PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Discussion on Lecture 29 -- Chap. 9 in F & W

Introduction to hydrodynamics

1. Newton's laws for fluids and the continuity equation
2. Irrotational and incompressible fluids
3. Irrotational and isentropic fluids
4. Approximate solutions in the linear limit – next time

Physics Colloquium

Thursday

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November 3, 2022

"What Do You Want to Be When You Grow Up?" How to Leverage Your Experiences for a Career in Biomedical Engineering and Technologies

In this talk, I will provide background and strategies for how students can utilize their skill sets to be attractive for landing a career in biomedical engineering, and being able to transition between academic and corporate enterprises. If you have a technical background and interested in learning about how to translate your skills into the biomedical workforce, this talk is for YOU!!! I will utilize examples from my experiences running a research lab in academia, working on problems pertaining to thrombosis due to disease, as well as experiences from my career in consulting in biomechanics, working with Fortune 500 companies to help them solve problems

Rodney Averett

Exponet, INC

4:00 pm - Olin 101*

*Link provided for those unable to attend in person.

Note: For additional information on the seminar or to obtain the video conference link, contact wfuphys@wfu.edu

Reception at 3:30pm - Olin Entrance

	Fri, 10/14/2022	No class	Fall break		
22	Mon, 10/17/2022	Chap. 7	Green's function methods	#16	10/19/2022
23	Wed, 10/19/2022	Chap. 7	Fourier and other transform methods	#17	10/21/2022
24	Fri, 10/21/2022	Chap. 7	Complex variables and contour integration	#18	10/24/2022
25	Mon, 10/24/2022	Chap. 5	Rigid body motion	#19	10/26/2022
26	Wed, 10/26/2022	Chap. 5	Rigid body motion	#20	10/28/2022
27	Fri, 10/28/2022	Chap. 8	Elastic two-dimensional membranes		
28	Mon, 10/31/2022	Chap. 9	Mechanics of 3 dimensional fluids	#21	11/02/2022
29	Wed, 11/02/2022	Chap. 9	Mechanics of 3 dimensional fluids	#22	11/04/2022
30	Fri, 11/04/2022	Chap. 9	Linearized hydrodynamics equations		
31	Mon, 11/07/2022	Chap. 9	Linear sound waves		
32	Wed, 11/09/2022	Chap. 9	Sound sources and scattering		
33	Fri, 11/11/2022	Chap. 9	Non linear effects in sound waves and shocks		
34	Mon, 11/14/2022	Chap. 10	Surface waves in fluids		
35	Wed, 11/16/2022	Chap. 10	Surface waves in fluids; soliton solutions		
36	Fri, 11/18/2022	Chap. 11 or 12	Heat conduction or Viscous effects on hydrodynamics		
37	Mon, 11/21/2022	Chap 1-12	Review		
	Wed, 11/23/2022		Thanksgiving Holiday		
	Fri, 11/25/2022		Thanksgiving Holiday		
	Mon, 11/28/2022		Presentations I		
	Wed, 11/30/2022		Presentations II		
	Fri, 12/02/2022		Presentations III		

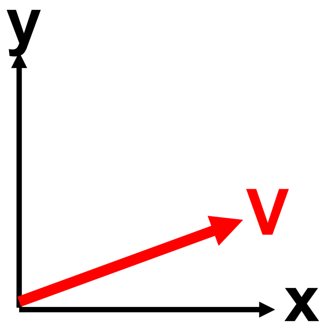
PHY 711 -- Assignment #22

Nov. 02, 2022

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Consider the example discussed in Lecture 29, concerning the flow of an incompressible fluid in the **z** direction in the presence of a stationary cylindrical log oriented in the **y** direction. For this homework problem, the log is replaced by a stationary sphere. Find the velocity potential for this case, using the center of the sphere as the origin of the coordinate system and spherical polar coordinates.

