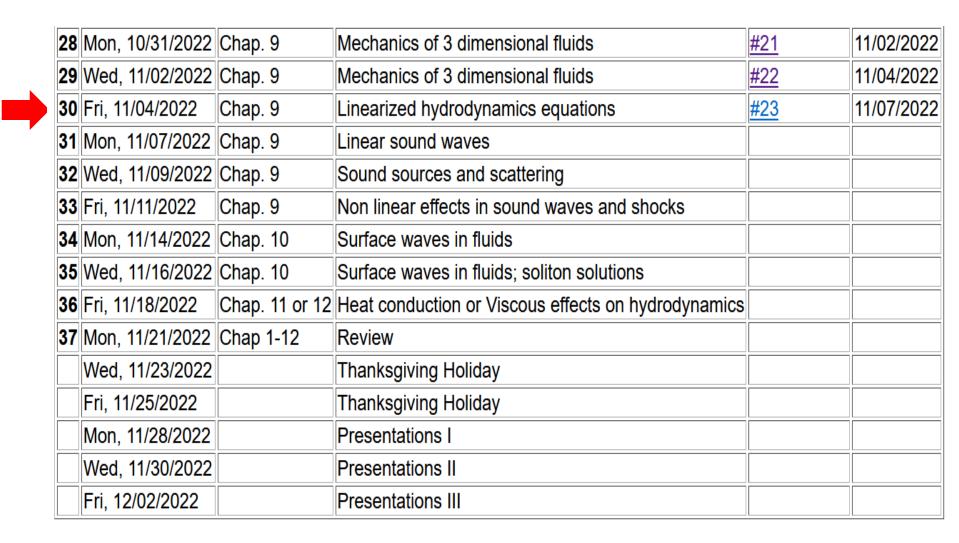


PHY 711 Classical Mechanics and Mathematical Methods 10-10:50 AM MWF in Olin 103

Notes for Lecture 30 -- Chap. 9 in F & W More hydrodynamics

- 1. Newton's laws for fluids and the continuity equation
- 2. Approximate solutions in the linear limit
- 3. Linear sound waves



PHY 711 -- Assignment #23

Nov. 04, 2022

Continue reading Chapter 9 in Fetter & Walecka.

1. Using the analysis covered in class, estimate the speed of sound in the fluid of He gas at 1 atmosphere of pressure and at 300K temperature.

Recall the basic equations of hydrodynamics

Basic variables: Density $\rho(\mathbf{r},t)$ Velocity $\mathbf{v}(\mathbf{r},t)$ Pressure $p(\mathbf{r},t)$

Newton-Euler equation of motion:

Continuity equation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

+ relationships among the variables due to principles of thermodynamics due to the particular fluid (In fact, we will focus on an ideal gas.)

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times \left(\nabla \times \mathbf{v} \right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Additional relationships among the variables apply, depending on the fluid material and on thermodynamics

At the moment we are interested in the case where there is no heat exchange.

A little thermodynamics First law of thermodynamics: $dE_{int} = dQ - dW$ For isentropic conditions: dQ = 0 $dE_{int} = -dW = -pdV$ Here W == work V == volume Solution of Euler's equation for fluids – isentropic (continued) $dE_{int} = -dW = -pdV$

In terms of mass density: $\rho = \frac{M}{V}$ For fixed *M* and variable *V*: $d\rho = -\frac{M}{V^2}dV$ $dV = -\frac{M}{\rho^2} d\rho$ Internal In terms in intensive variables: Let $E_{int} = M \mathcal{E}$ per unit Internal per unit mass $dE_{\rm int} = Md\varepsilon = -dW = -pdV = M\frac{p}{\rho^2}d\rho$ $\left(\frac{\partial \mathcal{E}}{\partial \rho}\right)_{d \rho = 0} = \frac{p}{\rho^2}$ $d\varepsilon = \frac{p}{\rho^2} d\rho$ 11/4/2022 6 Y 711 Fall 2022-- Lecture 30

Solution of Euler's equation for fluids – isentropic (continued)

Note: Under conditions of constant $\left(\frac{\partial \mathcal{E}}{\partial \rho}\right)_{i=0} = \frac{p}{\rho^2}$ entropy, we assume e can be expressed in terms of the density alone. Consider: $\nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dO=0} \nabla \rho = \frac{p}{\rho^2} \nabla \rho$ Rearranging: $\nabla \left(\varepsilon + \frac{p}{\rho} \right) = \frac{\nabla p}{\rho}$

Note that here we are assuming that we can write ε as $\varepsilon(\rho, s)$.

