PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Lecture 2 – Chap 1 in F&W
Two particle interactions and scattering theory

- **Announcements – Physics Colloquium – Thurs. @ 4 PM**
- **Brief comment about "quiz"**
- **One-on-one meetings and/or office hours**
- **Your questions**
- **Systematic discussion of the concept of "scattering theory"**

4 PM Thursday in Olin 101 and by zoom
Reception and Refreshments in Olin Lobby at 3:30 PM

- **Summer research presentations by undergraduate students – Part 1**
- **Welcoming overview, announcements, and information**

4 PM Thursdays in Olin 101 and via zoom

https://www.physics.wfu.edu/seminars-2022-fall/

WFU Physics Colloquium Schedule – Fall 2022

Previous and Future Colloquia

All colloquia will be held at 4 PM in Olin 101 (unless noted otherwise). Refreshments will be served at 3:30 PM in Olin Lobby prior to each seminar. For additional information contact wfuphys@wfu.edu.

Thurs. Aug. 25, 2022 — Welcome and Summer research presentations

Thurs. Sept. 1, 2022 — Research opportunities in Physics at WFU – Part 1

Thurs. Sept. 8, 2022 — Research opportunities in Physics at WFU – Part II

Thurs. Sept. 15, 2022 —Professor Yoeri van de Burgt, Eindhoven University of Technology, The Netherlands – “Organic Neuromorphic Electronics and Biohybrid Systems” – (host: O. Jurchescu)

Comment on one-on-one meetings and office hours

- Encourage scheduled or spontaneous one-on-one or group discussions on course material, software issues, etc. etc.
- Can use office hours or other times (Open Door)

Fall 2022 Schedule
for N. A. W. Holzwarth

<table>
<tr><th></th><th>Monday</th><th>Tuesday</th><th>Wednesday</th><th>Thursday</th><th>Friday</th></tr>
<tr><td>9:00-10:00</td><td>Lecture Preparation</td><td rowspan="5">Physics Research</td><td>Lecture Preparation</td><td rowspan="4">Physics Research</td><td>Lecture Preparation</td></tr>
<tr><td>10:00-11:00</td><td>Classical Mechanics
PHY711</td><td>Classical Mechanics
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PHY711</td></tr>
<tr><td>11:00-12:00</td><td>Office Hours</td><td>Office Hours</td><td>Office Hours</td></tr>
<tr><td>12:00-4:00</td><td rowspan="2">Physics Research</td><td rowspan="2">Physics Research</td><td rowspan="2">Physics Research</td></tr>
<tr><td>4:00-5:00</td><td>Physics
Colloquium</td></tr>
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Your questions –

From Lee -- ..question regarding the "hard sphere," two particle potential. Is this similar to the subatomic (quark) potential? I tried to visualize this, and potentials for subatomic particles came to mind.

From Katie –
1. Could you further explain the math on the slide where you define V_eff(r) and talk about the special cases as well (such as when r goes to infinity)?
2. Also can you explain the differential cross section and how we determine that for different cases?
3. Can we go over the logic behind the steps in the example problem with the collision of hard spheres? I understand the math that is shown on the slide just would like a further explanation of how it was determined.

From Evan -- On slide # 11, there is a graph showing potential energy diagrams for a two particle interaction. Are the blue and green curves just examples of what could possibly be seen from such an interaction or is it an example of a more specific scenario? Also, why is it more meaningful to show V(r)/V0 on the y-axis as opposed to simply V(r)?

Questions continued –

From Sam -- ..is the r^2 (dtheta/dt)^2 term that arises in the energy of the target particle a result of rotational energy? Also, in the preclass assessment the derivative of the definite integral (1st question), you plug in the bounds of the integral to the equation for x, adding the upper bound term and subtracting the lower bound term, in addition to taking the derivative with respect to t and then integrating. I am having trouble understanding why this is a necessary step mathematically.

From Zezong -- At the end of the slide you listed several assumptions we have to make about the scattering process analysis. I wonder in real experiments, to what extent these make a difference for the results.

Comment on quiz questions

1. $g(t)=\int_0^t\left(x^2+t\right)dx$

$$\frac{dg}{dt}=\int_0^t \frac{d\left(x^2+t\right)}{dt}dx+\left(x^2+t\right)\Big|_{x=t}$$

$$=\int_0^t dx+\left(t^2+t\right)=t^2+2t$$

2. Evaluate the integral $\oint \frac{dz}{z}$ for a closed contour about the origin.

Suppose that $z=e^{i\theta}$ $\quad dz=e^{i\theta}id\theta \quad \oint\frac{dz}{z}=\int_0^{2\pi}\frac{e^{i\theta}id\theta}{e^{i\theta}}=2\pi i$

3. $\frac{df}{dx}=f \quad \Rightarrow \frac{df}{f}=dx \quad \Rightarrow d\left(\ln f\right)=dx \quad \Rightarrow f(x)=Ae^x$

$$f(x=0)=A=1 \quad \Rightarrow A=1 \quad f(x)=e^x$$

4. $\sum_{n=1}^{N}a^n=\frac{a-a^{N+1}}{1-a}$ $\quad$ Let $S\equiv\sum_{n=1}^{N}a^n$ $\quad$ Note that $\quad aS-S=a^{N+1}-a$

$$\Rightarrow S=\frac{a^{N+1}-a}{a-1}$$

Some more details on Question #1:
Note that you can integrate first and then take the derivative and you should get the same answer

$$g(t)=\int_0^t \left(x^2+t\right)dx$$

$$g(t)=\int_0^t \left(x^2+t\right)dx=\frac{t^3}{3}+t^2 \qquad \frac{dg}{dt}=t^2+2t$$

Suppose $\quad G(t)=\int_{A(t)}^{B(t)} f(x,t)dx$

Then $\quad \frac{dG}{dt}=\int_{A(t)}^{B(t)} \frac{\partial f(x,t)}{\partial t}dx+\frac{dB}{dt}f(x=B(t),t)-\frac{dA}{dt}f(x=A(t),t)$

A more systematic discussion of these and other mathematical details will be discussed throughout the course.