Comment about HW #20 – Euler angles can be use in different ways:



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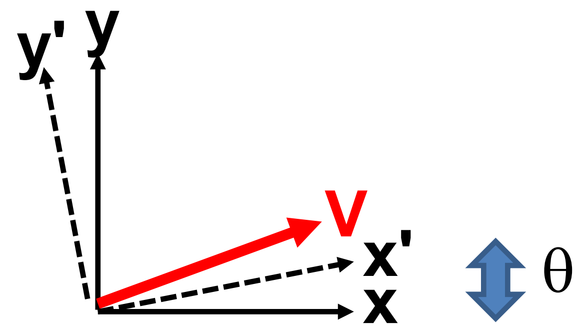

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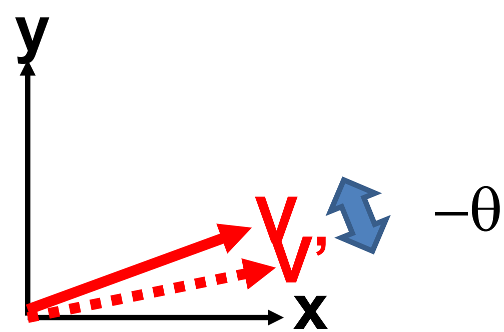


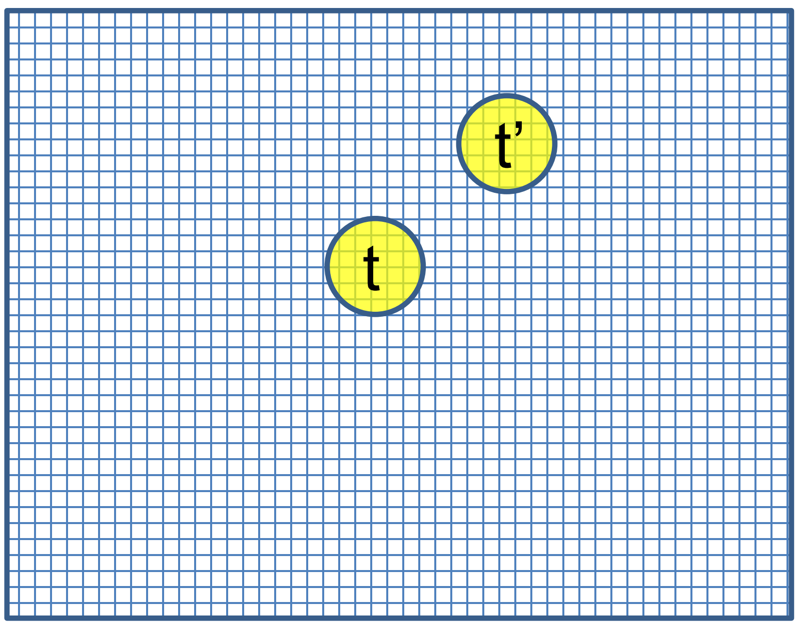
Newton's equations for fluids

Use **Euler** formulation; properties described in terms of stationary spatial grid

$$\text{Variables:} \quad \text{Density} \quad \rho(x,y,z,t)$$

$$\text{Pressure} \quad p(x,y,z,t)$$

$$\text{Velocity} \quad \mathbf{v}(x,y,z,t)$$



$$\text{Particle at } t: \quad \mathbf{r}, t$$

$$\text{Particle at } t': \quad \mathbf{r} + \mathbf{v}\,\delta t, t'$$

$$t' = t + \delta t$$

Euler analysis -- continued

Particle at t : $\quad \mathbf{r}, t$

Particle at t': $\quad \mathbf{r}+\mathbf{v}\delta t, t' \qquad$ where $\quad \delta t = t' - t$

For $f(\mathbf{r}, t)$:

$$\frac{df}{dt} = \lim_{\delta t \rightarrow 0} \left(\frac{f(\mathbf{r}, t') - f(\mathbf{r}, t)}{\delta t} + \frac{f(\mathbf{r}+\mathbf{v}\delta t, t) - f(\mathbf{r}, t)}{\delta t} \right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f$$

It can be shown that: $(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla\left(\frac{1}{2}v^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v})$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f$$

For $f \rightarrow v_x$ $$\frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + (\mathbf{v} \cdot \nabla) v_x$$

