PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 28 – Chap. 9 in F & W

Introduction to hydrodynamics

1. Motivation for topic
2. Newton's laws for fluids
3. Conservation relations

	Fri, 10/14/2022	No class	Fall break		
22	Mon, 10/17/2022	Chap. 7	Green's function methods for one-dimensional Sturm-Liouville equations	#16	10/19/2022
23	Wed, 10/19/2022	Chap. 7	Fourier and other transform methods	#17	10/21/2022
24	Fri, 10/21/2022	Chap. 7	Complex variables and contour integration	#18	10/24/2022
25	Mon, 10/24/2022	Chap. 5	Rigid body motion	#19	10/26/2022
26	Wed, 10/26/2022	Chap. 5	Rigid body motion	#20	10/28/2022
27	Fri, 10/28/2022	Chap. 8	Elastic two-dimensional membranes		
28	Mon, 10/31/2022	Chap. 9	Mechanics of 3 dimensional fluids	#21	11/02/2022
29	Wed, 11/02/2022	Chap. 9	Mechanics of 3 dimensional fluids		

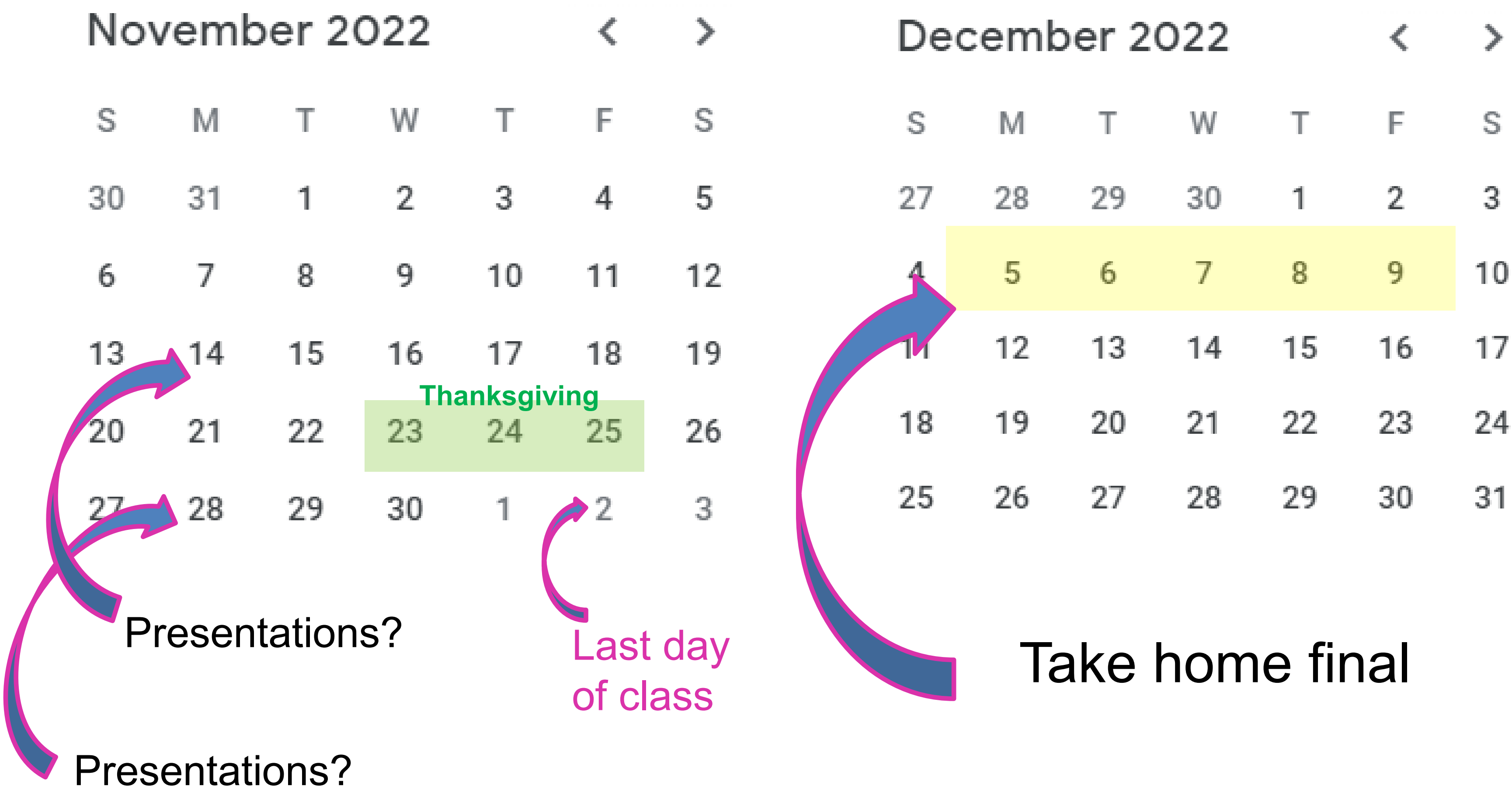
PHY 711 -- Assignment #21

Oct. 31, 2022

Start reading Chapter 9 in **Fetter & Walecka**.

1. Approximate the ocean as an incompressible fluid and ignore effects of fluid motion to estimate the pressure difference at a height of 100 meters below the sea relative to the pressure at the sea surface. Please mention the density of sea water you assume for your estimate.

Now is a good time to start thinking about your projects --



Presentation expectations

- Prepare with powerpoint (or equivalent) for ~ 10 minutes and expect ~ 5 minutes for discussion (3 presentations per day)
- To accommodate all students, we will need 3 days….
- Details listed on webpage
  http://users.wfu.edu/natalie/f22phy711/info/computational.html

Project

The purpose of this assignment is to provide an opportunity for you to study a topic of your choice in greater depth. The general guideline for your choice of project is that it should have something to do with classical mechanics, and there should be some degree of of analytic or numerical computation associated with the project. The completed project will include a short write-up and a presentation to the class. You may design your own project or use one of the following list (which will be updated throughout the term).