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 AM	OPL 103	http://www.wfu.edu/~natalie/f22phy711/

Instructor: Natalie Holzwarth	Office:300 OPL	e-mail:natalie@wfu.edu

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Course schedule

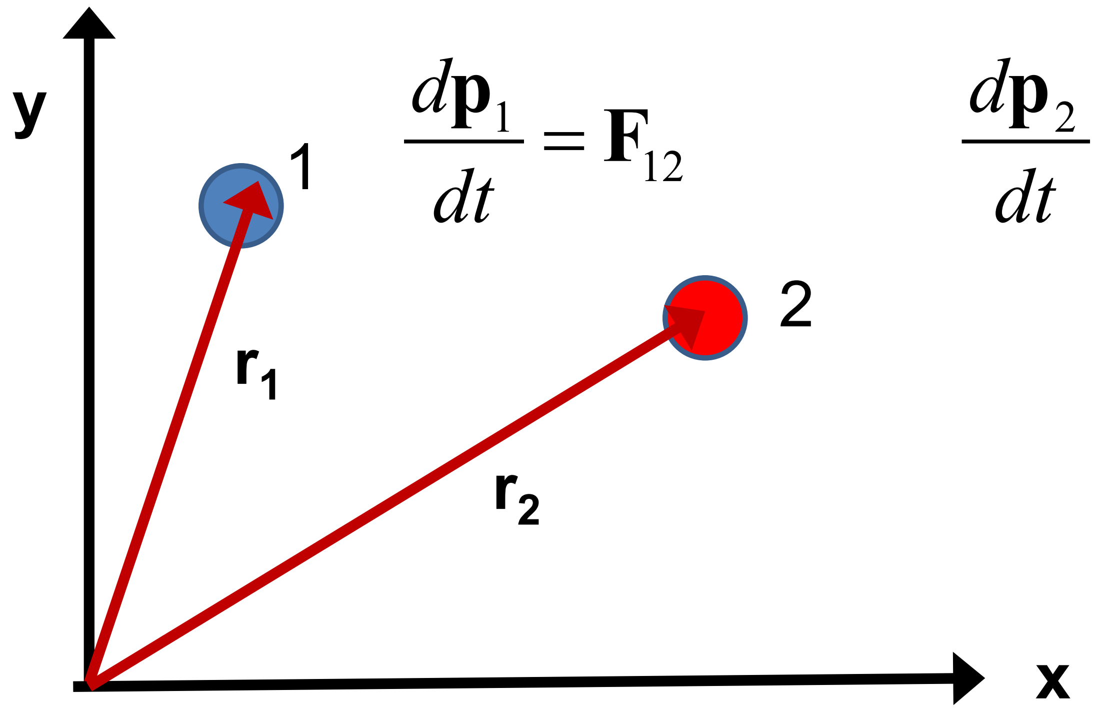
(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/22/2022		Introduction	#1	8/26/2022
2	Wed, 8/24/2022	Chap. 1	Scattering theory		
3	Fri, 8/26/2022	Chap. 1	Scattering theory	#2	8/29/2022
4	Mon, 8/29/2022	Chap. 1	Scattering theory		

Introduction to the analysis of the energy and forces between two particles --

First consider fundamental picture of particle interactions

Classical mechanics of a conservative 2-particle system.



$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21}$$

$$\mathbf{F}_{12} = -\nabla_1 V\left(\mathbf{r}_1 - \mathbf{r}_2\right) \qquad \Rightarrow E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

For this discussion, we will assume that $V(\mathbf{r})=V(r)$ (a central potential).

Digression on two particles in one dimension – from the qualifier exam:

A point particle of mass m with initial velocity $\mathbf{v}_0 = v_0\hat{\mathbf{x}}$ has a head-on elastic collision with a point particle of mass M, initially at rest. After the collision the velocity of the m particle is $\mathbf{v}_1 = v_1\hat{\mathbf{x}}$ and the velocity of the M particle is $\mathbf{w}_1 = w_1\hat{\mathbf{x}}$.

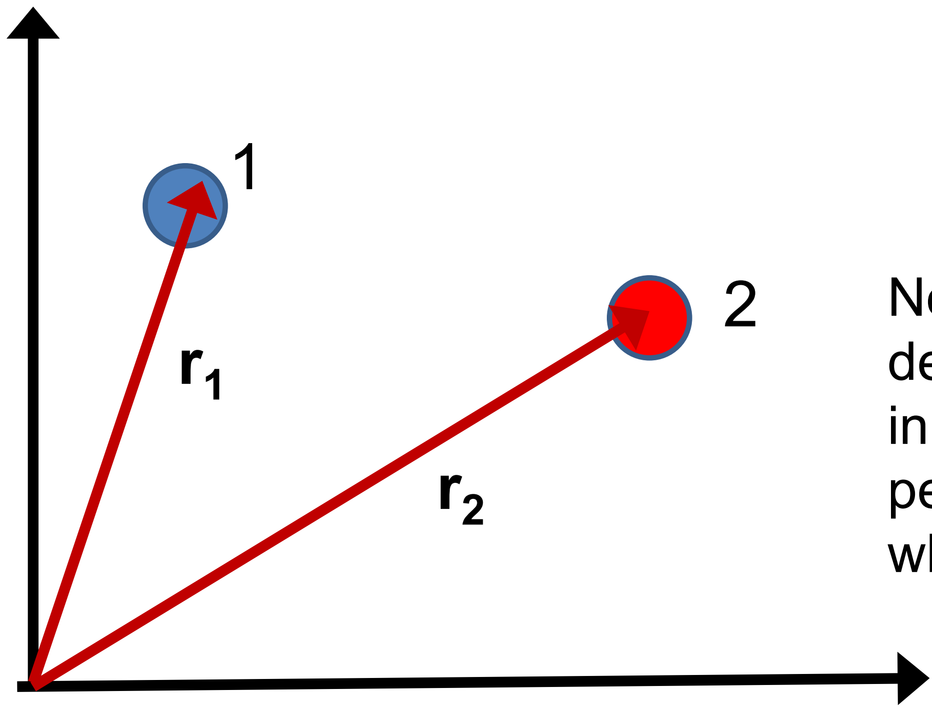
(a) (14/20) Determine v_1 and w_1 as a function of v_0, m, and M.