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times \left(\nabla \times \mathbf{v} \right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\mathbf{v}p}{\rho} = \nabla \left(\mathcal{E} + \frac{p}{\rho} \right)$$

if $\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \mathbf{v} = -\nabla \Phi$

 ∇n (n)

$$\mathbf{f}_{applied} = -\nabla U$$

$$\frac{\partial \left(-\nabla \Phi\right)}{\partial t} + \nabla \left(\frac{1}{2}v^{2}\right) = -\nabla U - \nabla \left(\varepsilon + \frac{p}{\rho}\right)$$
$$\Rightarrow \nabla \left(\varepsilon + \frac{p}{\rho} + U + \frac{1}{2}v^{2} - \frac{\partial \Phi}{\partial t}\right) = 0$$

For isentropic and irrotational fluid.

Some details --

$$(\nabla \times \mathbf{v}) = 0 \quad \text{"irrotational flow"} \qquad \Rightarrow \mathbf{v} = -\nabla \Phi$$

Check: $(\nabla \times \mathbf{v}) = -(\nabla \times \nabla \Phi) = ?$
 $(\nabla \times \nabla \Phi) \Big|_{x} = \frac{\partial^{2} \Phi}{\partial y \partial z} - \frac{\partial^{2} \Phi}{\partial z \partial x}$

Summary: For isentropic and irrotational fluid with internal energy per unit mass ϵ :

$$\nabla \left(\varepsilon + \frac{p}{\rho} + U + \frac{1}{2}v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

Here ϵ is the internal energy of the fluid per unit mass. For an ideal gas fluid, it has a relatively simple form.

11/4/2022



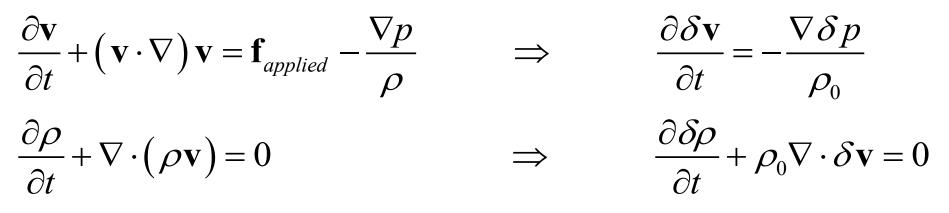
Now consider the fluid to be air near equilibrium

Near equilibrium: $\rho = \rho_0 + \delta \rho$ $p = p_0 + \delta p$ $\mathbf{v} = \mathbf{0} + \delta \mathbf{v}$ $\mathbf{f}_{applied} = \mathbf{0}$

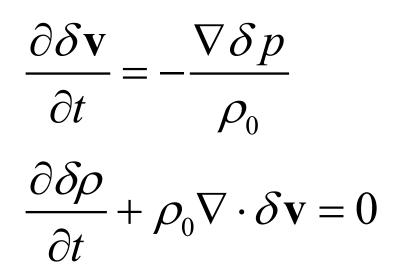
 ρ_0 represents the average air density p_0 represents the average air pressure (usually \approx 1 atmosphere)

$$\mathbf{v}_0 = 0$$
 average velocity

Equations to lowest order in perturbation:



Coupled equations to be solved:



Need to reconcile the interdependencies --

Expressing pressure in terms of the density assuming constant entropy: $p = p(s, \rho) = p_0 + \delta p$ where s denotes the (constant) entropy $p_0 = p(s, \rho_0)$ $\delta p = \left(\frac{\partial p}{\partial \rho}\right)_{s} \delta \rho \equiv c^{2} \delta \rho$ Here $c^{2} = \left(\frac{\partial p}{\partial \rho}\right)$ $\nabla \left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0} \right) = 0 \qquad \Rightarrow -\frac{\partial \Phi}{\partial t} + c^2 \frac{\delta \rho}{\rho_0} = (\text{constant})$ $\Rightarrow -\frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho} \frac{\partial \delta \rho}{\partial t} = 0$ $\frac{\partial \delta \rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \implies \frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$
 $\mathbf{v} = -\nabla \Phi$

Note that, we also have :

$$\frac{\partial^2 \delta \rho}{\partial t^2} - c^2 \nabla^2 \delta \rho = 0$$
$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

How can we determine *c*?



Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Equation of state for ideal gas:

$$pV = NkT \qquad \qquad N = \frac{M}{M_0}$$

$$p = \frac{M}{V} \frac{k}{M_0} T = \rho \frac{k}{M_0} T$$

$$k = 1.38 \times 10^{-23} J / K$$

 M_0 = average mass of each molecule



Internal energy for ideal gas :

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2} \frac{k}{M_0} T = \frac{f}{2} \frac{p}{\rho}$$

In terms of specific heat ratio : $\gamma \equiv \frac{C_p}{C_V}$

$$dE = dQ - dW$$

$$C_{V} = \left(\frac{dQ}{dT}\right)_{V} = \left(\frac{\partial E}{\partial T}\right)_{V} = \frac{f}{2}\frac{Mk}{M_{0}}$$

$$C_{p} = \left(\frac{dQ}{dT}\right)_{p} = \left(\frac{\partial E}{\partial T}\right)_{p} + p\left(\frac{\partial V}{\partial T}\right)_{p} = \frac{f}{2}\frac{Mk}{M_{0}} + \frac{Mk}{M_{0}}$$

$$\gamma = \frac{C_{p}}{C_{V}} = \frac{\frac{f}{2} + 1}{\frac{f}{2}} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1}$$

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Digression

Internal energy for ideal gas: $f \equiv$ "degrees of freedom"

$$E = \frac{f}{2}NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

$$\frac{f}{2} = \frac{1}{\gamma - 1} \quad \Rightarrow \quad E = \frac{1}{\gamma - 1} NkT \qquad \varepsilon = \frac{1}{\gamma - 1} \frac{k}{M_0} T = \frac{1}{\gamma - 1} \frac{p}{\rho}$$

	f	γ
Spherical atom	3	1.66667
Diatomic molecule	5	1.40000



Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma - 1} \frac{k}{M_0} T = \frac{1}{\gamma - 1} \frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions:

$$d\varepsilon = -\frac{p}{M}dV = \frac{p}{\rho^2}d\rho$$
$$\left(\frac{\partial\varepsilon}{\partial\rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial\rho}\left(\frac{1}{\gamma-1}\frac{p}{\rho}\right)_s = \left(\frac{\partial p}{\partial\rho}\right)_s \frac{1}{(\gamma-1)\rho} - \frac{p}{(\gamma-1)\rho^2}$$
$$\Rightarrow \left(\frac{\partial p}{\partial\rho}\right)_s = \frac{p\gamma}{\rho}$$



Analysis of wave velocity in an ideal gas:

$$c^{2} = \left(\frac{\partial p}{\partial \rho}\right)_{s} = \frac{p\gamma}{\rho}$$

Linearized speed of sound

$$c_0^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s, p_0, \rho_0} = \frac{\gamma p_0}{\rho_0}$$

Evaluation using approximate data for air at room temperature:

$$c_0^2 \approx \frac{1.5 \cdot 1.013 \times 10^5 Pa}{1.3 kg / m^3}$$
 $c_0 \approx 340 \text{ m/s}$

More general case -- Isentropic or adiabatic equation of state:

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \qquad \qquad \Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

Density dependence of speed of sound for ideal gas:

$$c^{2} = \left(\frac{\partial p}{\partial \rho}\right)_{s} = \frac{\gamma p}{\rho}$$
$$\frac{p}{p_{0}} = \left(\frac{\rho}{\rho_{0}}\right)^{\gamma}$$
$$c^{2} = \frac{p_{0}\gamma}{\rho_{0}} \frac{p / p_{0}}{\rho / \rho_{0}} = c_{0}^{2} \left(\frac{\rho}{\rho_{0}}\right)^{\gamma-1} \text{ for } c_{0}^{2} \equiv \frac{p_{0}\gamma}{\rho_{0}}$$

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Summary of linearized hydrodynamic equations for isentropic fluid

In terms of the velocity potential:

$$\delta \mathbf{v} = -\nabla \Phi \qquad \qquad \frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi = 0 \qquad c_0^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,\rho_0}$$

In term of density fluctuation:

In term of pressure fluctuation:

$$\frac{\partial^2 \delta \rho}{\partial t^2} - c_0^2 \nabla^2 \delta \rho = 0$$
$$\frac{\partial^2 \delta p}{\partial t^2} - c_0^2 \nabla^2 \delta p = 0$$

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$
 $\mathbf{v} = -\nabla \Phi$

Note that, we also have :

$$\frac{\partial^2 \delta \rho}{\partial t^2} - c^2 \nabla^2 \delta \rho = 0$$
$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity V :

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta \mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

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