- Explain the details of a homework problem that was assigned or one you design, including the basic principles and the solution methods and results.
- Consider a scattering experiment in which you specify the spherically symetric interaction potential V(r). Write a computer program (using your favorite language) to evaluate the scattering cross section for your system. (Depending on your choice, you may wish to present your results either in the the center-of-mass or lab frames of reference.)
- Consider the Foucoult Pendulum. Analyze the equations of motion including both the horizontal and vertical motions. You can either solve the equations exactly or use perturbation theory. Compare the effects of the vertical motion to the effects of air friction.
- Consider a model system of 2 or more interacting particles with appropriate initial conditions, using numerical methods to find out how the system evolves in time and space. For few particles and special initial conditions this approach can be used to explore orbital mechanics. For many particles and random initial conditions, this approach can be used to explore statistical mechanics via molecular dynamics simulations.
- Examine the normal modes of vibration for a model system with 3 or more masses in 2 or 3 dimensions.
- Analyze the soliton equations beyond what was covered in class.

Hydrodynamic analysis

Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

Plan

1. Newton's laws for fluids (leaving out dissipative effects for now)
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

Newton's equations for fluids

Use Euler formulation; following "particles" of fluid

$$\text{Variables:} \quad \text{Density} \quad \rho(x,y,z,t)$$

$$\text{Pressure} \quad p(x,y,z,t)$$

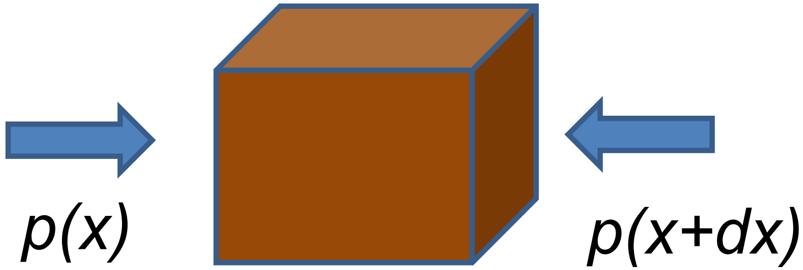
$$\text{Velocity} \quad \mathbf{v}(x,y,z,t)$$

