PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 31 -- Chap. 9 of F&W

Wave equation for sound in linear approximation

1. Wave equations for sound
2. Plane wave solutions
3. Standing wave solutions
4. Coupling of resonances to audible sound.

28	Mon, 10/31/2022	Chap. 9	Mechanics of 3 dimensional fluids	#21	11/02/2022
29	Wed, 11/02/2022	Chap. 9	Mechanics of 3 dimensional fluids	#22	11/04/2022
30	Fri, 11/04/2022	Chap. 9	Linearized hydrodynamics equations	#23	11/07/2022
31	Mon, 11/07/2022	Chap. 9	Linear sound waves	#24	11/09/2022
32	Wed, 11/09/2022	Chap. 9	Sound sources and scattering		
33	Fri, 11/11/2022	Chap. 9	Non linear effects in sound waves and shocks		
34	Mon, 11/14/2022	Chap. 10	Surface waves in fluids		
35	Wed, 11/16/2022	Chap. 10	Surface waves in fluids; soliton solutions		
36	Fri, 11/18/2022	Chap. 11 or 12	Heat conduction or Viscous effects on hydrodynamics		
37	Mon, 11/21/2022	Chap 1-12	Review		
	Wed, 11/23/2022		Thanksgiving Holiday		
	Fri, 11/25/2022		Thanksgiving Holiday		
	Mon, 11/28/2022		Presentations I		
	Wed, 11/30/2022		Presentations II		
	Fri, 12/02/2022		Presentations III		

PHY 711 -- Assignment #24

Nov. 07, 2022

Continue reading Chapter 9 in **Fetter & Walecka**.

1. Consider a cylindrical pipe of length 0.5 m and radius 0.05 m, open at both ends. For air at 300 K and atmospheric pressure in this pipe, find several of the lowest frequency resonances.

PHY 711 Presentation Schedule for Fall 2022

Monday, November 28, 2022

	Name	Title/Topic
10:00-10:15		
10:17-10:32		
10:35-10:50		

Wednesday, November 30,, 2022

	Name	Title/Topic
10:00-10:15		
10:17-10:32		
10:35-10:50		

Friday, December 2, 2022

	Name	Title/Topic
10:00-10:15		
10:17-10:32		
10:35-10:50		

Review –

Hydrodynamic equations for isentropic air + linearization about equilibrium → wave equation for air   (sound waves)

Which of the following things correctly describe the wave equation for sound in air and the wave equation for elastic media?

a. The wave velocity is different for sound in air and waves in elastic media.
b. The wave motion in elastic media can be either transverse or longitudinal.
c. The wave motion for sound in air can be either transverse or longitudinal.

Equations to lowest order in perturbation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} \quad \Rightarrow \quad \frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \quad \Rightarrow \quad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0$$

In terms of the velocity potential:

$$\delta\mathbf{v} = -\nabla\Phi$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \quad \Rightarrow \nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla\cdot\delta\mathbf{v} = 0 \quad \Rightarrow \frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2\Phi = 0$$

Expressing pressure in terms of the density assuming constant entropy:

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_{s,\rho_0} \delta\rho \equiv c_0^2 \; \delta\rho$ (strictly keeping to the linear approximation)

$$\nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \qquad \Rightarrow -\frac{\partial \Phi}{\partial t} + c_0^2 \frac{\delta\rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2 \Phi}{\partial t^2} + \frac{c_0^2}{\rho_0}\frac{\partial \delta\rho}{\partial t} = 0$$ (assuming we can adjust Φ accordingly)

$$\frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \qquad \Rightarrow \frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi = 0$$

$$\Rightarrow \delta\rho = \frac{\rho_0}{c_0^2}\frac{\partial \Phi}{\partial t} \qquad \delta p = \rho_0 \frac{\partial \Phi}{\partial t}$$

Wave equation for air:

$$\frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi = 0$$

Here, $c_0^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,\rho_0}$

$$\mathbf{v} = -\nabla \Phi$$

Note that, we also have:

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c_0^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c_0^2 \nabla^2 \delta p = 0$$

Solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution:

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

$$\delta\mathbf{v} = -\nabla\Phi = -i\mathbf{k}Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$$

$\Leftarrow$ Note this is a pure longitudinal wave

$$\delta\rho = \frac{\rho_0}{c_0^2}\frac{\partial\Phi}{\partial t} = -i\omega\frac{\rho_0}{c_0^2}Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\delta p = \rho_0\frac{\partial\Phi}{\partial t} = -i\omega\rho_0 Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