(b) (6/20) Describe the three different motions of the particles for $m < M$, for $m = M$, and for $m > M$.

Note that linear momentum and energy is conserved.

Also note that when we do our analysis we assume that the energy is kinetic energy.

Energy is conserved: $E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$



Note that while *V(r)* determines the trajectory, in scattering theory, we perform our evaluations where *V(r)=0*.

For a central potential $V(\mathbf{r})=V(r)$, angular momentum is conserved. For the moment we also make the simplifying assumption that $m_2 >> m_1$ so that particle 1 dominates the motion.

Typical two-particle interactions –

Central potential: $$V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$$

Hard sphere: $$V(r) = \begin{cases} \infty & r \leq a \\ 0 & r > a \end{cases}$$

Coulomb or gravitational: $$V(r) = \frac{K}{r}$$

Lennard-Jones: $$V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$$

More details of two particle interaction potentials

Central potential: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r)$

This means that the interaction only depends on the distance between the particles and not on the angle between them. This would typically be true of the particles are infinitesimal points without any internal structure such as two infinitesimal charged particles or two infinitesimal masses separated by a distance *r*:

$$V(r) = \frac{K}{r}$$

Example – Interaction between a proton and an electron. Note we are treating the interactions with classical mechanics; in some cases, quantum effects are non-trivial.

Other examples of central potentials --

<u>Example</u>

Hard sphere: $V(r) = \begin{cases} \infty & r \le a \\ 0 & r > a \end{cases}$ Two marbles

Note that sub-atomic particles might be modeled this way???
Particle physics has lots of clever ideas…

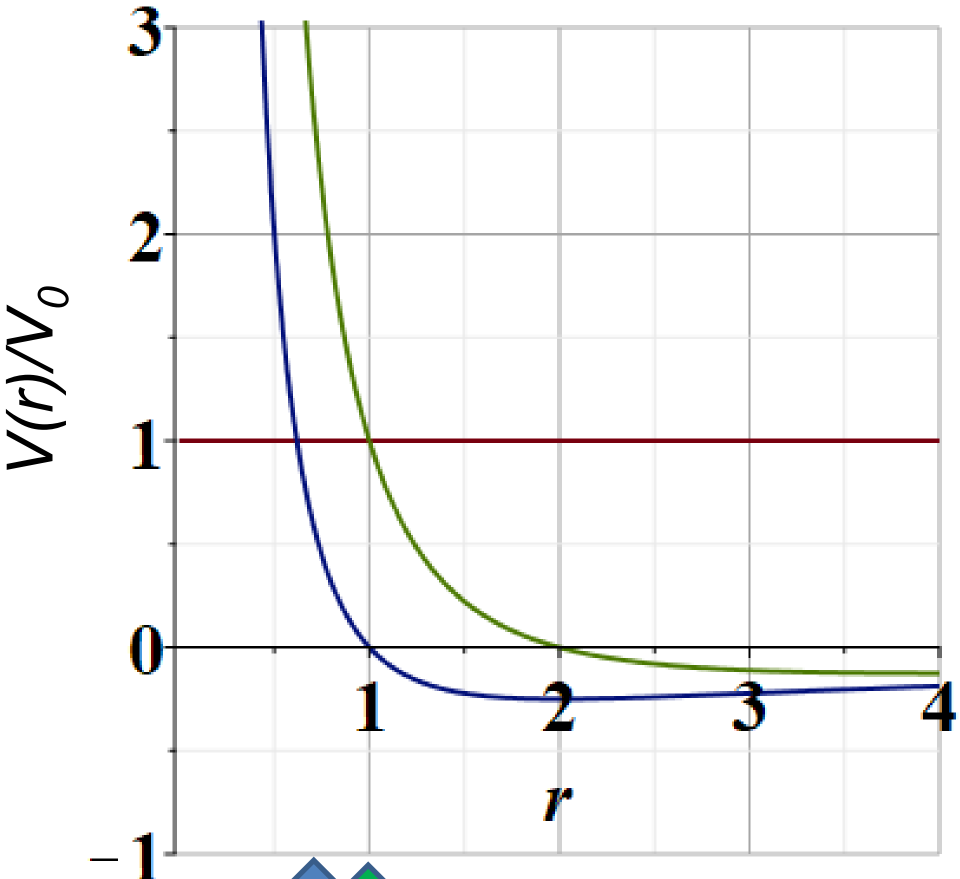
Lennard-Jones: $V(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$ Two Ar atoms

Note – not all systems are described by "central" potentials. Some counter examples:

1. Molecules (internal degrees of freedom)
2. Systems with more than two particles such as crystals

Representative plots of *V(r)* –
two different possibilities:

Why plot in scaled units?
It is often convenient…



E/V_0

Note that particles are not bound; can reach infinite separation

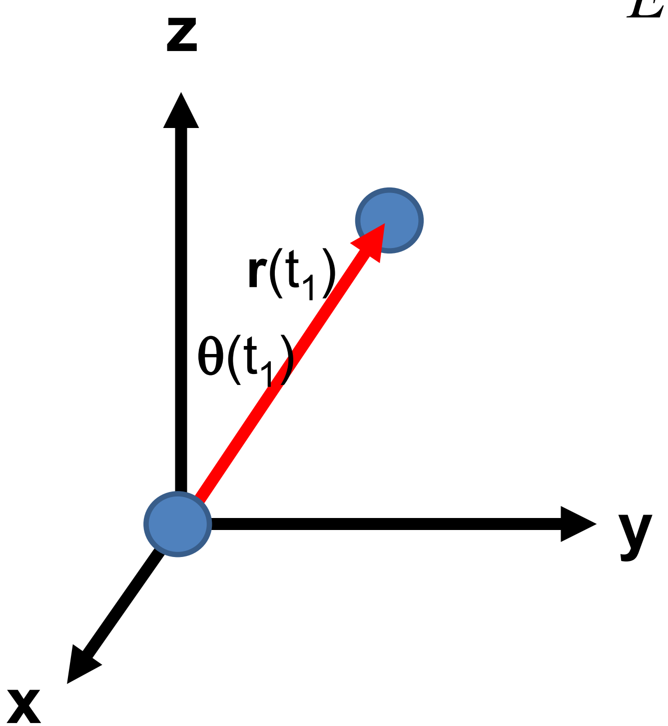
Distances of closest approach

Why?