$$m\mathbf{a} = \mathbf{F}$$

$$m \rightarrow \rho dV$$

$$\mathbf{a} \rightarrow \frac{d\mathbf{v}}{dt}$$

$$\mathbf{F} \rightarrow \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$



$$F_{pressure}\Big|_x = \big(-p(x+dx,y,z)+p(x,y,z)\big)dydz$$

$$=\frac{\big(-p(x+dx,y,z)+p(x,y,z)\big)}{dx}dxdydz$$

$$=-\frac{\partial p}{\partial x}dV$$

Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\,\rho dV - (\nabla p) dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

$$m = \rho dV$$

$$\mathbf{f}_{\text{applied}} = \frac{\mathbf{F}_{\text{applied}}}{m}$$

$$\mathbf{F}_{\text{pressure}} = -\nabla p dV$$

Detailed analysis of acceleration term:

$$\mathbf{v} = \mathbf{v}\left(x, y, z, t\right)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \left(\mathbf{v}\cdot\nabla\right)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Note that : $\mathbf{v} \equiv v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$

$$\frac{\partial \mathbf{v}}{\partial x}v_x + \frac{\partial \mathbf{v}}{\partial y}v_y + \frac{\partial \mathbf{v}}{\partial z}v_z = \nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right)$$

Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho\left(\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{applied} - \nabla p$$

$$\rho\left(\nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{applied} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Detail – What is irrotational flow?

Irrotational flow: $\nabla \times \mathbf{v} = 0$

$$\nabla \times \mathbf{v} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Which of the following vector functions have zero curl?

a. $\mathbf{v} = C\hat{\mathbf{x}}$ (C is a constant)

b. $\mathbf{v} = Cx\hat{\mathbf{x}}$

c. $\mathbf{v} = Cy\hat{\mathbf{x}}$

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v} \times \left(\nabla \times \mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla \times \mathbf{v}\right) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$ Φ is "velocity potential"

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho = \text{(constant)}$ incompressible fluid

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial \Phi}{\partial t}\right) = 0$$

Bernoulli's integral of Euler's equation for irrotational and incompressible fluid

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\quad \mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\big(\Phi(\mathbf{r},t)+C'(t)\big)$

$$\Rightarrow \frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0 \qquad \text{Bernoulli's theorem}$$

Examples of Bernoulli's theorem

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial \Phi}{\partial t} = 0$$

$$\text{Modified form; assuming } \frac{\partial \Phi}{\partial t} = 0$$

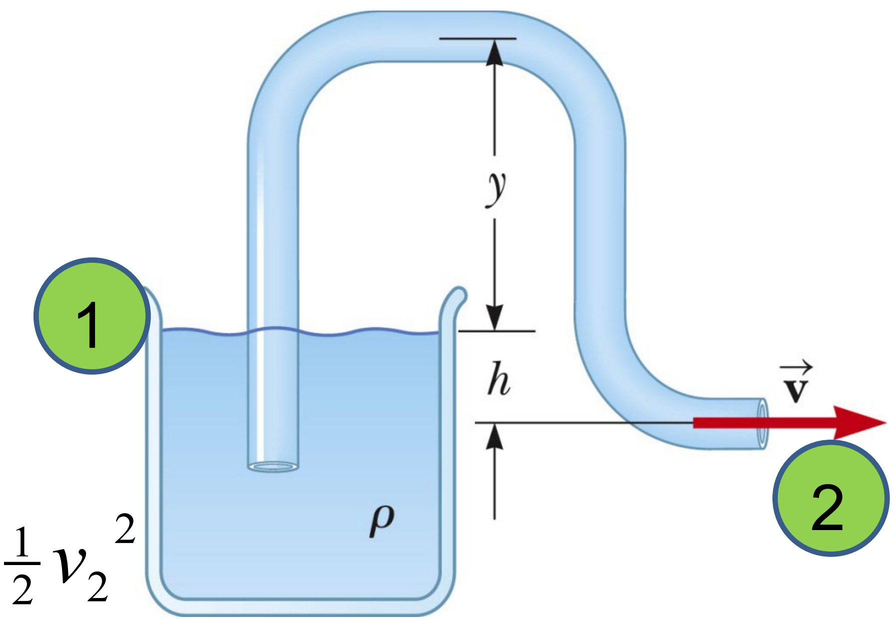
$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$

