PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 32: Chap. 9 of F&W

Linear and non-linear sound waves

1. Summary of linear sound phonenmena
2. Introduction to non-linear effects
3. Analysis of instability – shock phenomena

Physics Colloquium

Thursday

November 10th, 2022

A Tale of Cosmic Ecosystems: How galaxies evolve

In recent years, our understanding of how galaxies form and evolve have matured dramatically. The latest numerical simulations can successfully reproduce both the stellar content of galaxies and their large scale statistical properties. However, these models still fail to match the observed properties of the diffuse gas, which span hundreds of kiloparsecs beyond the visible stellar disks of the galaxies. Understanding the complex physical processes that dictate this circumgalactic space is a crucial next step towards creating a comprehensive model of galaxy evolution. I will highlight some of our recent results in characterizing this circumgalactic gas both in small and large scales. I will also relate how ubiquitously observed circumgalactic HI relates with the ionized reservoirs of circumgalactic gas around galaxies. These will provide empirical bedrocks to identify the dominant mechanisms that govern the circumgalactic gas in driving galaxy evolution. Lastly, I will highlight some exciting insights from latest James Webb Space Telescope observations and how it is reshaping our understanding of the early

Rongmon Bordoloi
Assistant Professor, NCSU

4:00 pm - Olin 101*

*Link provided for those unable to attend in person.

Note: For additional information on the seminar or to obtain the video conference link, contact wfuphys@wfu.edu

Reception at 3:30pm - Olin Entrance

Signup with name and topic at 9 AM on Friday 11/11/2022

PHY 711 Presentation Schedule for Fall 2022

Monday, November 28, 2022

	Name	Title/Topic
10:00-10:15		
10:17-10:32		
10:35-10:50		

Wednesday, November 30,, 2022

	Name	Title/Topic
10:00-10:15		
10:17-10:32		
10:35-10:50		

Friday, December 2, 2022

	Name	Title/Topic
10:00-10:15		
10:17-10:32		
10:35-10:50		

28	Mon, 10/31/2022	Chap. 9	Mechanics of 3 dimensional fluids	#21	11/02/2022
29	Wed, 11/02/2022	Chap. 9	Mechanics of 3 dimensional fluids	#22	11/04/2022
30	Fri, 11/04/2022	Chap. 9	Linearized hydrodynamics equations	#23	11/07/2022
31	Mon, 11/07/2022	Chap. 9	Linear sound waves	#24	11/09/2022
32	Wed, 11/09/2022	Chap. 9	Scattering of sound and non-linear effects	#25	11/11/2022
33	Fri, 11/11/2022	Chap. 10	Surface waves int fluids		
34	Mon, 11/14/2022	Chap. 10	Surface waves in fluids; soliton solutions		
35	Wed, 11/16/2022	Chap. 11	Heat conduction		
36	Fri, 11/18/2022	Chap. 12	Viscous effects on hydrodynamics		
37	Mon, 11/21/2022	Chap 1-12	Review		
	Wed, 11/23/2022		Thanksgiving Holiday		
	Fri, 11/25/2022		Thanksgiving Holiday		
	Mon, 11/28/2022		Presentations I		
	Wed, 11/30/2022		Presentations II		
	Fri, 12/02/2022		Presentations III		

PHY 711 -- Assignment #25

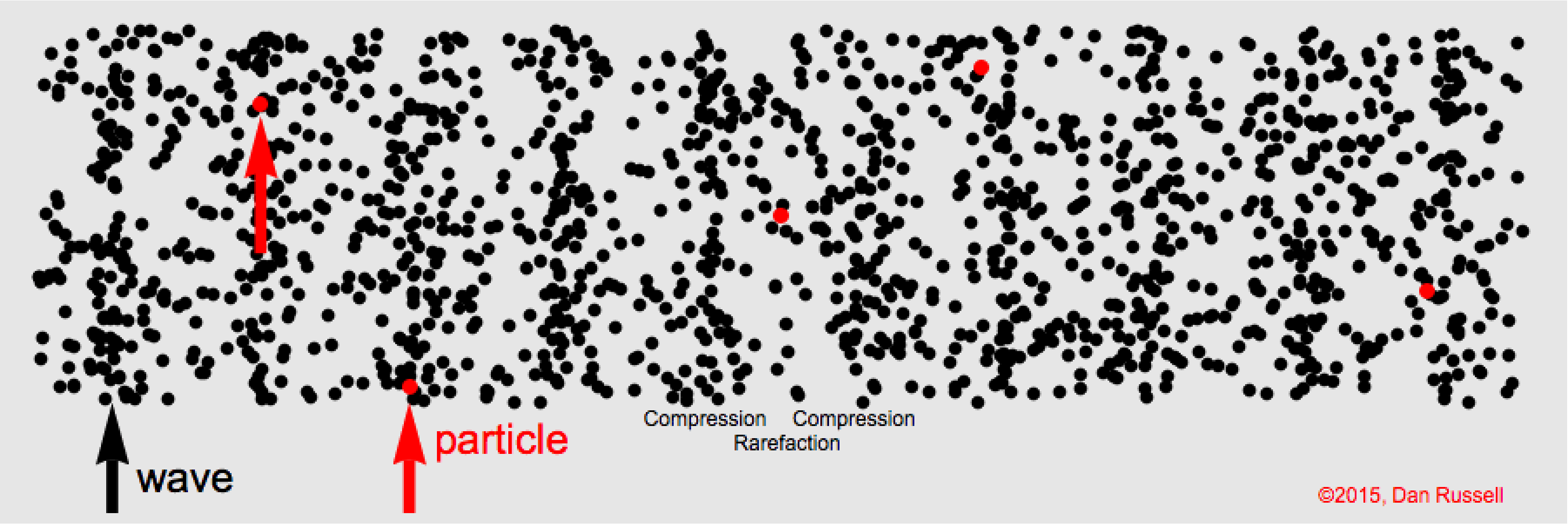
Nov. 09, 2022

Finish reading Chapter 9 in **Fetter & Walecka**.

