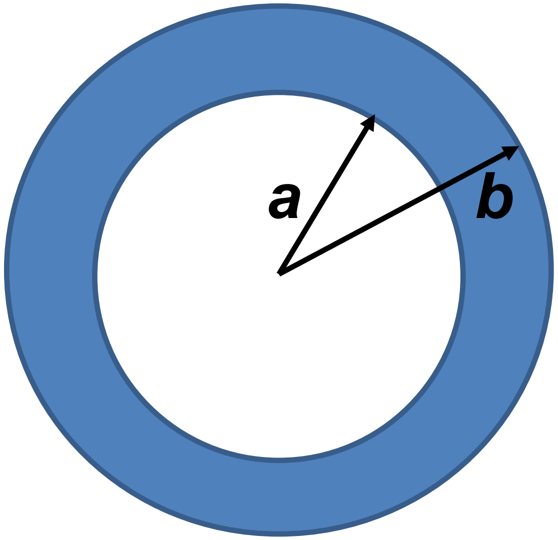


Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

General form of radial function: $f(r)=AJ_m(kr)+BN_m(kr)$

Normal modes of an annular membrane -- continued



Boundary conditions:

$f(a) = 0 \qquad f(b) = 0$

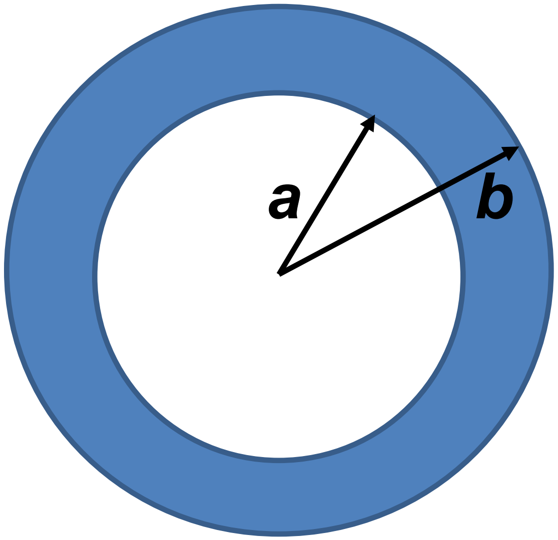
$$AJ_m(ka) + BN_m(ka) = 0$$

$$AJ_m(kb) + BN_m(kb) = 0$$

$\Rightarrow$ 2 equations and 2 unknowns -- k and $\frac{B}{A}$

$$\frac{B}{A} = \frac{-J_m(ka)}{N_m(ka)} = \frac{-J_m(kb)}{N_m(kb)} \quad \text{(transcendental equation for } k\text{)}$$

Normal modes of an annular membrane -- continued



Boundary conditions:

$$f(a) = 0 \qquad\qquad f(b) = 0$$

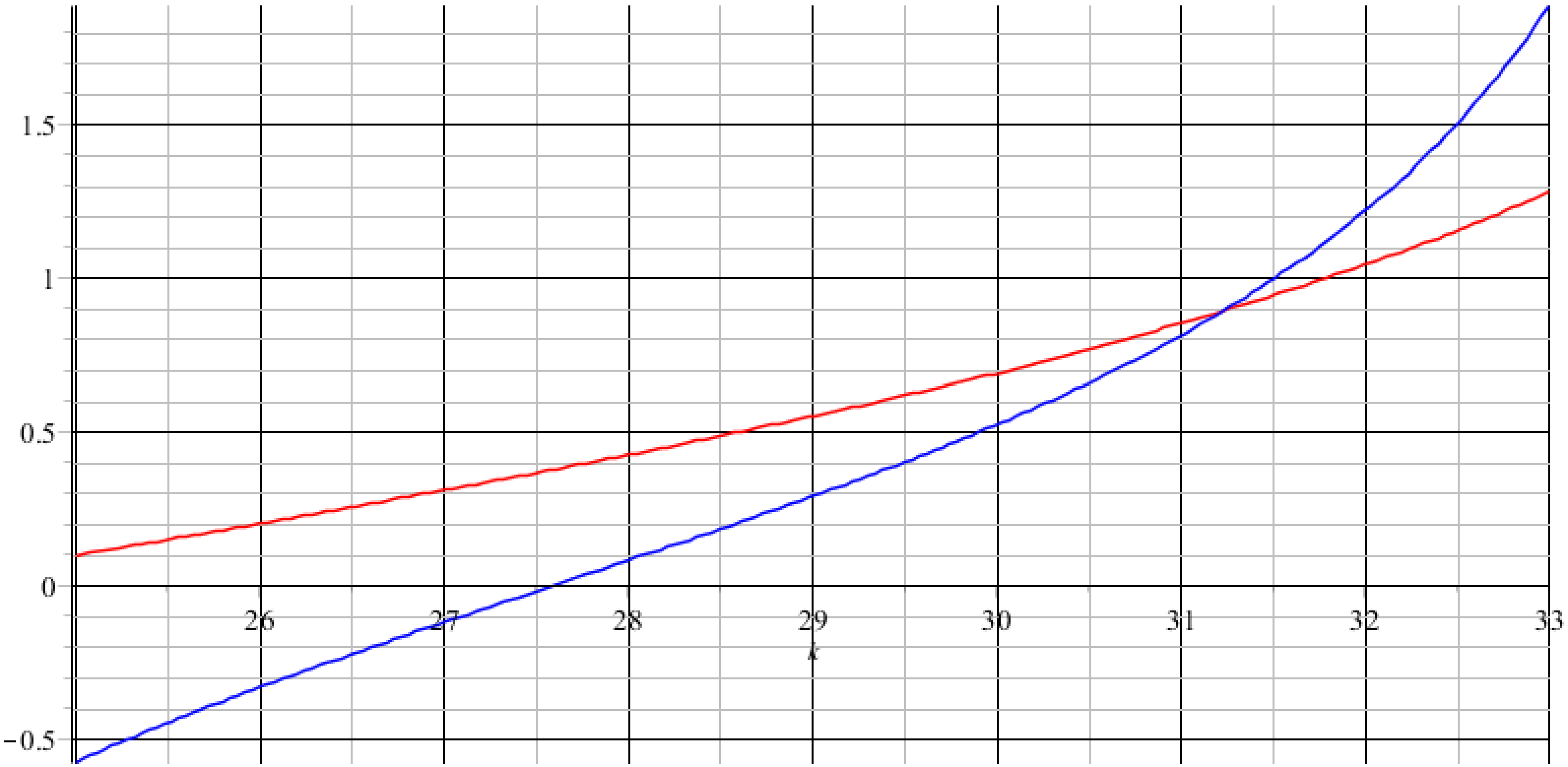
$$\frac{B}{A} = \frac{-J_m(ka)}{N_m(ka)} = \frac{-J_m(kb)}{N_m(kb)}$$ -- in terms of solution k_{mn} :

$$f(r) = A\left(J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r) \right)$$

Analysis for *m=0* and a=0.1, b=0.2:

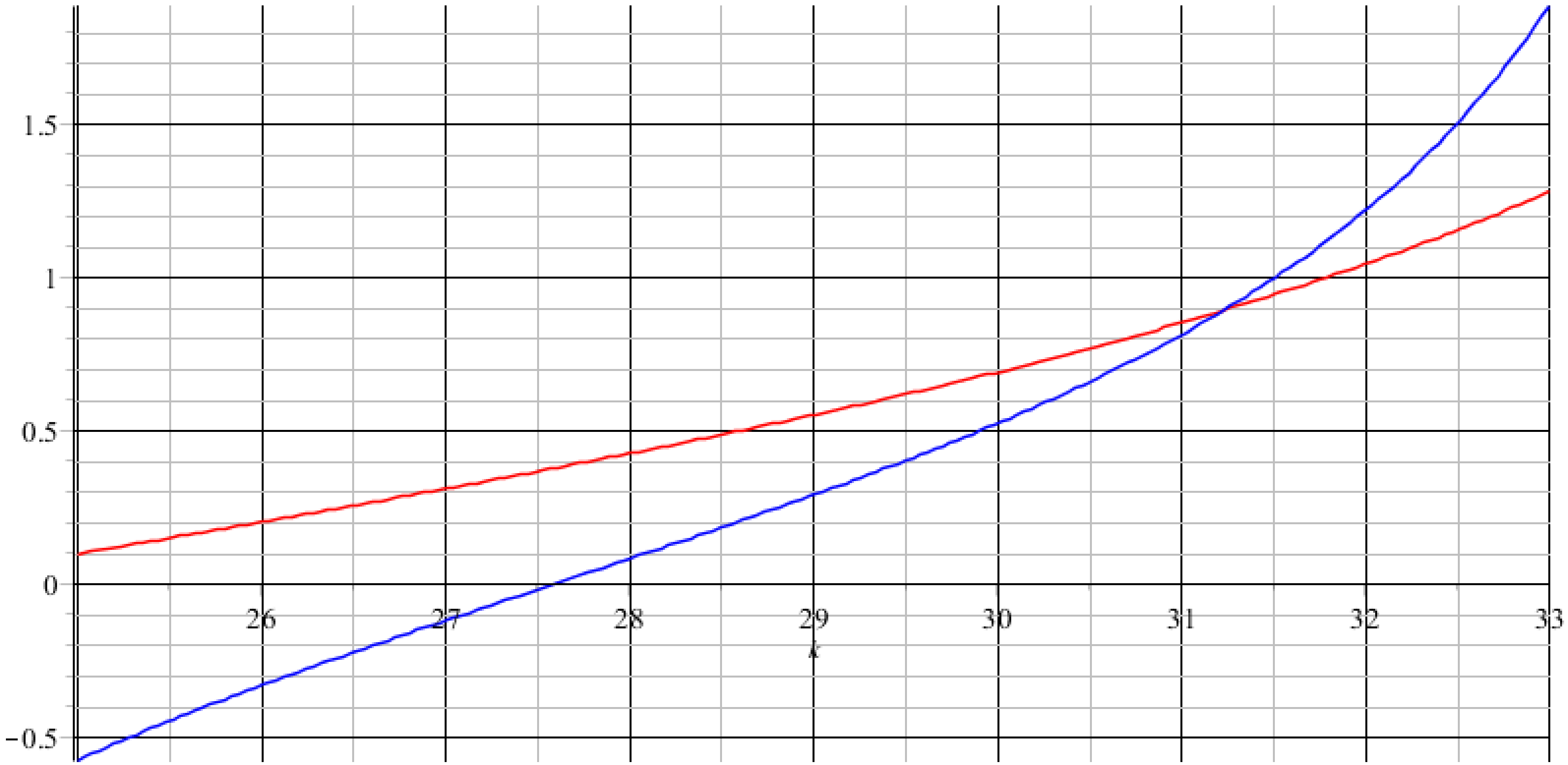
=

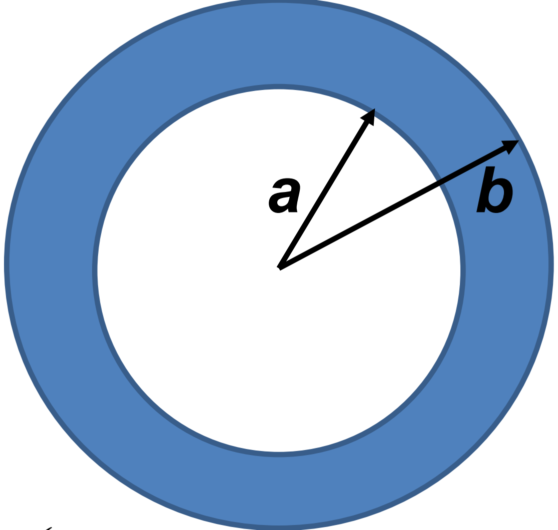
$$> \ plot\left(\left\{\frac{-\text{BesselJ}(0, 0.1\cdot k)}{\text{BesselY}(0, 0.1\cdot k)}, \frac{-\text{BesselJ}(0, 0.2\cdot k)}{\text{BesselY}(0, 0.2\cdot k)}\right\}, k = 25..33, color = [red, blue]\right);$$



> $fsolve\left(\frac{-\text{BesselJ}(0, 0.1\cdot k)}{\text{BesselY}(0, 0.1\cdot k)} = \frac{-\text{BesselJ}(0, 0.2\cdot k)}{\text{BesselY}(0, 0.2\cdot k)}, k, 30..33\right);$

31.23030920





$$f(r) = A\left(J_m(k_{mn}r) - \frac{J_m(k_{mn}a)}{N_m(k_{mn}a)} N_m(k_{mn}r) \right) \qquad k_{01} = 31.230309$$