$$p_1 = p_2 = p_{atm}$$

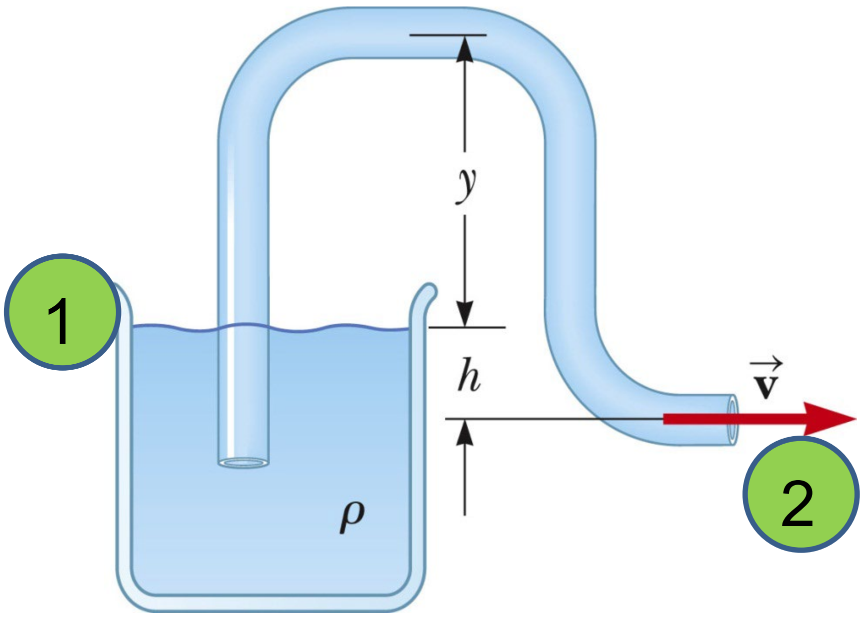
$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}{v_1}^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}{v_2}^2$$



Examples of Bernoulli's theorem -- continued



$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

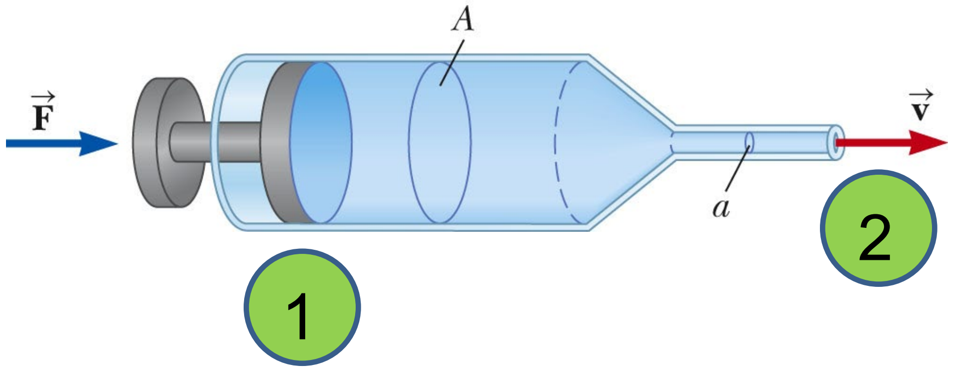
$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2} v_2^2$$

$$v_2 \approx \sqrt{2gh}$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2=\text{constant}$$



$$p_1=\frac{F}{A}+p_{atm} \qquad\qquad p_2=p_{atm}$$

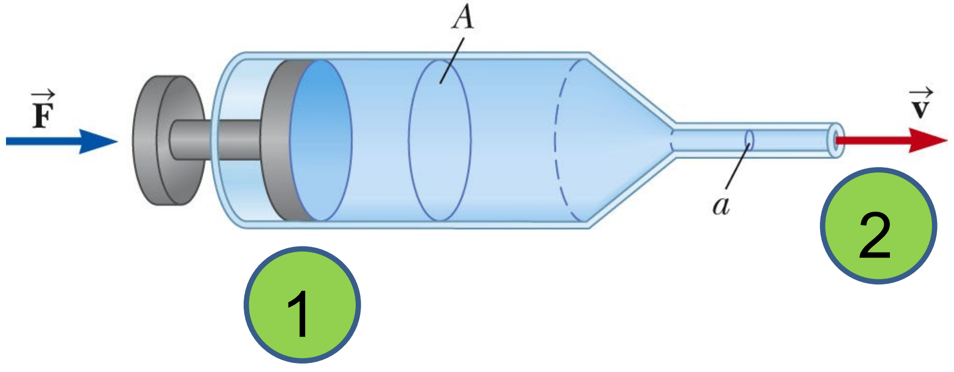
$$U_1=U_2$$

$$v_1A=v_2a \qquad \text{continuity equation}$$

$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}v_1^{\,2}=\frac{p_2}{\rho}+U_2+\tfrac{1}{2}v_2^{\,2}$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$\frac{2F}{A} = v_2^{\,2}\left(1 - \left(\frac{a}{A}\right)^2\right)$$