1. Assume the ideal gas law and adiabatic conditions for He gas, having an initial pressure of p_0= 101325 Pa (1 atm) and initial temperature of T_0=300K. Calculate the following when the pressure is changed $p_1=2p_0$.
   a. T_1.
   b. The change in the internal energy per unit mass $\Delta\ \varepsilon$.
   c. The change in the entropy per unit mass $\Delta\ s$

Visualization of longitudinal wave motion

From the website:
https://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html



Continuing to discuss linear sound waves – Scattering from a rigid cylinder

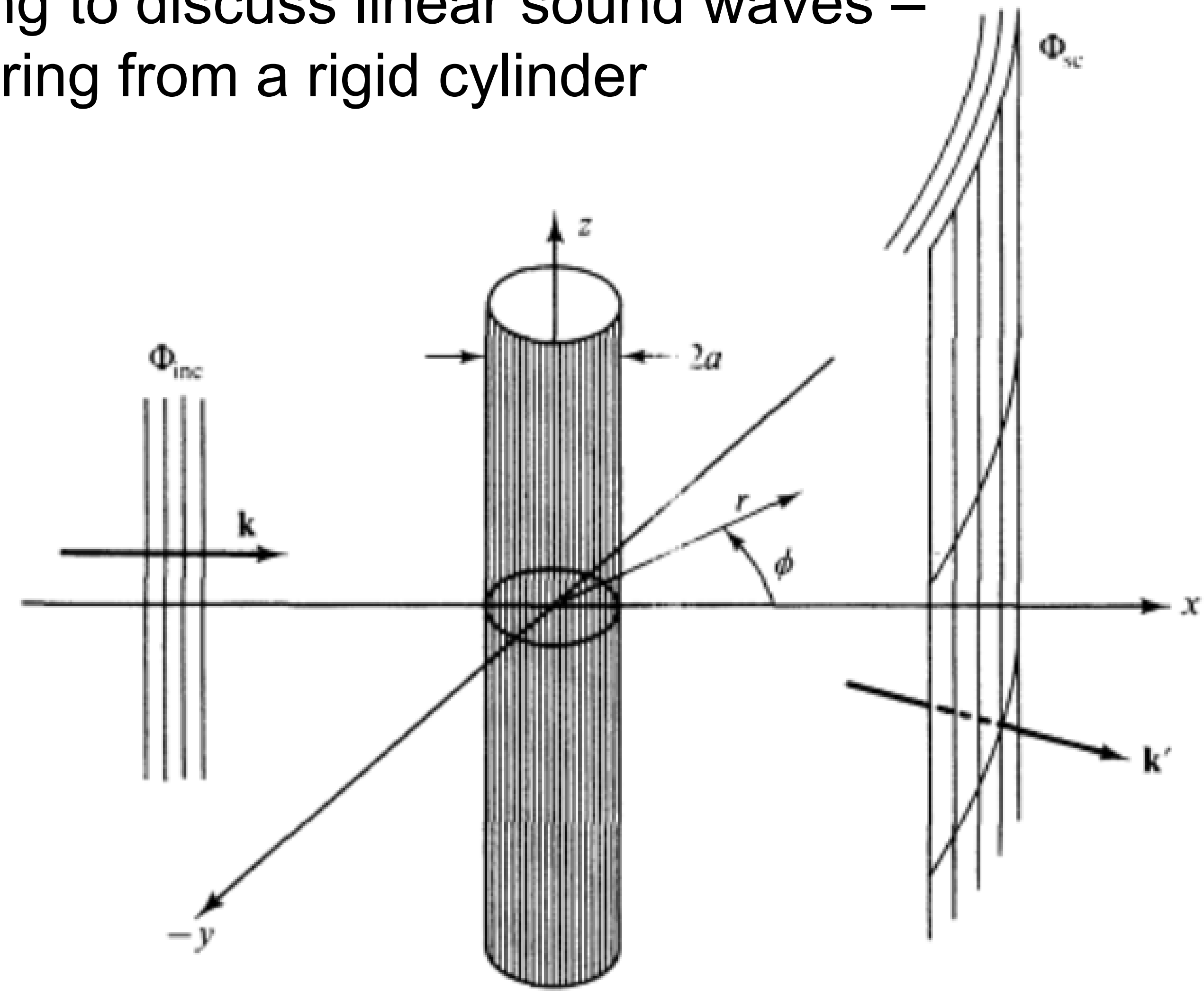


Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

Example of cylindrical scattering objects --

Suppose a trumpeter is playing near the columns. Maximal scattering occurs when
a. Facing toward the column b. Facing away from the column.

Scattering of sound waves – for example, from a rigid cylinder

Velocity potential --

$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \qquad \Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$$

Helmholz equation in cylindrical coordinates:

$$\left(\nabla^2 + k^2\right)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial z^2} + k^2\right)\Phi(\mathbf{r})$$

Assume: $\Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi} R_m(r)$

where $$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right) R_m(r) = 0$$

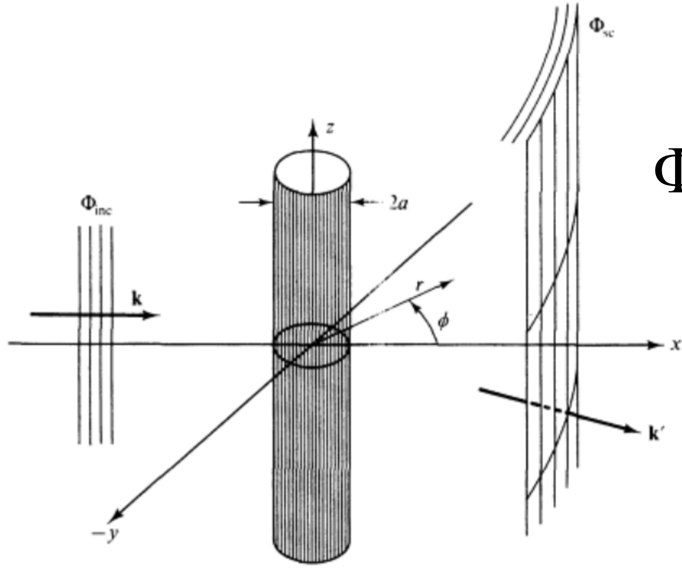


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

$$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr) \quad \text{where Hankel function}$$

represents an outgoing wave: $\quad H_m(kr) = J_m(kr) + iN_m(kr)$

Boundary condition at $r = a$: $\quad \left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \qquad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

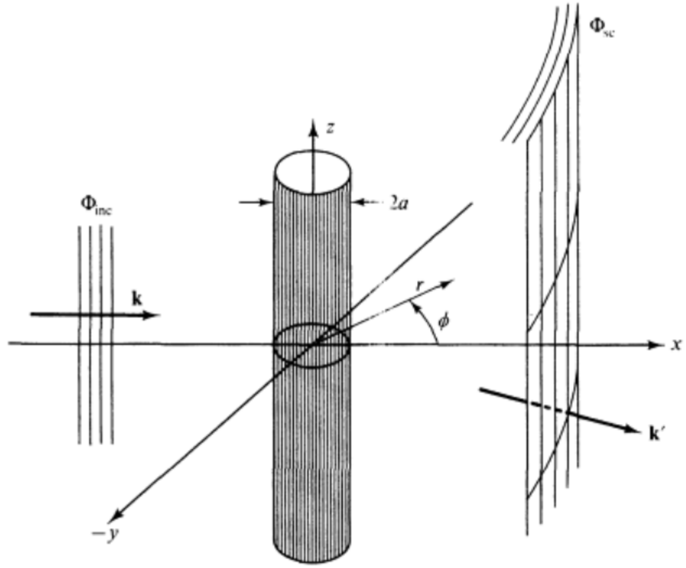


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = -\sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \underset{kr\to\infty}{\approx} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \underset{kr\to\infty}{\approx} f(\phi)\sqrt{\frac{1}{r}} e^{ikr} = -\sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Rightarrow f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi-\pi/4)}$$