Boundary values of wave equation

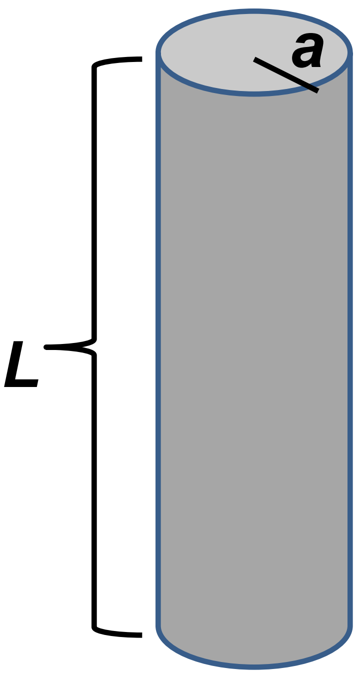
Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta\mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla\Phi$$

Free surface:

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

Time harmonic standing waves in a pipe



$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Boundary values:

At fixed surface: $\hat{\mathbf{n}} \cdot \nabla\Phi = 0$

At free surface: $\frac{\partial \Phi}{\partial t} = 0$

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \qquad \text{Define:} \quad k \equiv \frac{\omega}{c}$$

In cylindrical coordinates:

$$\Phi(r,\phi,z,t) = R(r)F(\phi)Z(z)e^{-i\omega t} \equiv R(r)F(\phi)Z(z)e^{-ikct}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Phi(r,\phi,z,t) = 0$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\Phi(r,\varphi,z,t)=0$$

$$\Phi(r,\varphi,z,t)=R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$F(\varphi)=e^{im\varphi};\;\; F(\varphi)=F(\varphi+2\pi N)\Rightarrow m=\text{integer}$$

$$Z(z)=e^{i\alpha z};\;\; \alpha=\text{real}\;\left(+\text{ other restrictions}\right)$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

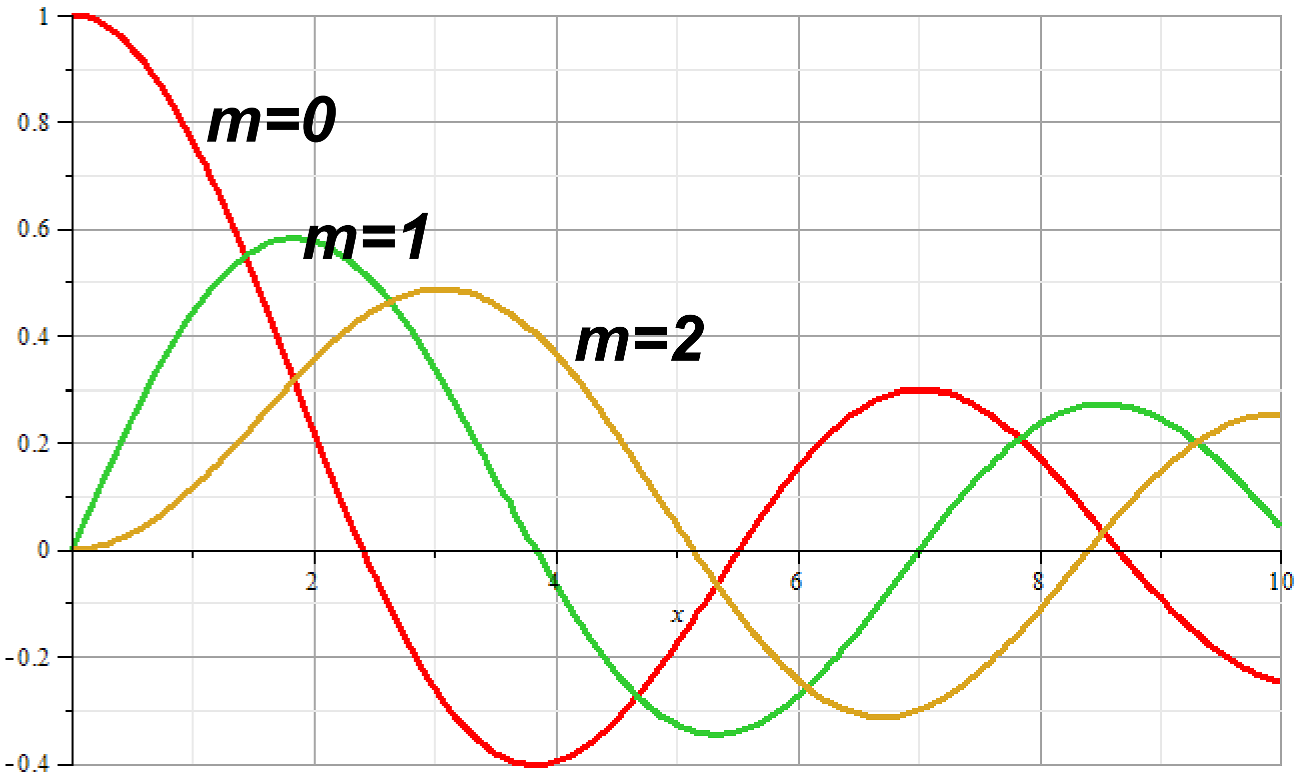
For $\quad k^2 \geq \alpha^2 \quad$ define $\quad \kappa^2 \equiv k^2-\alpha^2$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+\kappa^2\right)R(r)=0$$

