PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 30 -- Chap. 9 in F & W

More hydrodynamics

1. Newton's laws for fluids and the continuity equation
2. Approximate solutions in the linear limit
3. Linear sound waves

28	Mon, 10/31/2022	Chap. 9	Mechanics of 3 dimensional fluids	#21	11/02/2022
29	Wed, 11/02/2022	Chap. 9	Mechanics of 3 dimensional fluids	#22	11/04/2022
30	Fri, 11/04/2022	Chap. 9	Linearized hydrodynamics equations	#23	11/07/2022
31	Mon, 11/07/2022	Chap. 9	Linear sound waves		
32	Wed, 11/09/2022	Chap. 9	Sound sources and scattering		
33	Fri, 11/11/2022	Chap. 9	Non linear effects in sound waves and shocks		
34	Mon, 11/14/2022	Chap. 10	Surface waves in fluids		
35	Wed, 11/16/2022	Chap. 10	Surface waves in fluids; soliton solutions		
36	Fri, 11/18/2022	Chap. 11 or 12	Heat conduction or Viscous effects on hydrodynamics		
37	Mon, 11/21/2022	Chap 1-12	Review		
	Wed, 11/23/2022		Thanksgiving Holiday		
	Fri, 11/25/2022		Thanksgiving Holiday		
	Mon, 11/28/2022		Presentations I		
	Wed, 11/30/2022		Presentations II		
	Fri, 12/02/2022		Presentations III		

PHY 711 -- Assignment #23

Nov. 04, 2022

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Using the analysis covered in class, estimate the speed of sound in the fluid of He gas at 1 atmosphere of pressure and at 300K temperature.

Recall the basic equations of hydrodynamics

Basic variables: Density $\rho(\mathbf{r},t)$

Velocity $\mathbf{v}(\mathbf{r},t)$

Pressure $p(\mathbf{r},t)$

Newton-Euler equation of motion:

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}_{applied}-\frac{\nabla p}{\rho}$$

Continuity equation: $$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$$

+ relationships among the variables due to principles of thermodynamics due to the particular fluid (In fact, we will focus on an ideal gas.)

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial \mathbf{v}}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)-\mathbf{v}\times\left(\nabla\times\mathbf{v}\right)=\mathbf{f}_{applied}-\frac{\nabla p}{\rho}$$

Additional relationships among the variables apply, depending on the fluid material and on thermodynamics

At the moment we are interested in the case where there is no heat exchange.

A little thermodynamics

First law of thermodynamics: $dE_{\mathrm{int}}=dQ-dW$

For isentropic conditions: $dQ=0$

$dE_{\mathrm{int}}=-dW=-pdV$

Here *W* == work
V == volume

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}} = -dW = -p\,dV$$

In terms of mass density: $\rho = \dfrac{M}{V}$

For fixed M and variable V: $d\rho = -\dfrac{M}{V^2}dV$

$$dV = -\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}} = M\varepsilon$

Internal energy per unit mass

$$dE_{\text{int}} = M d\varepsilon = -dW = -p\,dV = M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon = \frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Note: Under conditions of constant entropy, we assume e can be expressed in terms of the density alone.

$$\text{Consider:} \quad \nabla\varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla\rho = \frac{p}{\rho^2}\nabla\rho$$

$$\text{Rearranging:} \quad \boxed{\nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}}$$

Note that here we are assuming that we can write ε as $\varepsilon(\rho,s)$.

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

if $\nabla\times\mathbf{v} = 0$ ➔ $\mathbf{v} = -\nabla\Phi$ $\mathbf{f}_{applied} = -\nabla U$

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

For isentropic and irrotational fluid.

Some details --

$$\left(\nabla \times \mathbf{v}\right) = 0 \qquad \text{"irrotational flow"} \qquad\qquad \Rightarrow \mathbf{v} = -\nabla\Phi$$

Check: $\left(\nabla \times \mathbf{v}\right) = -\left(\nabla \times \nabla\Phi\right) = ?$

$$\left(\nabla \times \nabla\Phi\right)\Big|_x = \frac{\partial^2\Phi}{\partial y\partial z} - \frac{\partial^2\Phi}{\partial z\partial x}$$

Summary: For isentropic and irrotational fluid with internal energy per unit mass ε:

$$\nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Here ε is the internal energy of the fluid per unit mass. For an ideal gas fluid, it has a relatively simple form.