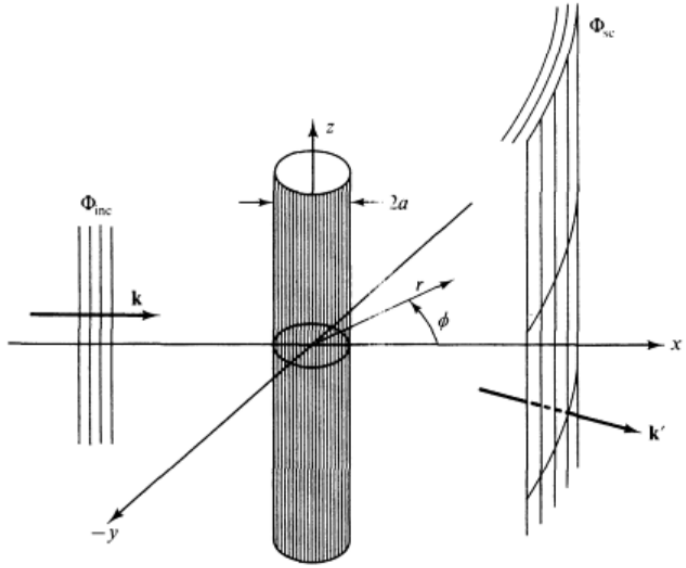
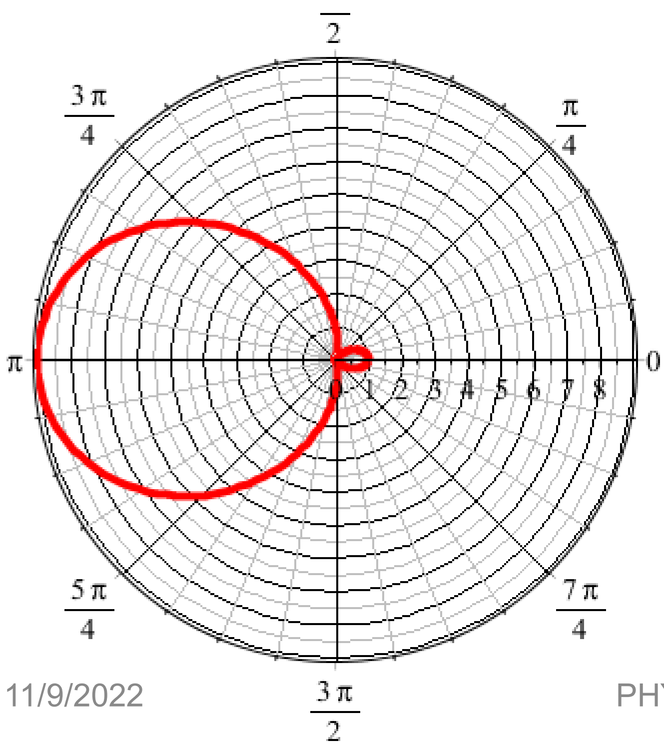


Figure 51.8 Scattering from a rigid cylinder.



$$\frac{d\sigma}{d\phi} = \left|f(\phi)\right|^2$$

$$f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi - \pi/4)}$$

For *ka* << *1*

$$\frac{d\sigma}{d\phi} = \left|f(\phi)\right|^2 \approx \frac{1}{8}\pi k^3 a^4 \left(1 - 2\cos\phi\right)^2$$

Revisiting the trumpeter question --

Conclusion – be careful when choosing a place to play your trumpet --

Now consider some non-linear effects in sound

Examples?

We will consider the simple case –

1. One dimension for motion
2. Fluid is assumed to be an ideal gas
3. Adiabatic conditions
4. All variables will be expressed in terms of the density $\rho(x,t)$

Effects of nonlinearities in fluid equations
-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Assume spatial variation confined to x direction ;

assume that $\mathbf{v} = v\hat{\mathbf{x}}$ and $\mathbf{f}_{applied} = 0.$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing p in terms of ρ: $\quad p = p(\rho)$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} \equiv c^2(\rho)\frac{\partial \rho}{\partial x} \qquad \text{where} \quad \frac{\partial p}{\partial \rho} \equiv c^2(\rho)$$

For adiabatic ideal gas: $\quad \frac{dp}{p} = \gamma\frac{d\rho}{\rho} \qquad p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}$

$$c^2(\rho) = \frac{\gamma p}{\rho} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \qquad \text{where} \quad c_0^2 \equiv \frac{\gamma p_0}{\rho_0}$$

Digression – What is gamma?

Internal energy for ideal gas: $pV = Nk_BT$

$E_{\text{int}} = \frac{f}{2} Nk_BT$ $f \equiv$ degrees of freedom; 3 for atom, 5 for diatomic molecule

In terms of specific heat ratio: $\gamma \equiv \frac{C_p}{C_V}$

$$dE_{\text{int}} = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2} Nk_B$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2} Nk_B + Nk_B$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} = 1 + \frac{2}{f} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1} \qquad E_{\text{int}} = \frac{1}{\gamma - 1} Nk_BT$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing variation of v in terms of $v\left(\rho\right)$:

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t} + v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} = 0$$

Some more algebra :

From Euler equation : $\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$

From continuity equation : $\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}=-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}$

Combined equation : $\frac{\partial v}{\partial \rho}\left(-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$

$$\Rightarrow \left(\frac{\partial v}{\partial \rho}\right)^2=\frac{c^2(\rho)}{\rho^2} \qquad \frac{\partial v}{\partial \rho}=\pm\frac{c}{\rho}$$

$$\Rightarrow \frac{\partial \rho}{\partial t}+(v\pm c)\frac{\partial \rho}{\partial x}=0$$

Assuming adiabatic process: $c^2 = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\qquad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$\Rightarrow \; c = c_0 \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}$$

$$\frac{\partial v}{\partial \rho} = \frac{dv}{d\rho} = \pm \frac{c}{\rho} \qquad \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho} \left(\frac{\rho'}{\rho_0}\right)^{(\gamma-1)/2} \frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm \frac{2c_0}{\gamma - 1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

Summary :

$$\frac{dv}{d\rho} = \pm \frac{c}{\rho}$$

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assuming adiabatic process : $c^2 = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\quad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$c = c_0 \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} \qquad v = \pm \frac{2c_0}{\gamma - 1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

Traveling wave solution:

Assume: $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Apparently: $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Traveling wave solution -- continued:

$$\frac{\partial\rho}{\partial t}+(v\pm c)\frac{\partial\rho}{\partial x}=0$$

$$\text{Assume:}\quad \rho=\rho_0+f(x-u(\rho)t)=\rho_0+f(x-(v\pm c)t)$$

For adiabatic ideal gas and + signs:

$$u=v+c=c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}-\frac{2}{\gamma-1}\right)$$