For $f \rightarrow v_y$ $$\frac{dv_y}{dt} = \frac{\partial v_y}{\partial t} + (\mathbf{v} \cdot \nabla) v_y$$

For $f \rightarrow v_z$ $$\frac{dv_z}{dt} = \frac{\partial v_z}{\partial t} + (\mathbf{v} \cdot \nabla) v_z$$

In vector form $$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v}$$

Note that $$(\mathbf{v} \cdot \nabla) \mathbf{v} = \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) \left(v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}} \right)$$

$\mathbf{v} = \mathbf{v}(x, y, z, t)$

In vector form $\dfrac{d\mathbf{v}}{dt} = \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}$

Note that $(\mathbf{v} \cdot \nabla)\mathbf{v} = \left(v_x \dfrac{\partial}{\partial x} + v_y \dfrac{\partial}{\partial y} + v_z \dfrac{\partial}{\partial z} \right)\left(v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}} \right)$

$$= \frac{1}{2} \nabla |\mathbf{v}|^2 - \mathbf{v} \times (\nabla \times \mathbf{v})$$

For example, applying this analysis to Newton's equation of motion for fluids:

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure} \qquad m = \rho dV$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\, \rho dV - (\nabla p) dV \qquad \mathbf{f}_{\text{applied}} = \frac{\mathbf{F}_{\text{applied}}}{m}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p \qquad \mathbf{F}_{\text{pressure}} = -\nabla p dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = \rho \mathbf{f}_{applied} - \nabla p$$

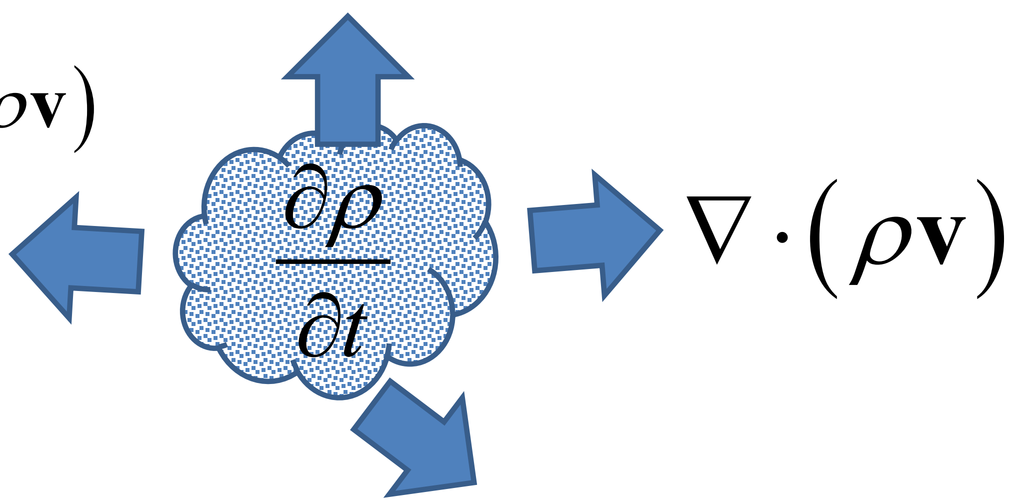
Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

The notion of the continuity is a common feature of continuous closed systems. Here we assume that there are no mechanisms for creation or destruction of the fluid.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$



Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

For incompressible fluid: $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

velocity potential

Irrotational flow: $\nabla \times \mathbf{v} = 0$ $\quad \Rightarrow \mathbf{v} = -\nabla \Phi$