Some more details --

Here we are assuming that the target particle is stationary and $m_1 \equiv m$.

The origin of our coordinate system is taken at the position of the target particle.



Conservation of energy:

$$E = \frac{1}{2} m \left(\frac{d\mathbf{r}}{dt} \right)^2 + V(r)$$

$$= \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \right) + V(r)$$

Conservation of angular momentum:

$$L = mr^2 \frac{d\theta}{dt}$$

Comment on spherical polar coordinates

Conservation of energy:

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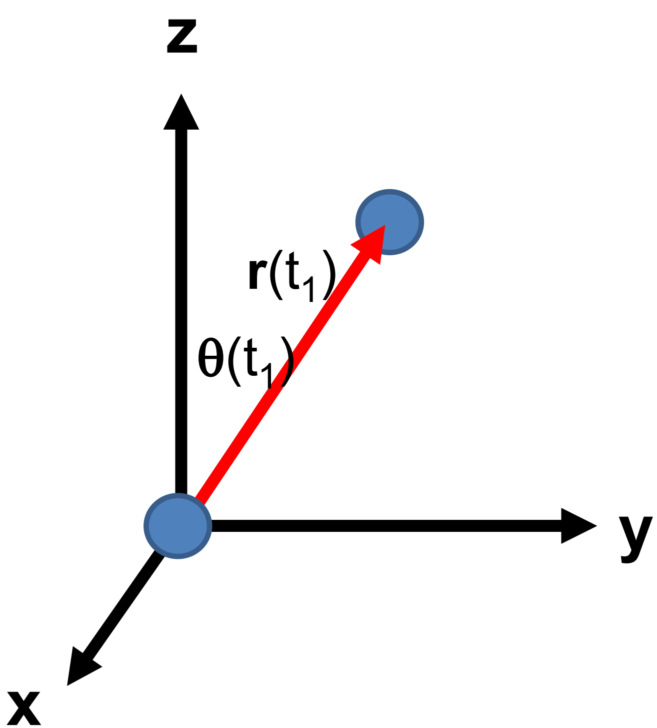


For general spherical polar coordinates:

$$\mathbf{r}=r\sin\theta\cos\phi\,\hat{\mathbf{x}}+r\sin\theta\sin\phi\,\hat{\mathbf{y}}+r\cos\theta\,\hat{\mathbf{z}}$$

However, knowing that for a central potential the motion is within a plane, we can choose $\phi=0$

$$\Rightarrow \mathbf{r}=r\sin\theta\,\hat{\mathbf{x}}+r\cos\theta\,\hat{\mathbf{z}}$$

$$\mathbf{r}=r\sin\theta\,\hat{\mathbf{x}}+r\cos\theta\,\hat{\mathbf{z}}$$

$$\frac{d\mathbf{r}}{dt}=\frac{dr}{dt}\left(\sin\theta\,\hat{\mathbf{x}}+\cos\theta\,\hat{\mathbf{z}}\right)+r\frac{d\theta}{dt}\left(\cos\theta\,\hat{\mathbf{x}}-\sin\theta\,\hat{\mathbf{z}}\right)$$

$$\left|\frac{d\mathbf{r}}{dt}\right|^2=\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\theta}{dt}\right)^2$$

Comments continued --

Conservation of energy:

$$E = \frac{1}{2} m \left(\frac{d\mathbf{r}}{dt} \right)^2 + V(r)$$

$$= \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \right) + V(r)$$

$$= \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \boxed{\frac{L^2}{2mr^2} + V(r)}$$

← $V_{eff}(r)$

Conservation of angular momentum:

$$L = mr^2 \frac{d\theta}{dt}$$

This is useful because L is a constant for each trajectory.

Also note that when $r \to \infty$, $V(r) \to 0$

$$\mathbf{L} \equiv \mathbf{r} \times m \frac{d\mathbf{r}}{dt} \qquad L = b\sqrt{2mE}$$

For $r \to \infty$, $\frac{d\mathbf{r}}{dt} \to v_\infty = \sqrt{\frac{2E}{m}}$

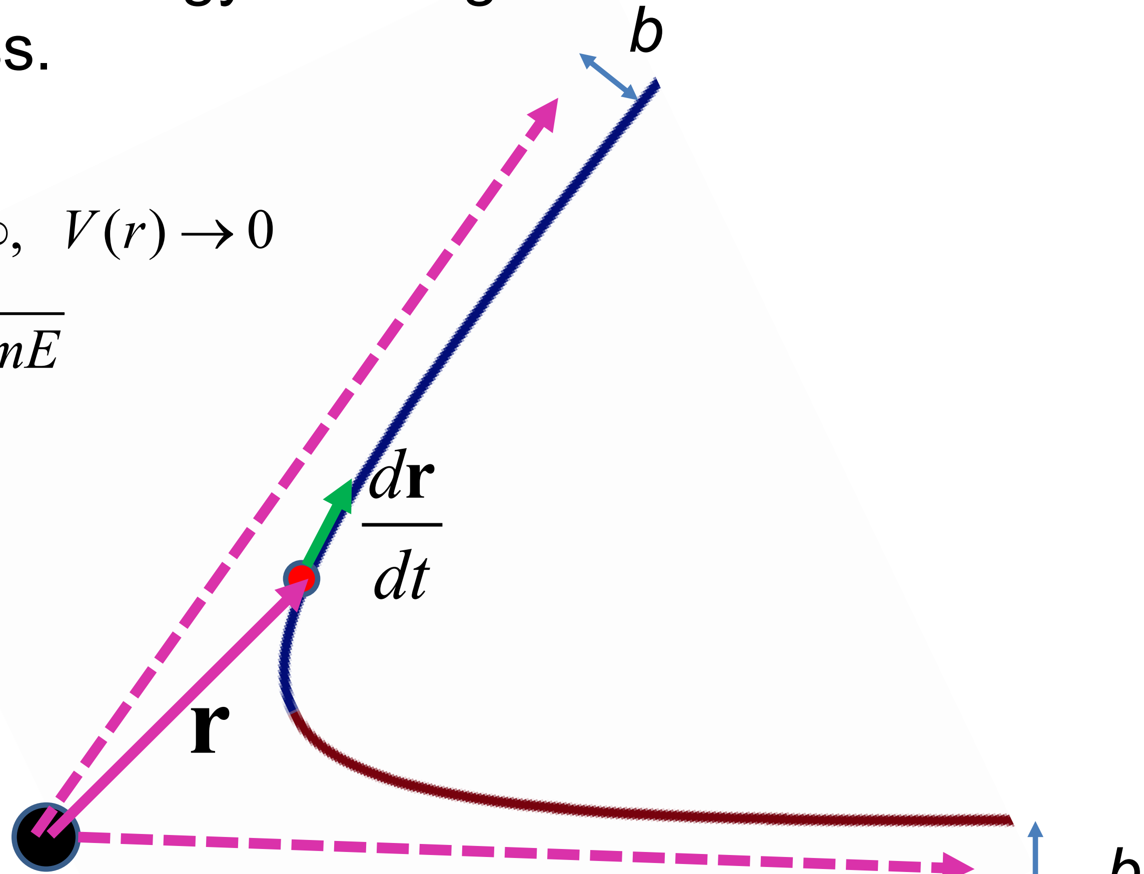
$$E = \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{b^2 E}{r^2} + V(r)$$