Solution in linear approxiation:

$$u=v+c\approx v_0+c_0=c_0\left(\frac{\gamma+1}{\gamma-1}-\frac{2}{\gamma-1}\right)=c_0$$

$$\Rightarrow \rho=\rho_0+f\left(x-c_0t\right)$$

Some details

Assume: $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Apparently: $u(\rho) \Leftrightarrow v \pm c$

Note that for $u = v + c$ (choice of + solution)

$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0$ is satisfied by a function of the form

$$\rho(x,t) = \rho_0 + f(x - u(\rho(x,t))t)$$

Let $w \equiv x - u(\rho(x,t))t$

$$\frac{df}{dw}\frac{\partial w}{\partial t} + u\frac{df}{dw}\frac{\partial w}{\partial x} = \frac{df}{dw}(-u + u) = 0$$

Traveling wave solution -- full non-linear case:

Visualization for particular waveform: $\rho = \rho_0 + f(\underbrace{x - u(\rho)t}_{w})$

Assume: $f(w) \equiv \rho_0 s(w)$

$$\frac{\rho}{\rho_0} = 1 + s(x - ut)$$

For adiabatic ideal gas:

$$u = c_0 \left(\frac{\gamma+1}{\gamma-1} \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

$$u = c_0 \left(\frac{\gamma+1}{\gamma-1} \left(1 + s(x - ut) \right)^{(\gamma-1)/2} - \frac{2}{\gamma-1} \right)$$

Visualization continued:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Plot $s(x-ut)$ for fixed t, as a function of x:

Let $w = x - ut$

$$x = w + ut = w + u(w)t \equiv x(w,t)$$

$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations:

plot $s(w)$ vs $x(w,t)$ for range of w at each t

Summary

$$\frac{\partial \rho}{\partial t} + u(\rho)\frac{\partial \rho}{\partial x} = 0$$

Solution: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0\big(1 + s(x - u(\rho)t)\big)$

For linear case: $u(\rho) = c_0$

For non-linear case: $u(\rho) = c_0\left(\frac{\gamma+1}{\gamma-1}\big(1 + s(x - ut)\big)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$

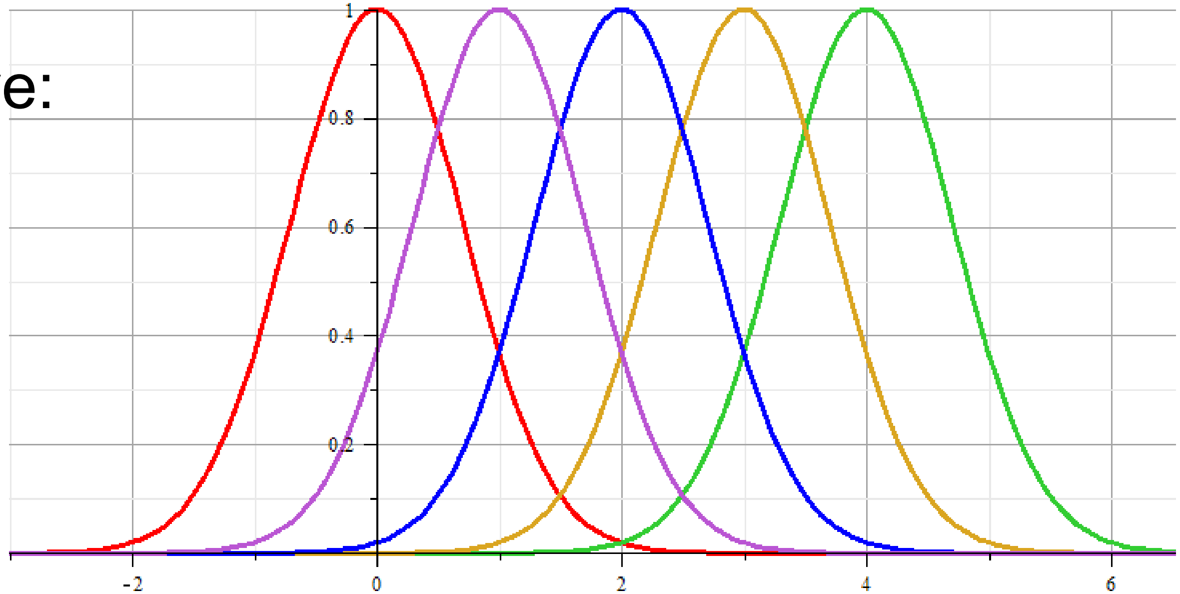
Plot $s(x - ut)$ for fixed t, as a function of x:

Let $w = x - ut \quad \Rightarrow \quad x = w + ut = w + u(w)t \equiv x(w,t)$

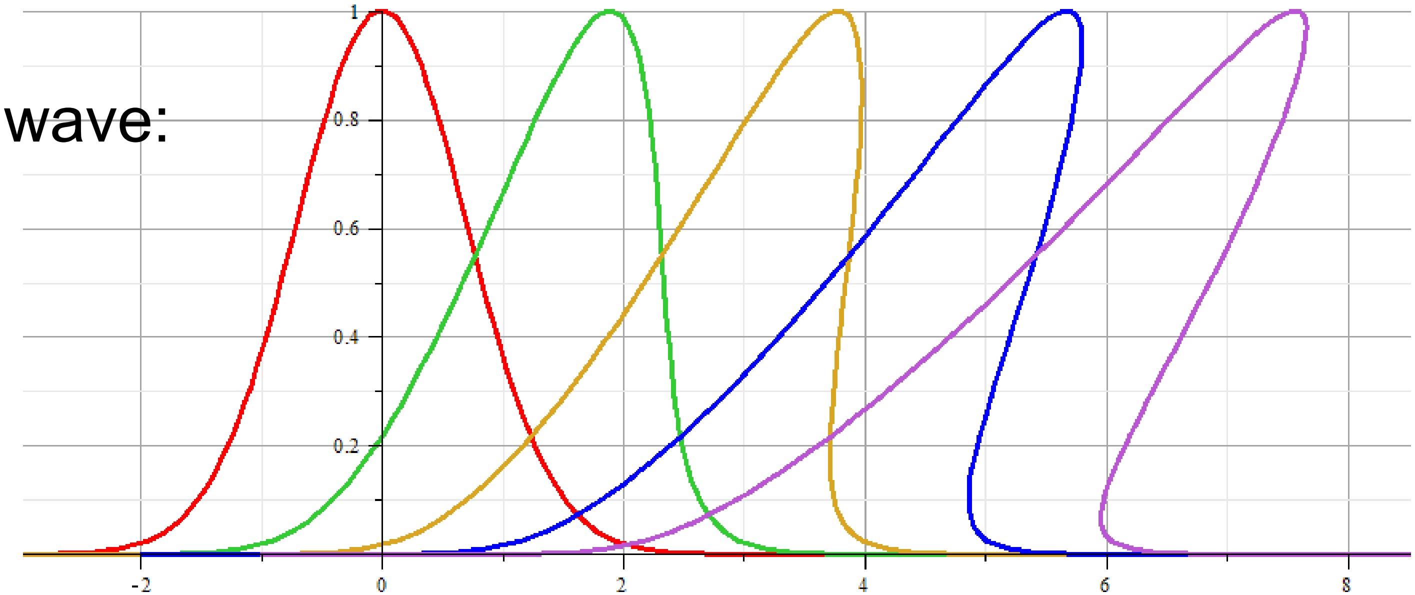
$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\big(1 + s(w)\big)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Parametric equations: plot $s(w)$ vs $x(w,t)$ for range of w