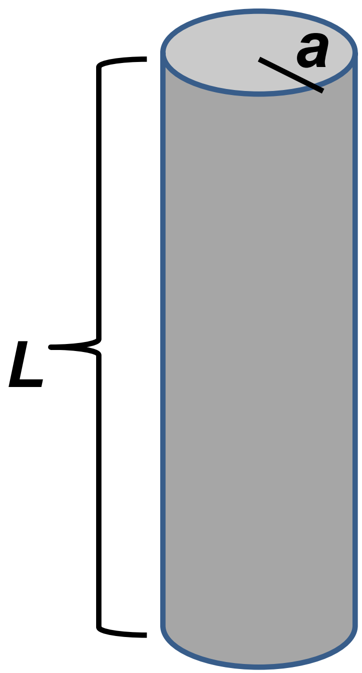
Cylinder surface boundary conditions: $\left.\frac{dR}{dr}\right|_{r=a}=0$

$\Rightarrow R(r)=J_m(\kappa r) \quad$ where for $\frac{dJ_m(x'_{mn})}{dx}=0, \quad \kappa_{mn}=\frac{x'_{mn}}{a}$

Bessel functions : $J_m(x)$



Now recall the boundary conditions



Boundary values:

At fixed surface: $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$

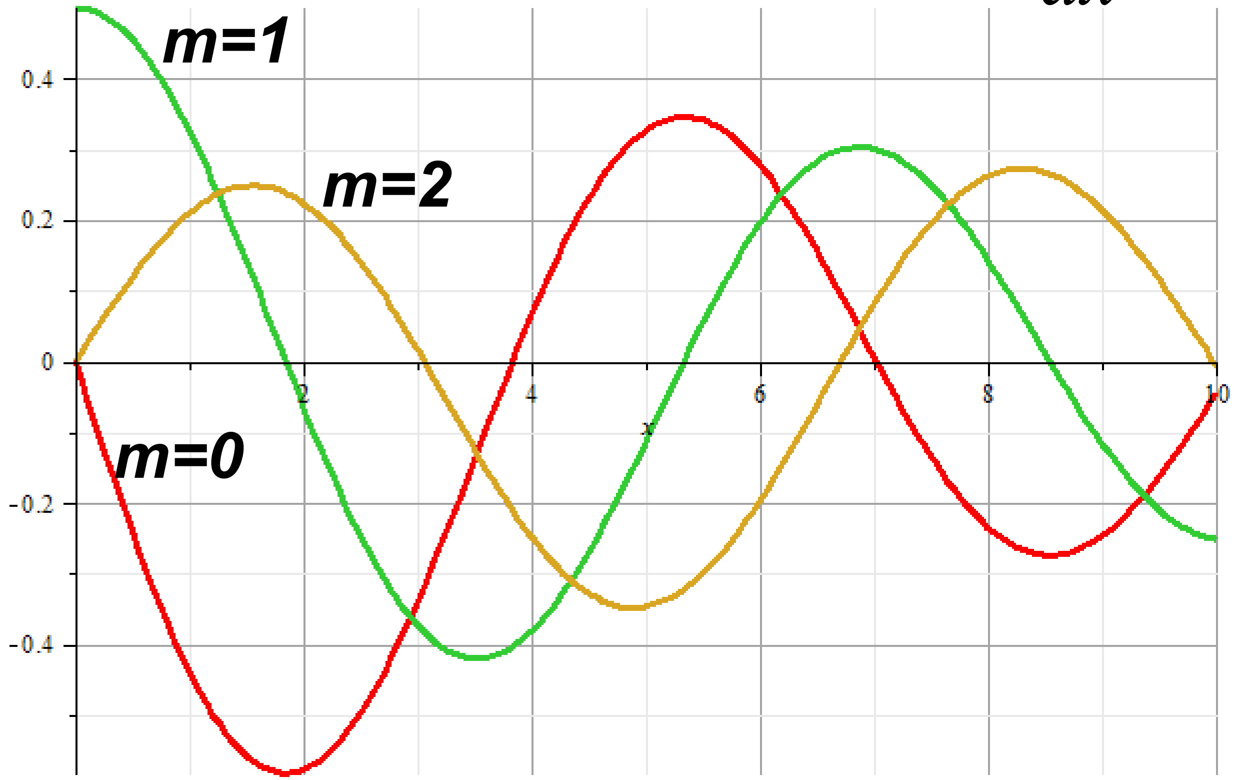
At free surface: $\dfrac{\partial \Phi}{\partial t} = 0$