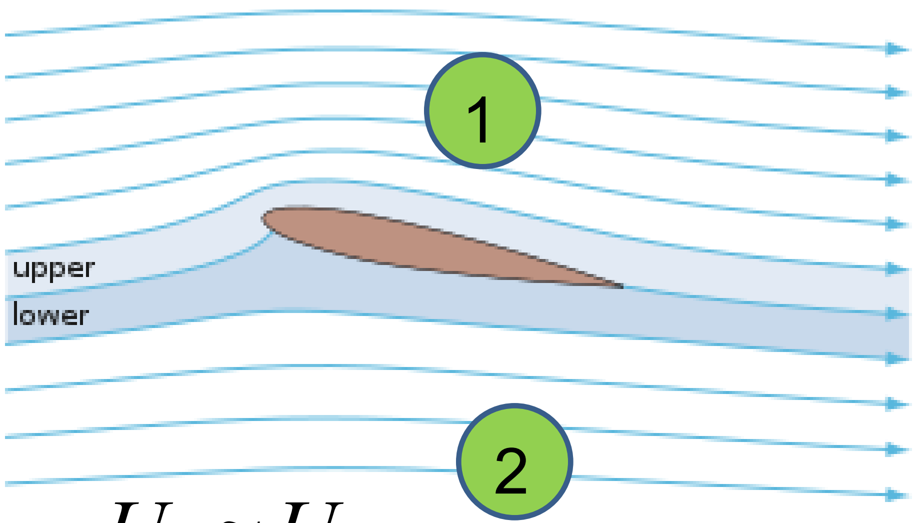
$$v_2 = \sqrt{\frac{2F/A}{1 - \left(\frac{a}{A}\right)^2}}$$

Examples of Bernoulli's theorem – continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2} v_1^{\,2} = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2} v_2^{\,2}$$

$$p_2 - p_1 = \tfrac{1}{2}\rho\left(v_1^{\,2} - v_2^{\,2}\right)$$

Your question -- What aspects do over simplified Bernoulli's equation not include in studying fluid dynamics?

According to a Scientific American article, the conclusion that $v_2 > v_1$ because of the shape of the airplane wing is not quite true. Numerical modeling reveal a more complicated picture.

https://www.scientificamerican.com/article/no-one-can-explain-why-planes-stay-in-the-air/

At NASA Ames Fluid Mechanics Laboratory, streamlines of dye in a water channel interact with a model airplane. Credit: *Ian Allen* (copied from Scientific American page mentioned above).

Continuity equation connecting fluid density and velocity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