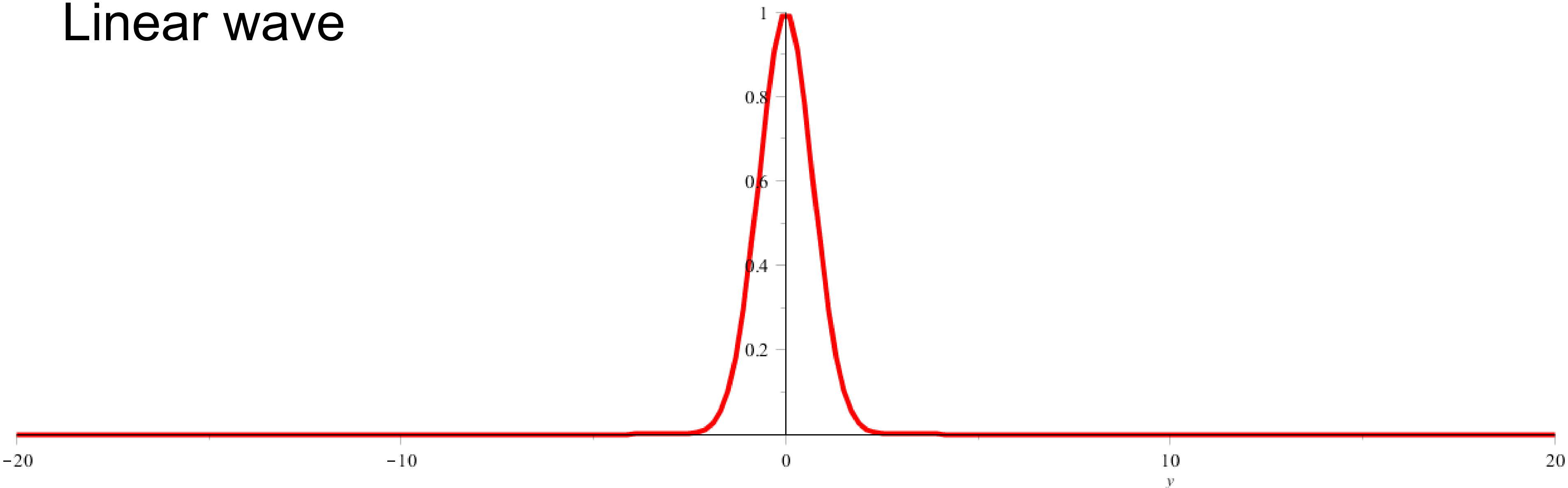
Linear wave:



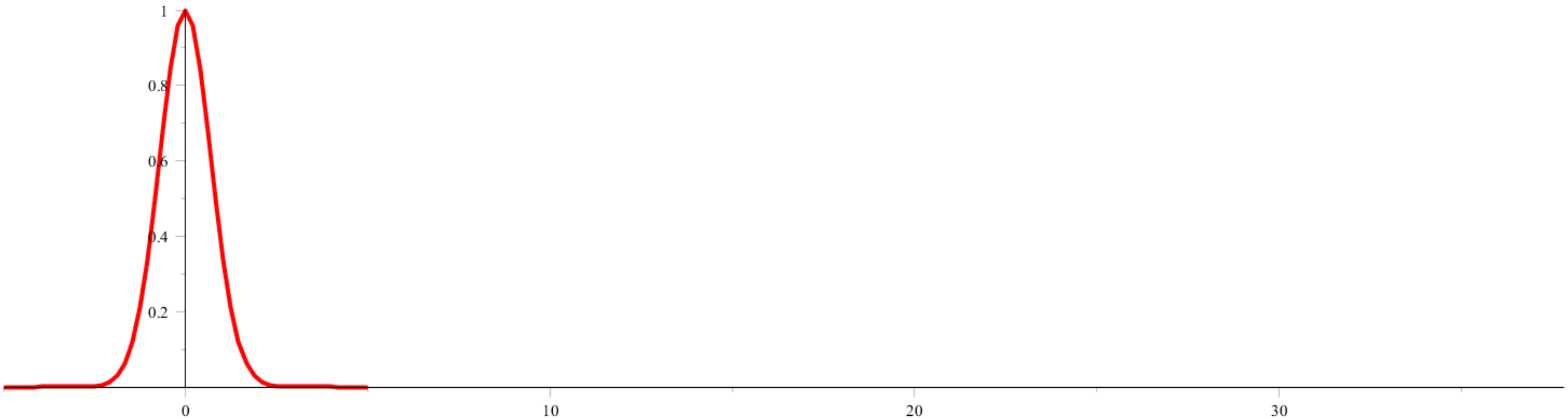
Non-linear wave:



Linear wave

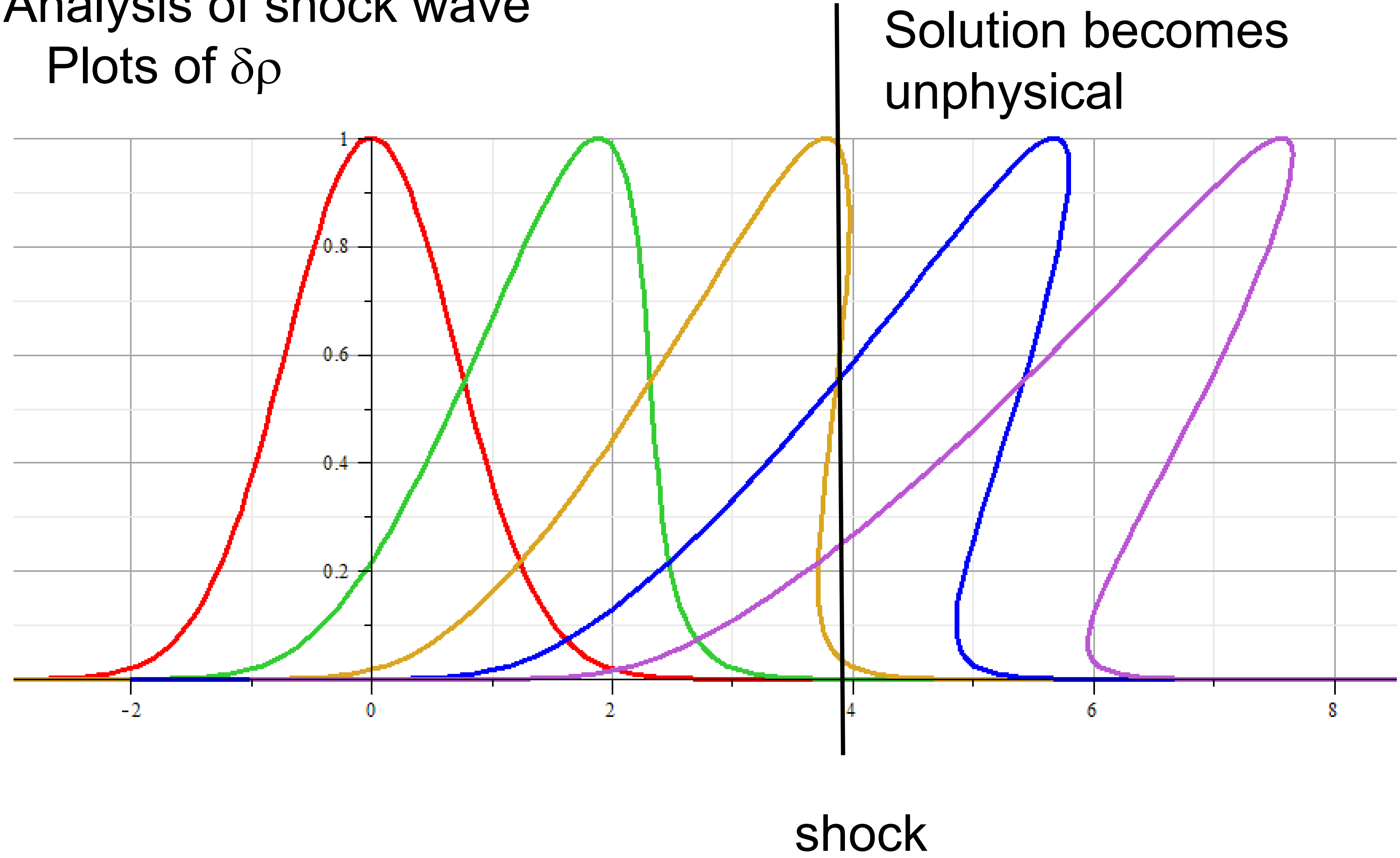


Non-linear wave

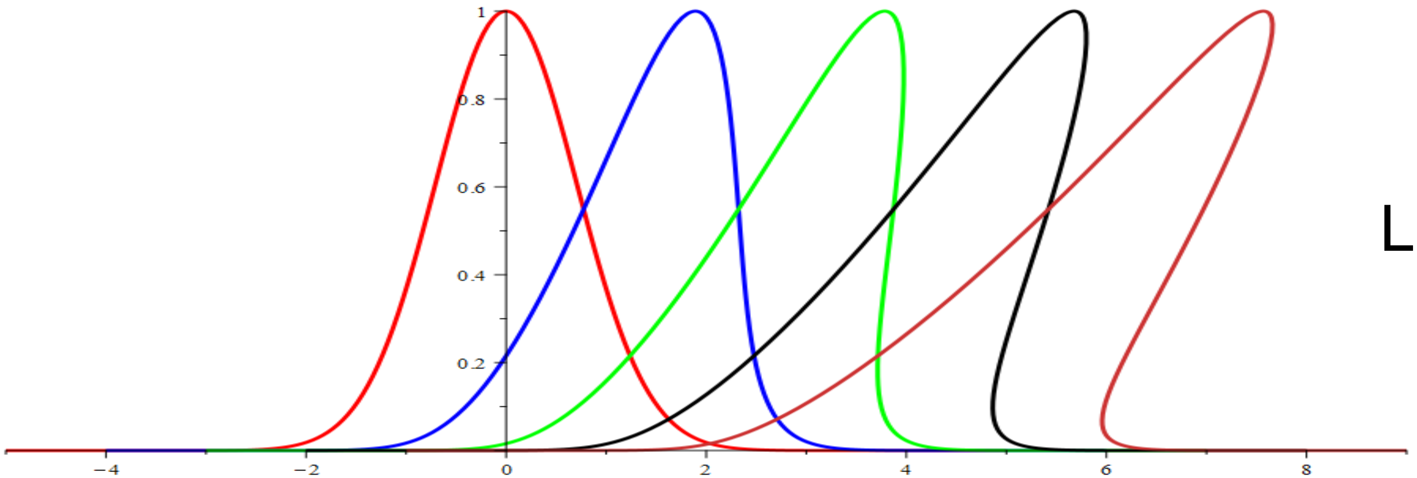


Analysis of shock wave
Plots of $\delta\rho$

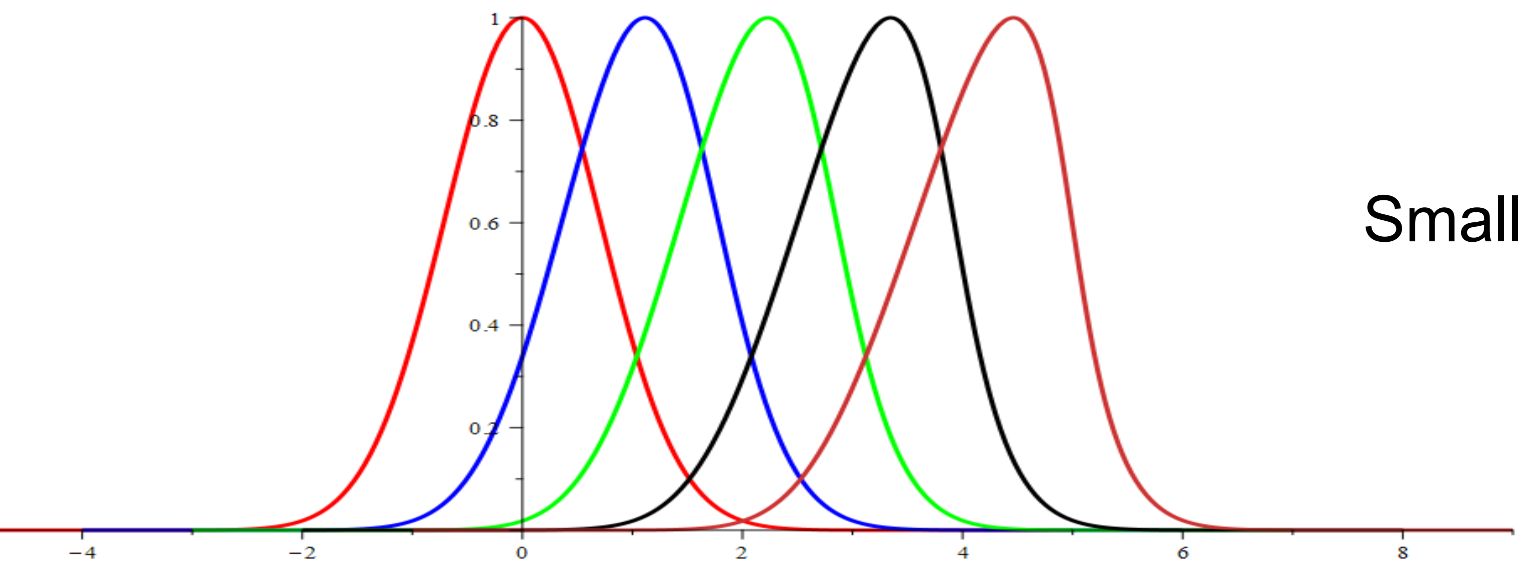
Solution becomes unphysical



Effects of amplitude of $\delta\rho$