$$\Phi(r,\varphi,z,t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$= J_m(\kappa r)e^{im\varphi}e^{i\alpha z}e^{-i\omega t}$$

$$\text{For} \quad r = a, \quad \left.\frac{\partial \Phi(r,\varphi,z,t)}{\partial r}\right|_{r=a} = 0$$

Bessel function derviatives : $\frac{dJ_m(x)}{dx}$

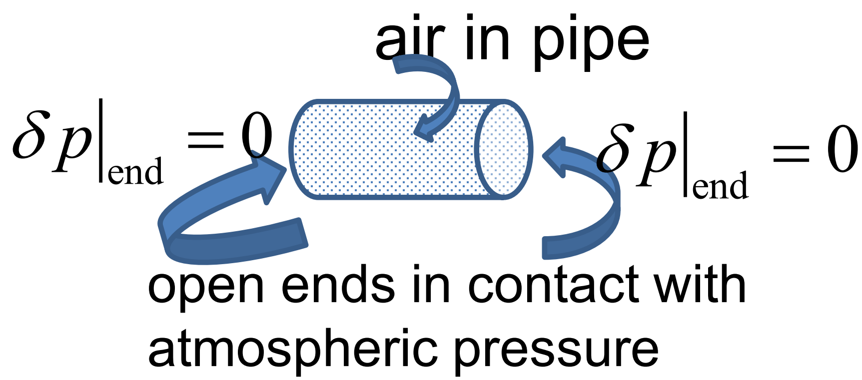


Zeros of derivatives: m=0: 0.00000, 3.83171, 7.01559
m=1: 1.84118, 5.33144, 8.53632
m=2: 3.05424, 6.70613, 9.96947

Some details on the open pipe boundary conditions --

Comment --

1. Open pipe boundary condition



Boundary condition for z=0, z=L:

For open - open pipe :

$$Z(0)=Z(L)=0 \qquad \Rightarrow \; Z(z)=\sin\left(\frac{p\pi z}{L}\right)$$

$$\Rightarrow \alpha_p=\frac{p\pi}{L}, \qquad p=1,2,3...$$

Resonant frequencies:

$$\frac{\omega^2}{c^2}=k^2=\kappa_{mn}^2+\alpha_p^2\equiv k_{mnp}^2$$

$$k_{mnp}^2=\left(\frac{x'_{mn}}{a}\right)^2+\left(\frac{\pi p}{L}\right)^2$$

Resonant frequencies: $\omega = ck_{mnp}$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2$$

Comment about units of frequency --

$$\Phi(\mathbf{r},t) = f(\mathbf{r})e^{-i\omega t} = f(\mathbf{r})e^{-2\pi i\nu t}$$

Note that ω has units of radians/sec

ν has units of cycles/sec (Hz)

$$\nu = \frac{\omega}{2\pi}$$

Example of open pipe of length L and radius a:

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2 = \left(\frac{\pi p}{L}\right)^2 \left(1 + \left(\frac{L}{a}\right)^2 \left(\frac{x'_{mn}}{\pi p}\right)^2\right)$$

$$\pi p = 3.14, 6.28, 9.42....$$

$$x'_{mn} = 0.00, 1.84, 3.05.... \quad \text{for } x'_{00}, x'_{10}, x'_{20}$$

$$\Phi(r,\varphi,z,t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$= J_m\left(\frac{x'_{mn}}{a} r\right) e^{im\varphi} \sin\left(\frac{p\pi}{L} z\right) e^{-i\omega t}$$