Now consider the fluid to be air near equilibrium

Near equilibrium:

$$\rho = \rho_0 + \delta\rho$$

$$p = p_0 + \delta p$$

$$\mathbf{v} = 0 + \delta\mathbf{v}$$

$$\mathbf{f}_{applied} = 0$$

ρ_0 represents the average air density

p_0 represents the average air pressure

(usually $\approx$ 1 atmosphere)

$\mathbf{v}_0 = 0$ average velocity

Equations to lowest order in perturbation:

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}_{applied}-\frac{\nabla p}{\rho} \quad \Rightarrow \quad \frac{\partial \delta\mathbf{v}}{\partial t}=-\frac{\nabla\delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 \quad \Rightarrow \quad \frac{\partial \delta\rho}{\partial t}+\rho_0\nabla\cdot\delta\mathbf{v}=0$$

Coupled equations to be solved:

$$\frac{\partial \delta\mathbf{v}}{\partial t}=-\frac{\nabla\delta p}{\rho_0}$$

$$\frac{\partial \delta\rho}{\partial t}+\rho_0\nabla\cdot\delta\mathbf{v}=0$$

Need to reconcile the interdependencies --

Expressing pressure in terms of the density assuming constant entropy:

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2 \delta\rho \qquad \text{Here} \quad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

$$\nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \qquad \Rightarrow -\frac{\partial \Phi}{\partial t} + c^2 \frac{\delta\rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho_0}\frac{\partial \delta\rho}{\partial t} = 0$$

$$\frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \quad \Rightarrow \boxed{\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0}$$

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$

$$\mathbf{v} = -\nabla \Phi$$

Note that, we also have :

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

How can we determine *c*?

Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Equation of state for ideal gas:

$$pV = NkT \qquad N = \frac{M}{M_0}$$

$$p = \frac{M}{V}\frac{k}{M_0}T = \rho\frac{k}{M_0}T$$

$$k = 1.38\times 10^{-23} J/K$$

$M_0 =$ average mass of each molecule

Internal energy for ideal gas :

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

In terms of specific heat ratio : $\gamma \equiv \frac{C_p}{C_V}$

$$dE = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2}\frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2}\frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1}$$

Digression

Internal energy for ideal gas: $f \equiv$ "degrees of freedom"

$$E = \frac{f}{2} NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

$$\frac{f}{2} = \frac{1}{\gamma - 1} \quad \Rightarrow \quad E = \frac{1}{\gamma - 1} NkT \qquad \varepsilon = \frac{1}{\gamma - 1}\frac{k}{M_0}T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

	f	γ
Spherical atom	3	1.66667
Diatomic molecule	5	1.40000

Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma - 1}\frac{k}{M_0}T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions:

$$d\varepsilon = -\frac{p}{M} dV = \frac{p}{\rho^2} d\rho$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)_s = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{1}{(\gamma - 1)\rho} - \frac{p}{(\gamma - 1)\rho^2}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Linearized speed of sound

$$c_0{}^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,p_0,\rho_0} = \frac{\gamma p_0}{\rho_0}$$

Evaluation using approximate data for air at room temperature:

$$c_0{}^2 \approx \frac{1.5 \cdot 1.013 \times 10^5 Pa}{1.3 kg / m^3} \qquad c_0 \approx 340 \text{ m/s}$$

More general case -- Isentropic or adiabatic equation of state:

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \qquad \Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

Density dependence of speed of sound for ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$c^2 = \frac{p_0 \gamma}{\rho_0} \frac{p / p_0}{\rho / \rho_0} = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1} \quad \text{for } c_0^2 \equiv \frac{p_0 \gamma}{\rho_0}$$

Summary of linearized hydrodynamic equations for isentropic fluid

In terms of the velocity potential:

$$\delta \mathbf{v} = -\nabla \Phi \qquad \frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi = 0 \qquad c_0^2 = \left(\frac{\partial p}{\partial \rho} \right)_{s,\rho_0}$$

In term of density fluctuation: $$\frac{\partial^2 \delta\rho}{\partial t^2} - c_0^2 \nabla^2 \delta\rho = 0$$

In term of pressure fluctuation: $$\frac{\partial^2 \delta p}{\partial t^2} - c_0^2 \nabla^2 \delta p = 0$$

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

$$\mathbf{v} = -\nabla \Phi$$

Note that, we also have :

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta \mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$