What is the impact parameter?

Briefly, a convenient distance that depends on the conserved energy and angular momentum of the process.

Also note that when $r \to \infty, \quad V(r) \to 0$

$$\mathbf{L} \equiv \mathbf{r} \times m \frac{d\mathbf{r}}{dt} \qquad L = b\sqrt{2mE}$$



Which of the following are true:

a. The particle moves in a plane.
b. For any interparticle potential the trajectory can be determined/calculated.
c. Only for a few special interparticle potential forms can the trajectory be determined.

Why should we care about this?

a. We shouldn't really care.
b. It is only of academic interest
c. It is of academic interest but can be measured.
d. Many experiments can be analyzed in terms of the particle trajectory.

Scattering theory:

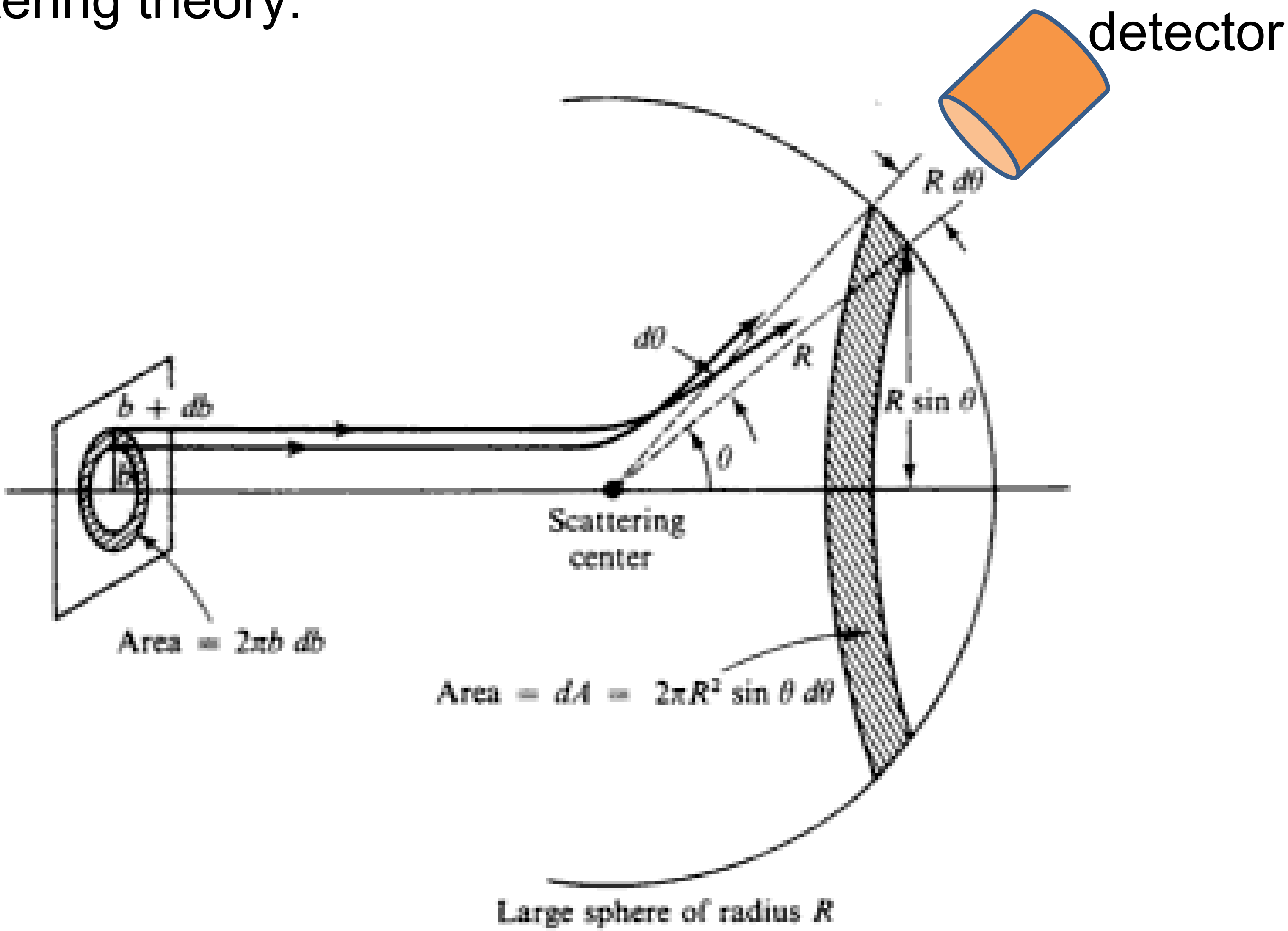


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Scattering theory:

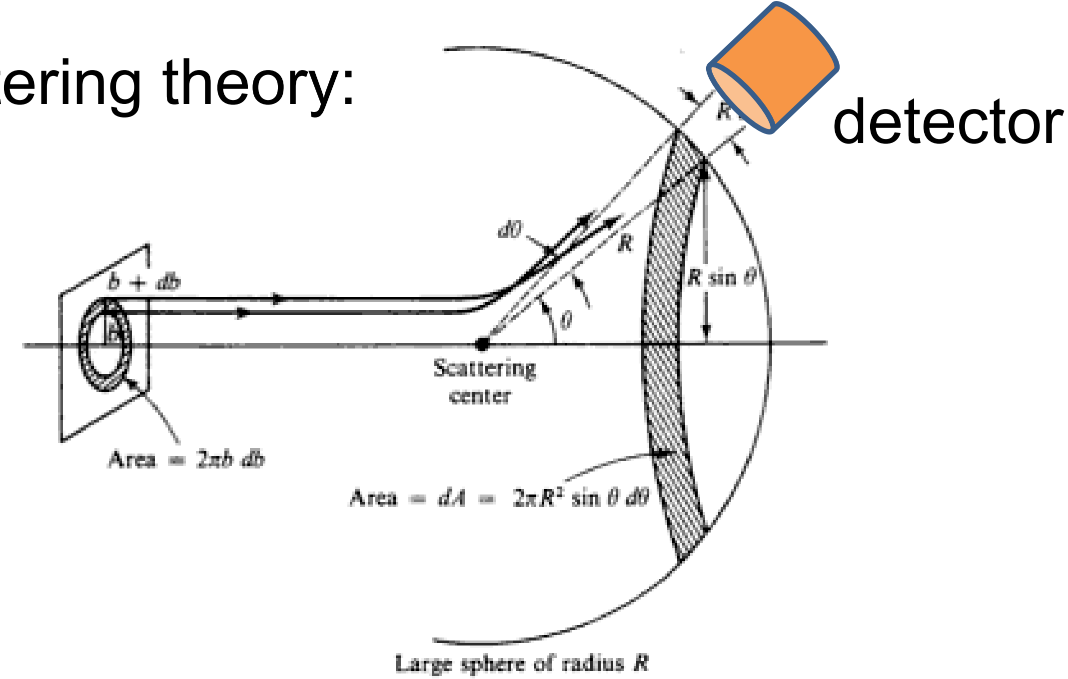


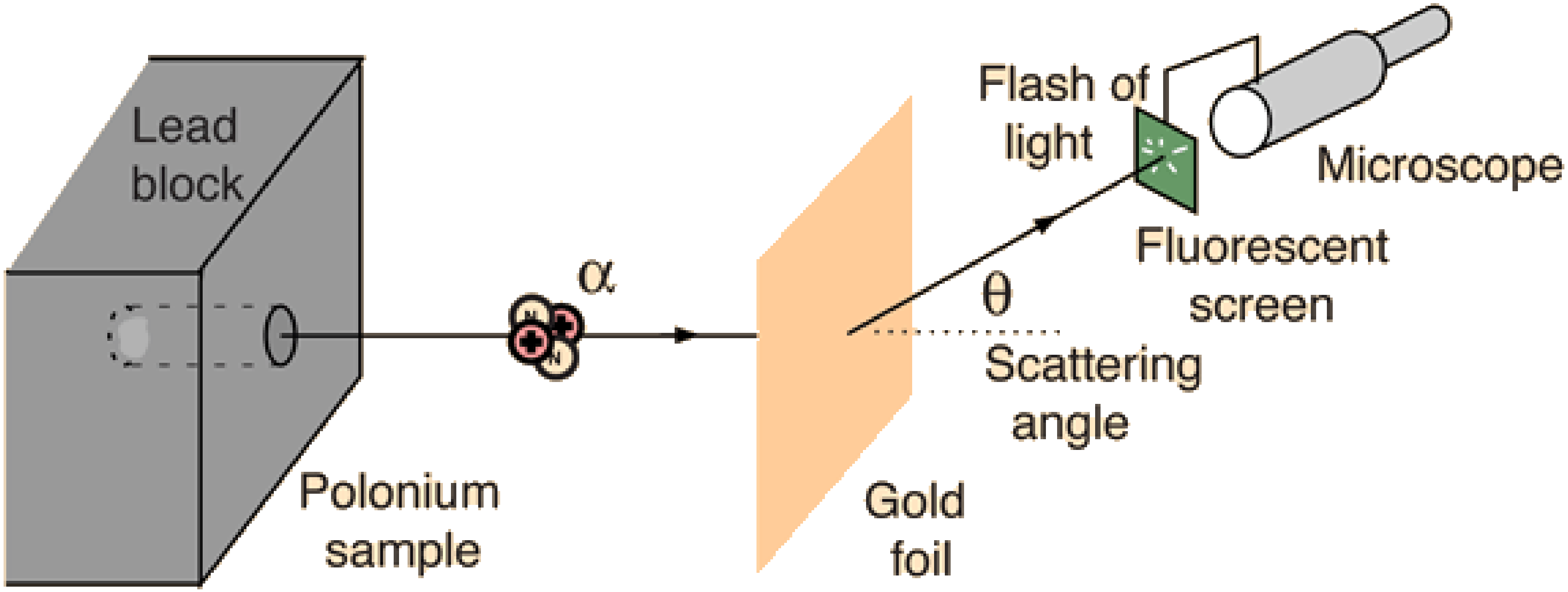
Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Some reasons that scattering theory is useful:

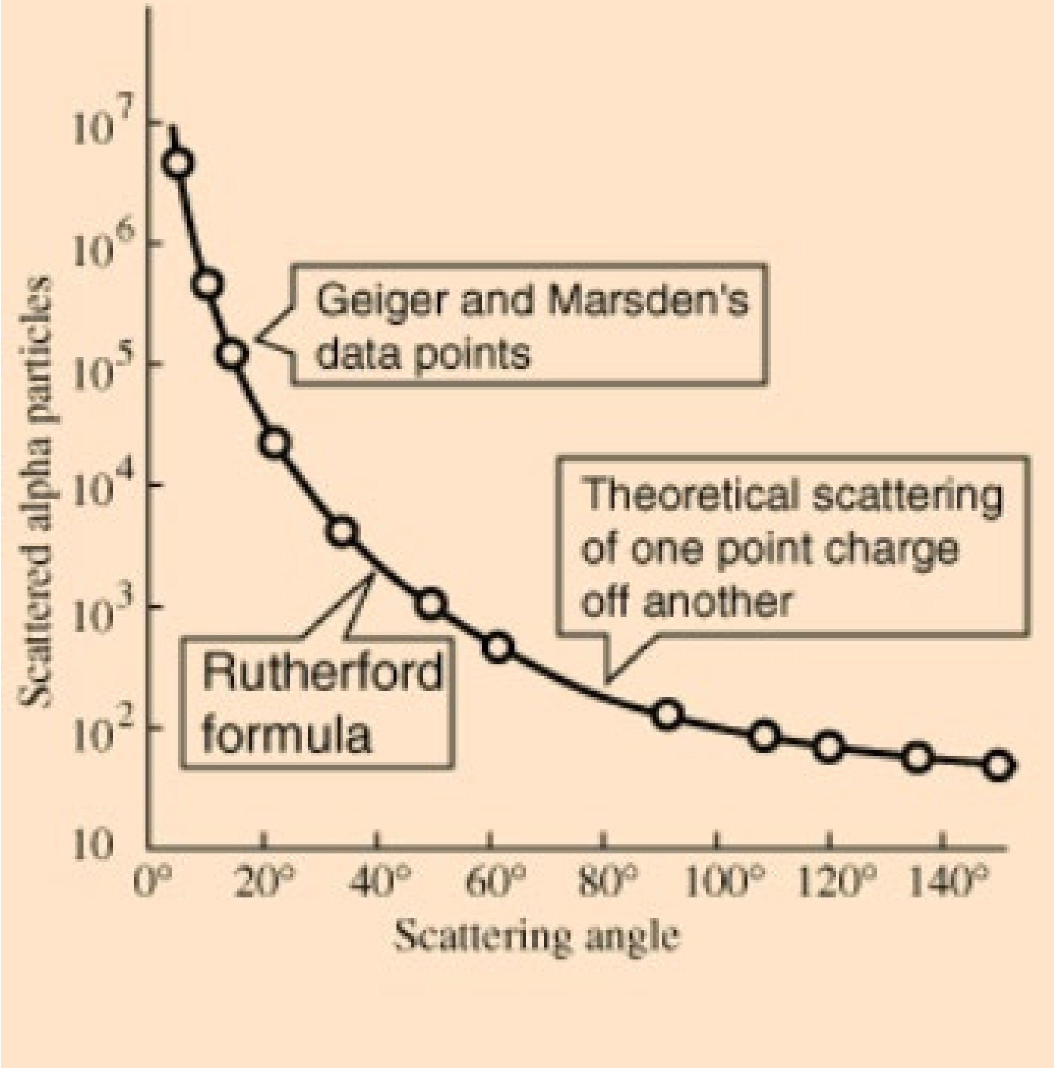
1. It allows comparison between measurement and theory
2. The analysis depends on knowledge of the scattering particles when they are far apart
3. The scattering results depend on the interparticle interactions

Example: Diagram of Rutherford scattering experiment

http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Graph of data from scattering experiment



From website: http://hyperphysics.phy-astr.gsu.edu/hbase/Nuclear/rutsca2.html

Standardization of scattering experiments --

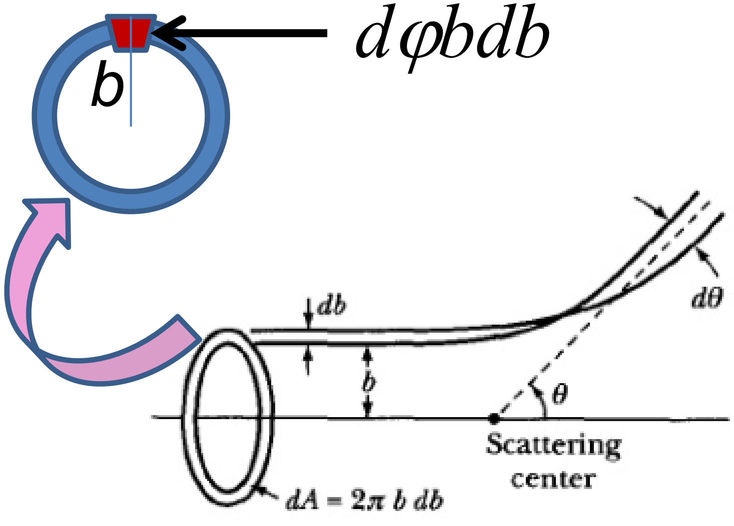
Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$$= \text{Area of incident beam that is scattered into detector at angle } \theta$$

Impact parameter: *b*

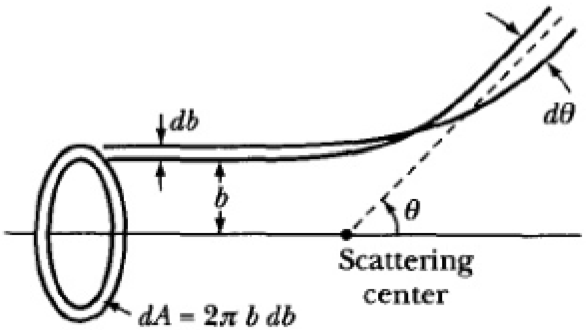
$$d\varphi b db$$



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.