Alternate boundary condition for z=0, z=L:

For open - closed pipe :

$$\frac{dZ(0)}{dz} = Z(L) = 0 \qquad \Rightarrow \; Z(z) = \cos\left(\frac{(2p+1)\pi z}{2L}\right)$$

$$\Rightarrow \alpha_p = \frac{(2p+1)\pi}{2L}, \qquad p = 0,1,2,3...$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi(2p+1)}{2L}\right)^2$$

The above analysis pertains to resonant air waves within a cylindrical pipe. As previously mentioned, you can hear these resonances if you put your ear close to such a pipe. The same phenomenon is the basis of several musical instruments such as organ pipes, recorders, flutes, clarinets, oboes, etc.

Question – what about a trumpet, trombone, French horn, etc?

a. Same idea?
b. Totally different?

But for musical instruments, you do not want to put your ear next to the device – additional considerations must apply. Basically, you want to couple these standing waves to produce traveling waves.

Modifications needed for the pandemic --

Image from the Winston-Salem Journal 11/1/2020

For other instruments, the resonance is initiated by another resonant device which couples to air --

In order to understand how audible sound couples to sound wave resonances, consider the following simple model of a sound amplifier --

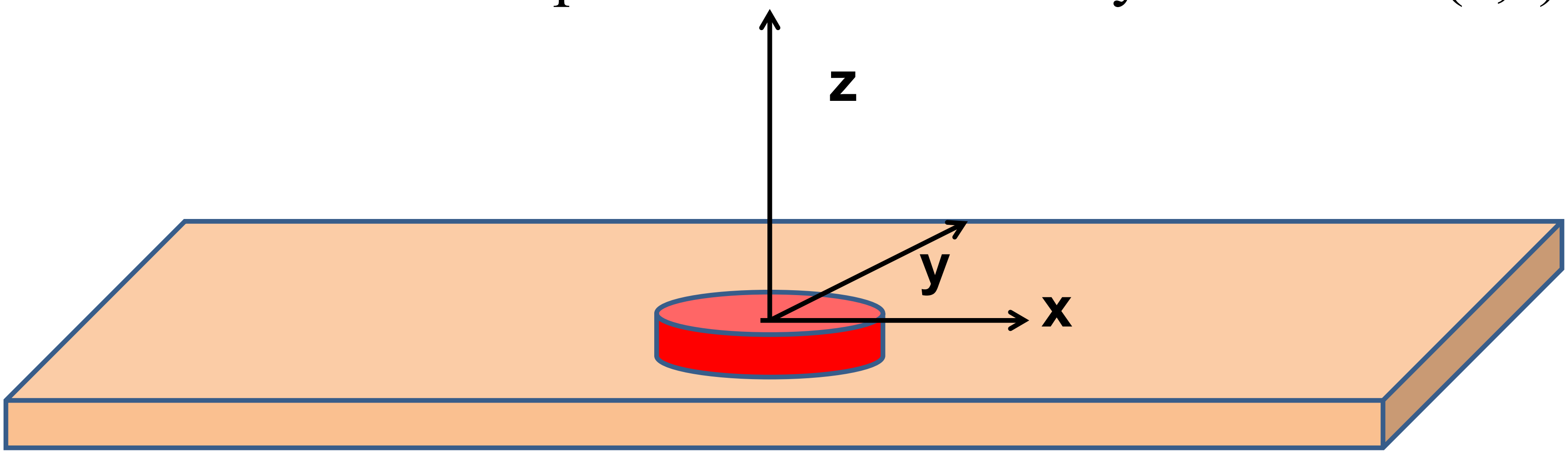
Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example:

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$

can be represented as boundary value of $\Phi(\mathbf{r},t)$



Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Solution in terms of Green's function:

$$\Phi(\mathbf{r},t) = \Phi_0(\mathbf{r},t) + \int d^3r' \int dt' \, G(\mathbf{r}-\mathbf{r}', t-t') f(\mathbf{r}',t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi_0(\mathbf{r},t) = 0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Recall that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}d\omega$$

Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that: $\int_V \left(h\nabla^2 g - g\nabla^2 h\right) d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot \hat{\mathbf{n}}\, d^2r$

$$\nabla^2\tilde{\Phi} + k^2\tilde{\Phi} = -\tilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

$$h \leftrightarrow \tilde{\Phi}; \qquad g \leftrightarrow \tilde{G}$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}}\, d^2r$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r},\omega) \right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega) \right) \cdot \hat{\mathbf{n}} d^2r$$