For irrotational flow of an incompressible fluid: $\nabla^2 \Phi = 0$

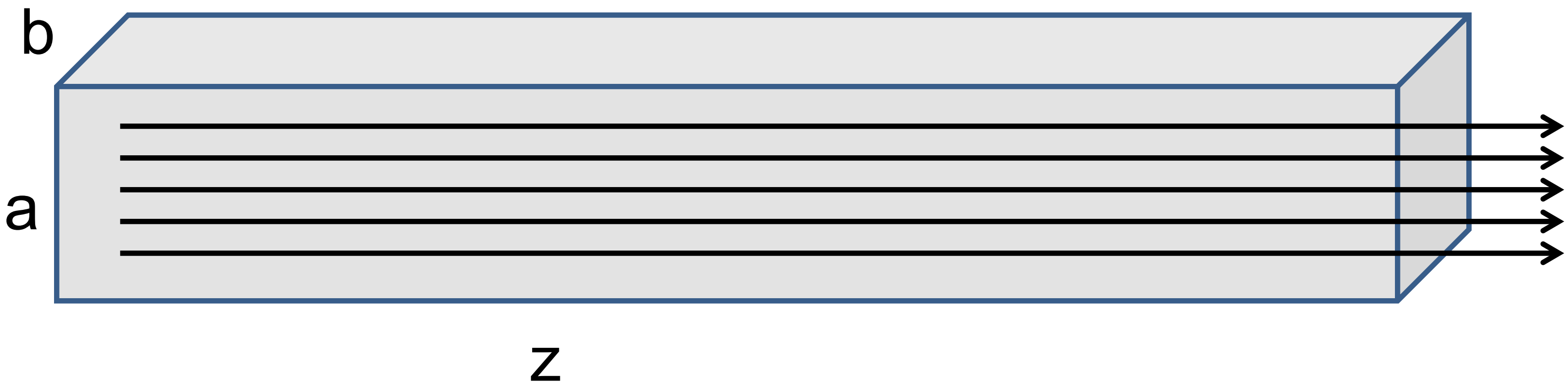
Checking --

Why does $\nabla \times \mathbf{v} = 0$ imply that $\mathbf{v} = -\nabla\Phi$?

Consider: $\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\Phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}}$

$\nabla \times \left(\nabla\Phi\right)\Big|_x = \frac{\partial^2\Phi}{\partial y \partial z} - \frac{\partial^2\Phi}{\partial z \partial y} = 0$ Similar results for other directions.

Example of irrotational flow of an incompressible fluid – uniform flow



$$\nabla^2 \Phi = 0$$

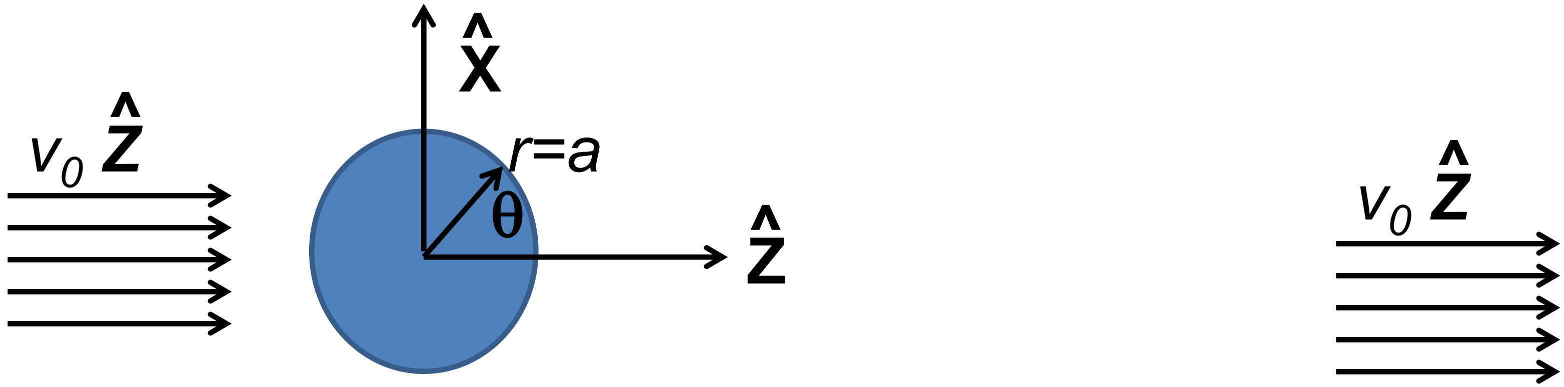
$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla \Phi = v_o \hat{\mathbf{z}}$$

