PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 34: Chapter 10 in F & W

Surface waves

- **Summary of linear surface wave solutions**
- **Non-linear contributions and soliton solutions**

This material is covered in Chapter 10 of your textbook using similar notation.

28	Mon, 10/31/2022	Chap. 9	Mechanics of 3 dimensional fluids	#21	11/02/2022
29	Wed, 11/02/2022	Chap. 9	Mechanics of 3 dimensional fluids	#22	11/04/2022
30	Fri, 11/04/2022	Chap. 9	Linearized hydrodynamics equations	#23	11/07/2022
31	Mon, 11/07/2022	Chap. 9	Linear sound waves	#24	11/09/2022
32	Wed, 11/09/2022	Chap. 9	Scattering of sound and non-linear effects	#25	11/11/2022
33	Fri, 11/11/2022	Chap. 10	Surface waves in fluids	#26	11/16/2022
34	Mon, 11/14/2022	Chap. 10	Surface waves in fluids; soliton solutions		
35	Wed, 11/16/2022	Chap. 11	Heat conduction		
36	Fri, 11/18/2022	Chap. 12	Viscous effects on hydrodynamics		
37	Mon, 11/21/2022	Chap 1-12	Review		
	Wed, 11/23/2022		Thanksgiving Holiday		
	Fri, 11/25/2022		Thanksgiving Holiday		
	Mon, 11/28/2022		Presentations I		
	Wed, 11/30/2022		Presentations II		
	Fri, 12/02/2022		Presentations III		

Note: No new HW assignments. Please use your extra time to prepare for your presentations, complete any outstanding assignments, and reviewing the material

PHY 711 Presentation Schedule for Fall 2022

Monday, November 28, 2022

	Name	Title/Topic
10:00-10:15	Lee Pryor	Foucault Pend. on a spinning torus
10:17-10:32		
10:35-10:50		

Wednesday, November 30,, 2022

	Name	Title/Topic
10:00-10:15	Katie Koch	2D Wave Equation
10:17-10:32	Banasree Sarkar Mou	Moment of inertia tensor of rigid body and the dynamics of spinning top
10:35-10:50	Arezoo Nameny	Foucault pendulum

Friday, December 2, 2022

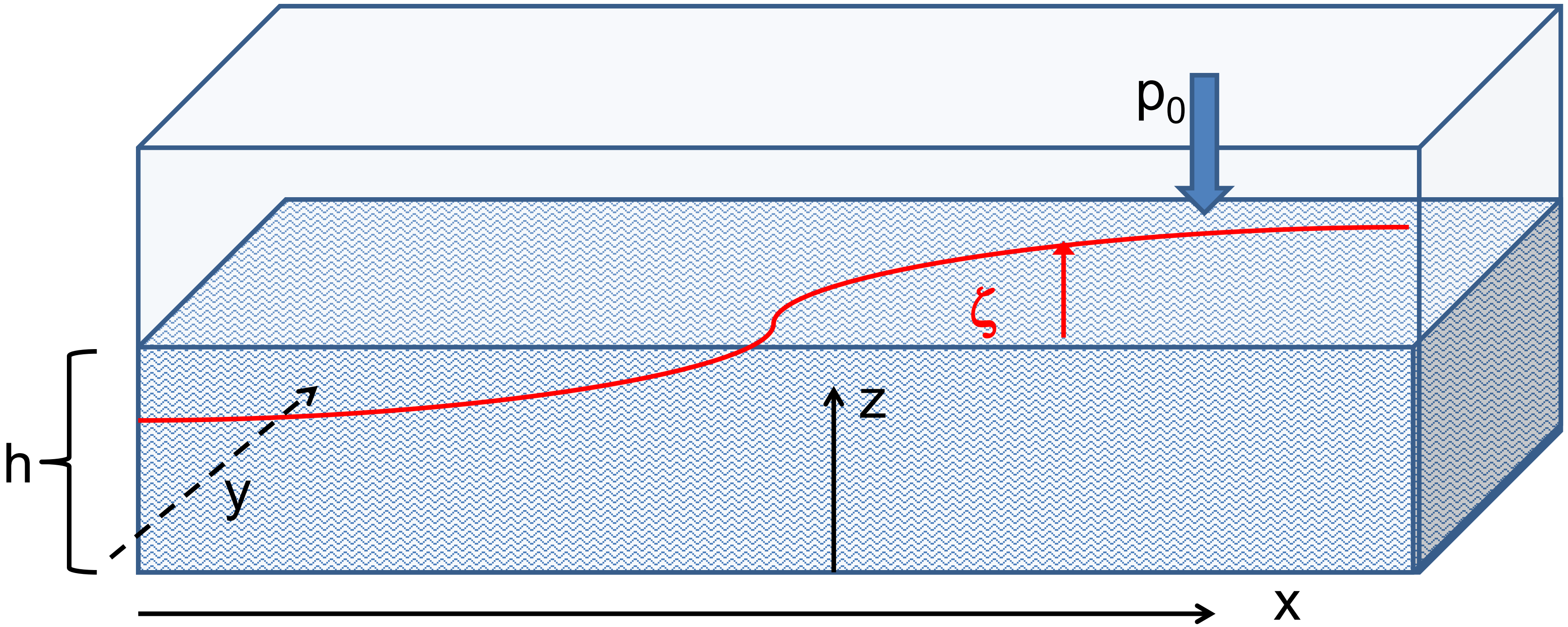
	Name	Title/Topic
10:00-10:15	Zezhong Zhang	Acoustic Tweezer
10:17-10:32	Athul Prem	Green's function methods
10:35-10:50	Evan Kumar	Fourier/Laplace Transforms