Exchanging $\mathbf{r} \leftrightarrow \mathbf{r}'$:

$$\int_V \left(\tilde{\Phi}(\mathbf{r}',\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega) \right) d^3r' =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega) \right) \cdot \hat{\mathbf{n}} d^2r'$$

If the integration volume V includes the point $\mathbf{r} = \mathbf{r}'$:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega) d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega) \right) \cdot \hat{\mathbf{n}} d^2r'$$

➔extra contributions from boundary

Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\tilde{f}(\mathbf{r}',\omega)d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

Boundary values for our example:

$$\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)_{z=0} = \begin{cases} 0 & \text{for } x^2+y^2>a^2 \\ i\omega\varepsilon a & \text{for } x^2+y^2<a^2 \end{cases}$$

Note: Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}' = -z'; \quad z>0$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)_{z'=0} = \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}; \quad z > 0$$

Some more details --

Note: Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}+\frac{e^{ik|\mathbf{r}-\overline{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\overline{\mathbf{r}}'|} \qquad \text{where } \overline{z}'=-z'; \quad z>0$$

Note that $|\mathbf{r}-\mathbf{r}'|\equiv\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$

$$|\mathbf{r}-\overline{\mathbf{r}}'|\equiv\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}$$

Fourier transform of velocity potential:

$$\tilde{\Phi}(\mathbf{r},\omega)=\int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)f(\mathbf{r}',\omega)d^3r'+$$

$$\oint_S\left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)-\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}'d^2r'$$

Need this term to vanish at z'=0

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$= -i\omega\varepsilon a\int_0^a r'\,dr'\int_0^{2\pi} d\phi'\left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$

Integration domain: $x' = r'\cos\varphi'$

$y' = r'\sin\varphi'$

For $r >> a$; $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\varphi = \frac{\pi}{2}$

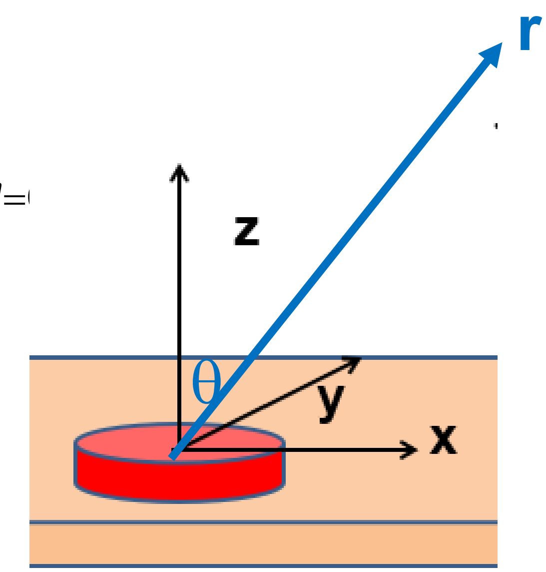
$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\varphi'$$

More details

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:z'=0} \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z'}dx'dy'$$

$$= -i\omega\varepsilon a\int_0^a r'dr'\int_0^{2\pi} d\varphi'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\Bigg|_{z'=0}$$



Integration domain: $x' = r'\cos\varphi'$

$y' = r'\sin\varphi'$

For $r >> a$; $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\varphi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\varphi'$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon a}{2\pi}\frac{e^{ikr}}{r}\int_0^a r'dr'\int_0^{2\pi} d\phi' e^{-ikr'\sin\theta\sin\phi'}$$

Note that : $$\frac{1}{2\pi}\int_0^{2\pi} d\phi' e^{-iu\sin\phi'} = J_0(u)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a\frac{e^{ikr}}{r}\int_0^a r'dr' J_0(kr'\sin\theta)$$

$$\int_0^w udu J_0(u) = wJ_1(w)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a^3\frac{e^{ikr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Energy flux : $\mathbf{j}_e = \delta\mathbf{v}p$

Taking time average : $\left\langle \mathbf{j}_e \right\rangle = \frac{1}{2}\Re\left(\delta\mathbf{v}p^*\right)$

$$= \frac{1}{2}\rho_0 \Re\left(\left(-\nabla\Phi\right)\left(-i\omega\Phi\right)^*\right)$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \left\langle \mathbf{j}_e \right\rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2}\rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka \sin\theta)}{ka \sin\theta} \right|^2$$