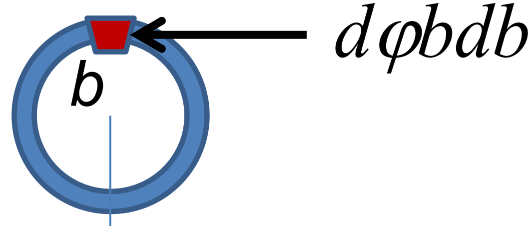
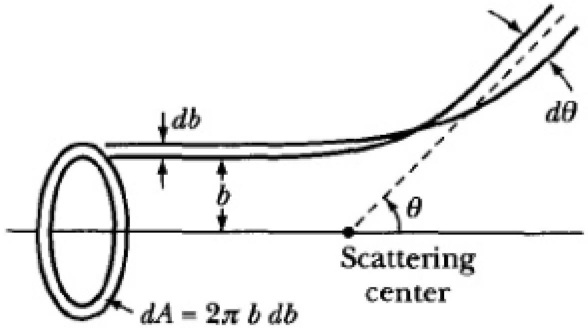


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

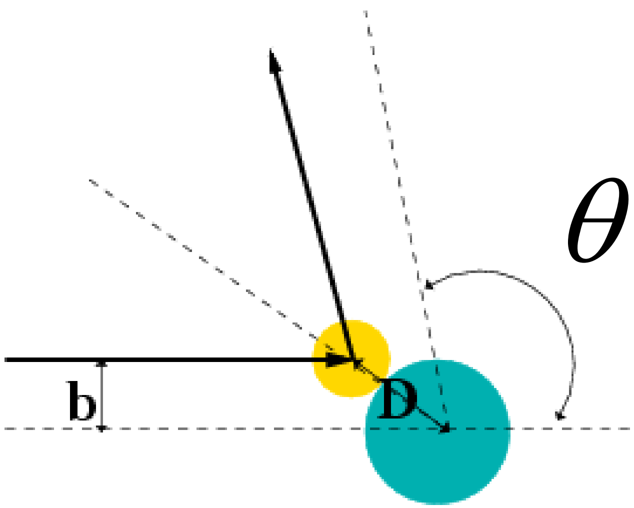
Note: We are assuming that the process is isotropic in φ

Simple example – collision of hard spheres having mutual radius D; very large target mass



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Microscopic view:

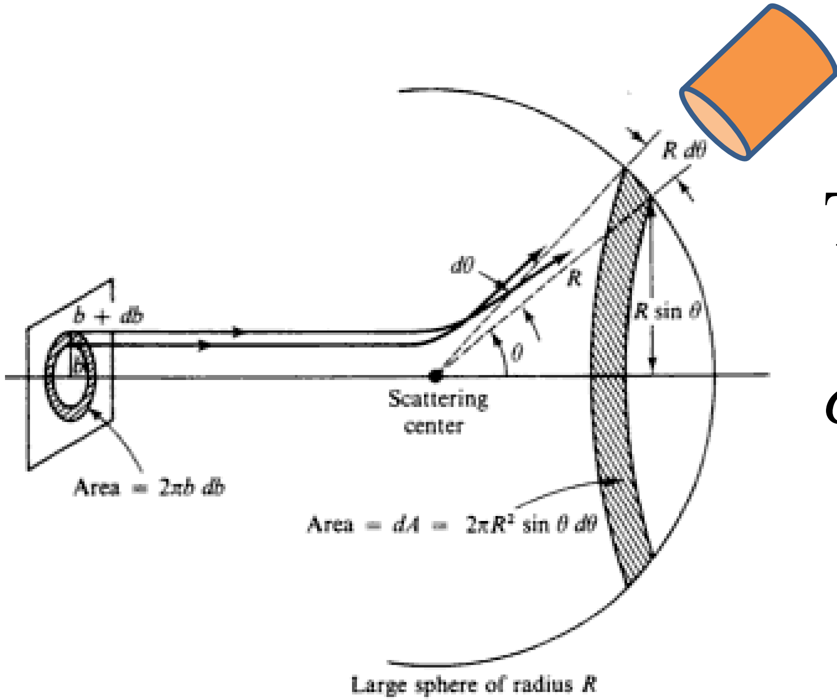


$$b(\theta) = ?$$

$$b(\theta) = D\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$

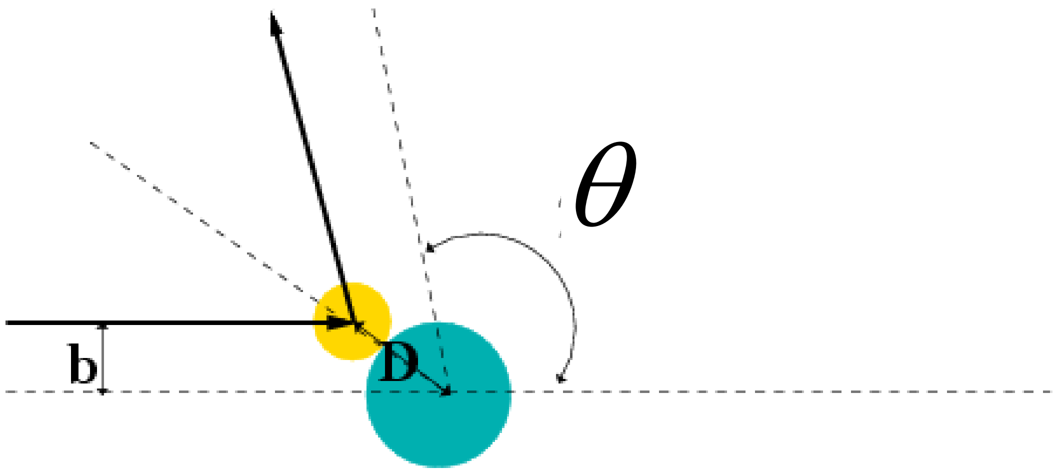
Simple example – collision of hard spheres -- continued



Total scattering cross section:

$$\sigma = \int \left(\frac{d\sigma}{d\Omega} \right) d\Omega$$

Hard sphere:



$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{D^2}{4}$$

$$\sigma = \pi D^2$$

More details of hard sphere scattering –

Hidden in the analysis are assumptions about the scattering process such as:

- No external forces ➔ linear momentum is conserved
- No dissipative phenomena ➔ energy is conserved
- No torque on the system ➔ angular momentum is conserved
- Target particle is much more massive than scattering particle
- Other assumptions??

Note that for quantum mechanical hard spheres at low energy the total cross section is 4 times as large.