Consider: $$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \mathbf{v}$$

$$\Rightarrow \frac{d\rho}{dt} + \rho (\nabla \cdot \mathbf{v}) = 0$$

alternative form

of continuity equation

Some details on the velocity potential

Continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

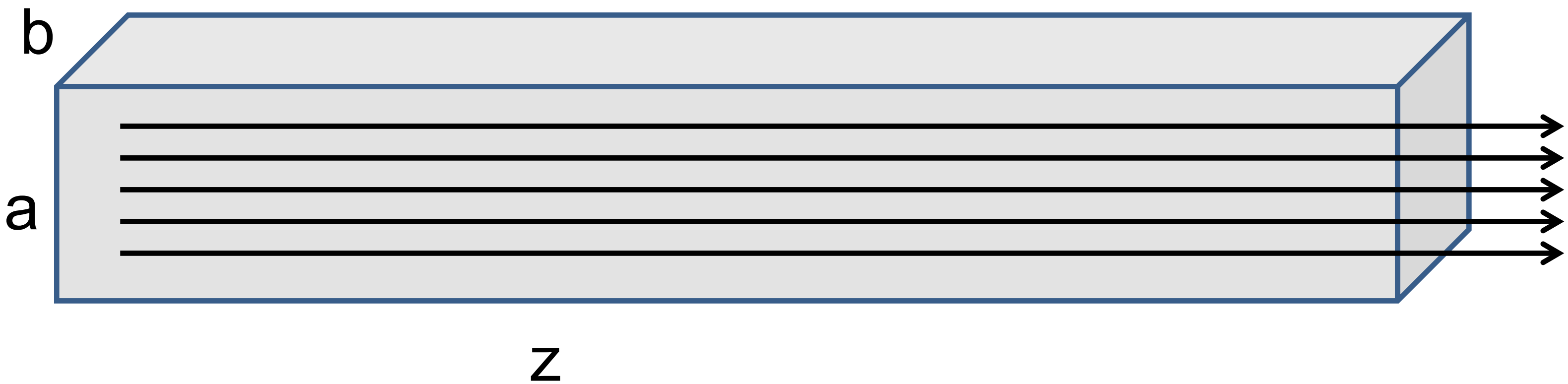
For incompressible fluid : $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

Irrotational flow : $\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \mathbf{v} = -\nabla \Phi$

$$\Rightarrow \nabla^2 \Phi = 0$$

Example – uniform flow



$$\nabla^2\Phi = 0$$

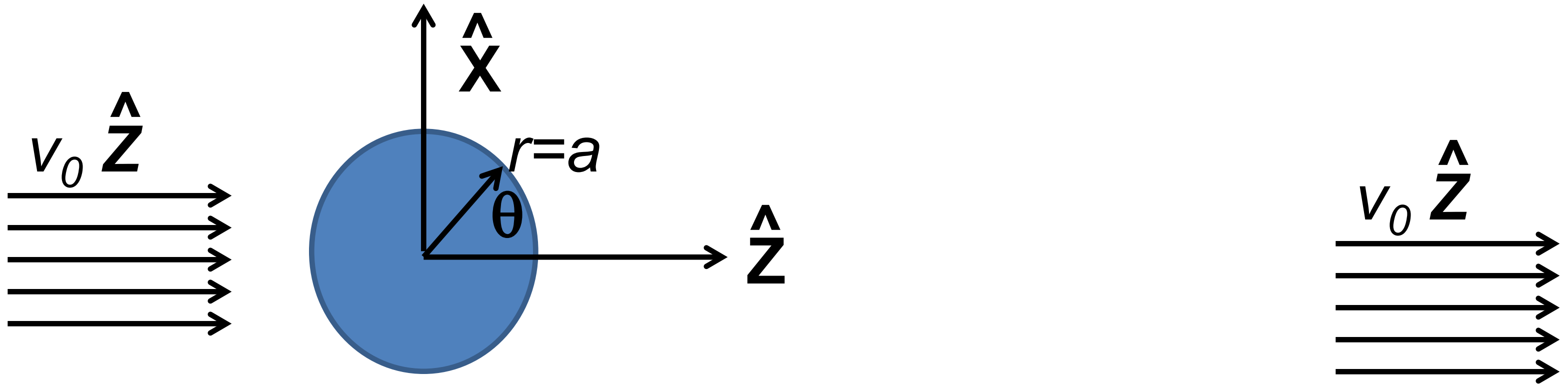
$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla\Phi = v_o \hat{\mathbf{z}}$$

Example – flow around a long cylinder (oriented in the *Y* direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

($r,\theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r \to \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \to \infty) = -v_0 \qquad \Rightarrow \Phi(r \to \infty, \theta) = -v_0 r \cos\theta$$

Note that : $\dfrac{\partial^2 \cos\theta}{\partial \theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f}{\partial r}-\frac{1}{r^2}f=0$$

$$f(r)=Ar+\frac{B}{r}$$ where A, B are constants

Boundary condition on cylinder surface:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a}=0$$

$$\frac{df}{dr}(r=a)=0=A-\frac{B}{a^2}$$

$$\Rightarrow B=Aa^2$$

Boundary condition at ∞: $\Rightarrow A=-v_0$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$v_r = -\frac{\partial \Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial \varphi^2}$$

to be continued …