Example – flow around a long cylinder (oriented in the *Y* direction)



$$\nabla^2 \Phi = 0$$

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

($r, \theta,$ defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r \to \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \to \infty) = -v_0 \qquad \Rightarrow \Phi(r \to \infty, \theta) = -v_0 r \cos\theta$$

Note that : $\dfrac{\partial^2 \cos\theta}{\partial \theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{df}{dr}(r = a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition as $r \to \infty$: $\Rightarrow A = -v_0$

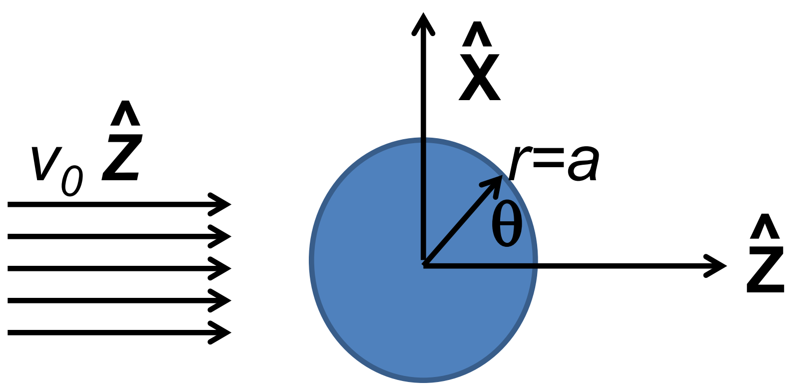
$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

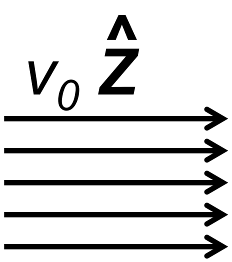
$$v_r = -\frac{\partial \Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For $r \to \infty$

$$\mathbf{v} \to v_0\cos\theta\,\hat{\mathbf{r}} - v_0\sin\theta\,\hat{\boldsymbol{\theta}} = v_0\hat{\mathbf{z}}$$





Now consider the case of your homework problem --

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Spherical system continued:

Laplacian in spherical polar coordinates:

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

In terms of spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi)$$

In our case:

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\Phi(r,\theta,\phi) = f(r)Y_{lm}(\theta,\phi)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{l(l+1)}{r^2}f = 0$$

(Continue analysis for homework)

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla\times\mathbf{v}\right) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla\Phi$$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho = (\text{constant})$ incompressible fluid

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

For incompressible fluid

Bernoulli's integral of Euler's equation for constant ρ

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\quad \mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\big(\Phi(\mathbf{r},t)+C(t)\big)$

It is convenient to modify $\Phi(\mathbf{r},t)\rightarrow\Phi(\mathbf{r},t)+\int^{t}C(t')dt'$

$$\Rightarrow\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0 \qquad \text{Bernoulli's theorem}$$

Extension of these ideas to some compressible fluids – now assuming conditions of constant entropy (no heat transfer).

Under what circumstances can there be no heat transfer?

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $\left(\nabla\times\mathbf{v}\right) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho \neq$ (constant) isentropic fluid

A little thermodynamics

Comment about sign convention

First law of thermodynamics: $dE_{\text{int}} = dQ - dW$

For isentropic conditions: $dQ = 0$

$$dE_{\text{int}} = -dW = -pdV$$

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}} = -dW = -pdV$$

In terms of mass density: $\rho = \frac{M}{V}$

For fixed M and variable V: $d\rho = -\frac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$

$$dE_{\text{int}} = Md\varepsilon = -dW = -pdV = M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

$$\text{Consider:} \quad \nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$$

$$\text{Rearranging:} \quad \nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

Is this useful?

a. Yes

b. No

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\nabla\times\mathbf{v} = 0 \qquad \mathbf{v} = -\nabla\Phi \qquad \mathbf{f}_{applied} = -\nabla U$$

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Summary of Bernoulli's results

For incompressible fluid

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

For isentropic fluid with internal energy density ε

$$\nabla\left(\varepsilon+\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Here ε is the internal energy of the fluid per unit mass. For an ideal gas fluid, it has a relatively simple form.