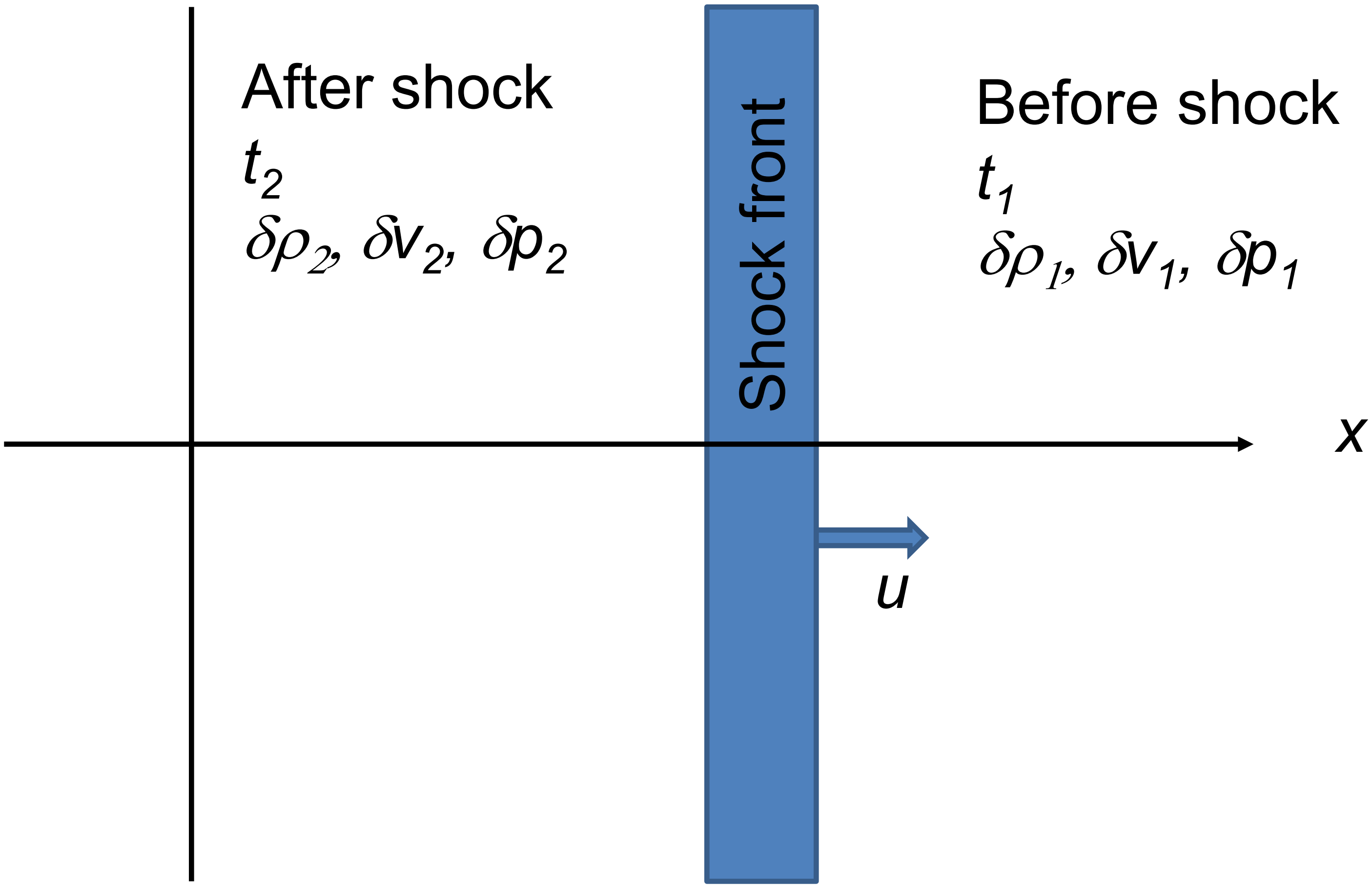


Large amplitude



Smaller amplitude

Analysis of shock wave – assumed to moving at velocity u

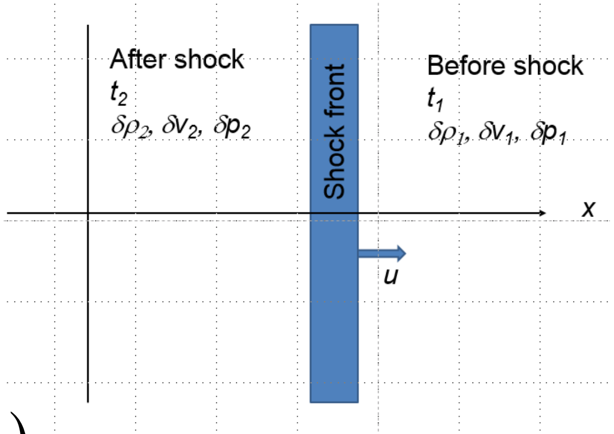


Note that in this case u is assumed to be a given parameter of the system.

Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2

Assume
$$\rho(x,t)=\rho(x-ut)$$
$$p(x,t)=p(x-ut)$$
$$v(x,t)=v(x-ut)$$



Continuity equation:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0=\frac{\partial(\rho v-\rho u)}{\partial x} \quad \Rightarrow (v_2-u)\rho_2=(v_1-u)\rho_1$$

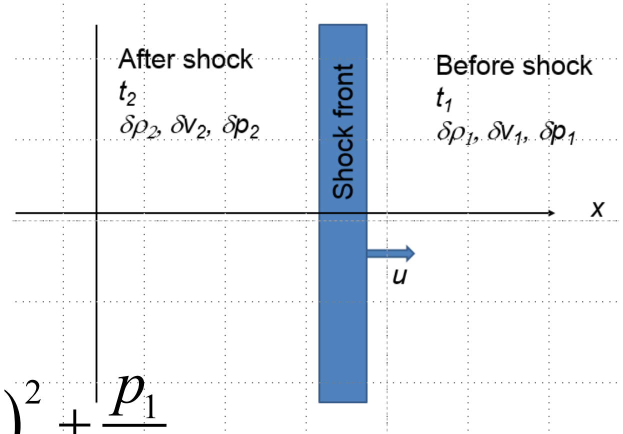
Conservation of energy and momentum:

$$\Rightarrow p_2+\rho_2(v_2-u)^2=p_1+\rho_1(v_1-u)^2$$

$$\Rightarrow \epsilon_2+\frac{1}{2}(v_2-u)^2+\frac{p_2}{\rho_2}=\epsilon_1+\frac{1}{2}(v_1-u)^2+\frac{p_1}{\rho_1}$$

Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2



Summary of equations

$$\Rightarrow (v_2 - u)\rho_2 = (v_1 - u)\rho_1$$

$$\Rightarrow p_2 + \rho_2 (v_2 - u)^2 = p_1 + \rho_1 (v_1 - u)^2$$

$$\Rightarrow \epsilon_2 + \frac{1}{2}(v_2 - u)^2 + \frac{p_2}{\rho_2} = \epsilon_1 + \frac{1}{2}(v_1 - u)^2 + \frac{p_1}{\rho_1}$$

Assume that within each regions (1 & 2), the ideal gas equations apply

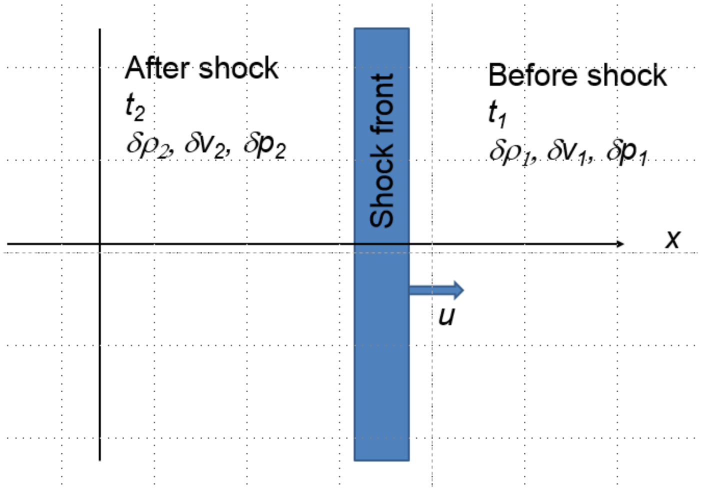
$$\epsilon_1 + \frac{p_1}{\rho_1} = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} \qquad \epsilon_2 + \frac{p_2}{\rho_2} = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}$$

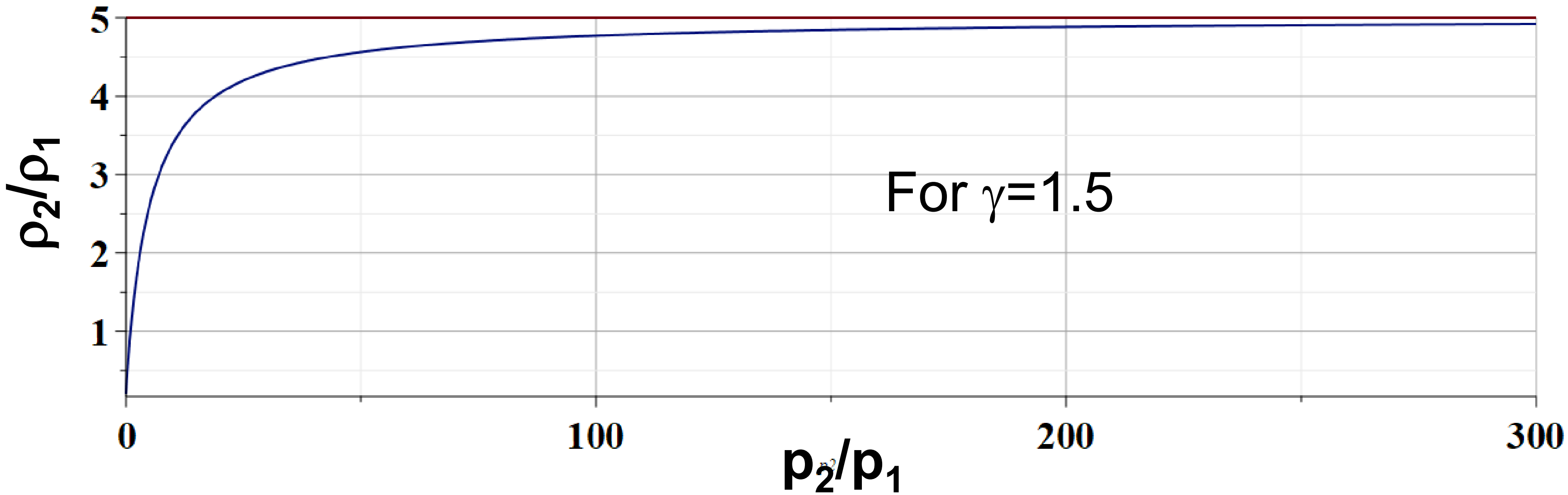
It follows that $$\frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}(v_2 - u)^2 = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}(v_1 - u)^2$$

Analysis of shock wave – continued

For adiabatic ideal gas, also considering energy and momentum conservation:

$$\frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}+1}{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}} \leq \frac{\gamma+1}{\gamma-1}$$





Analysis of shock wave – continued

For adiabatic ideal gas, entropy considerations::

Ideal gas law: $\frac{p}{\rho} = \frac{k_B T}{M_0}$ Adiabatic ideal gas: $\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma}$

Internal energy density: $\varepsilon \equiv \frac{E_{int}}{M} = \frac{p}{(\gamma - 1)\rho} = \frac{k_B T}{(\gamma - 1) M_0} \equiv c_V T$

First law of thermo: $d\varepsilon = T ds - p d\left(\frac{1}{\rho}\right)$

$$ds = \frac{1}{T}\left(d\left(\frac{p}{(\gamma-1)\rho}\right) + p d\left(\frac{1}{\rho}\right)\right) = \frac{p}{(\gamma-1)\rho T}\left(\frac{dp}{p} - \gamma\frac{d\rho}{\rho}\right) = c_V d \ln\left(\frac{p}{\rho^\gamma}\right)$$

$$s = c_V \ln\left(\frac{p}{\rho^\gamma}\right) + (\text{constant})$$

$$s_2 - s_1 = c_V \ln\left(\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right) \qquad 0 < s_2 - s_1 < c_V\left(\ln\left(\frac{p_2}{p_1}\right) - \gamma \ln\left(\frac{\gamma+1}{\gamma-1}\right)\right)$$