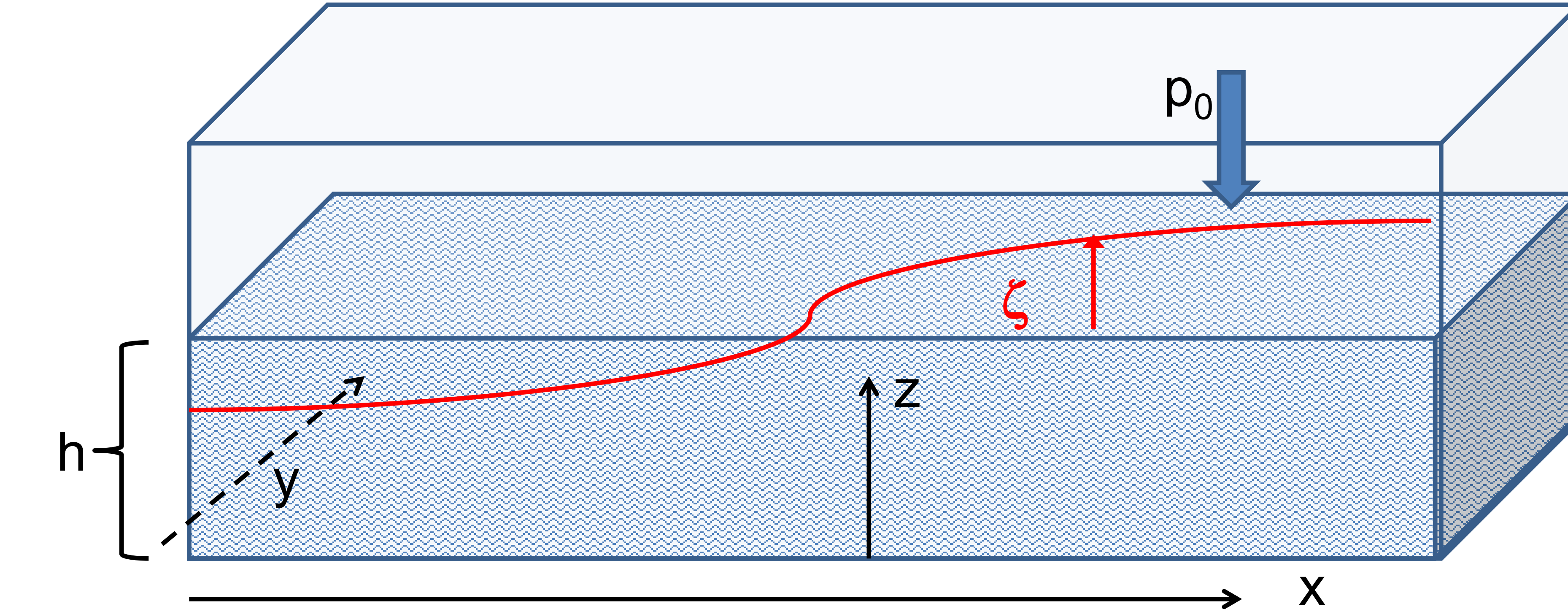
Note about presentations

- Please consult with me if you have any questions.
- Please prepare a ~ 10 minute presentation using powerpoint or equivalent software, leaving ~ 5 minutes for questions
- Please turn in your presentation, your preparation notes, mathematica or maple work if appropriate. If your topic follows a paper or write-up from the literature, please also include a copy of that.

Consider a container of water with average height h and surface h+$\zeta(x,y,t)$

Atmospheric pressure p_0 is in equilibrium at the surface





Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -\nabla U - \frac{\nabla p}{\rho}$$

Continuity equation within the fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{v} = 0$$

For irrotational flow -- $\mathbf{v} = -\nabla \Phi$

Linearized equation: $\nabla\left(-\frac{\partial \Phi}{\partial t} + g(z-h) + \frac{p}{\rho}\right) = 0$

At surface: $z = h + \zeta \qquad -\frac{\partial \Phi}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

Keep only linear terms and assume that horizontal variation is only along *x*:

For $0 \le z \le h+\zeta$: $\nabla^2\Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}\right)\Phi(x,z,t) = 0$

Consider and periodic waveform: $\Phi(x,z,t) = Z(z)\cos\big(k(x-ct)\big)$

$$\Rightarrow \left(\frac{d^2}{dz^2} - k^2\right)Z(z) = 0$$

Boundary condition at bottom of tank: $v_z(x,0,t) = 0$

$$\Rightarrow \frac{dZ}{dz}(0) = 0 \qquad Z(z) = A\cosh(kz)$$

At surface: $z = h+\zeta$ $\quad \frac{\partial \zeta}{\partial t} = v_z\left(x, h+\zeta, t\right) = -\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial z}$

Also: $\quad -\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0$

$$\Rightarrow -\frac{\partial^2 \Phi\left(x,h+\zeta,t\right)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi\left(x,h+\zeta,t\right)}{\partial t^2} - g\frac{\partial \Phi\left(x,h+\zeta,t\right)}{\partial z} = 0$$

Velocity potential: $\Phi(x,z,t) = A\cosh(kz)\cos\left(k(x-ct)\right)$

At surface: $\Phi(x,\left(h+\zeta\right),t) = A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)$

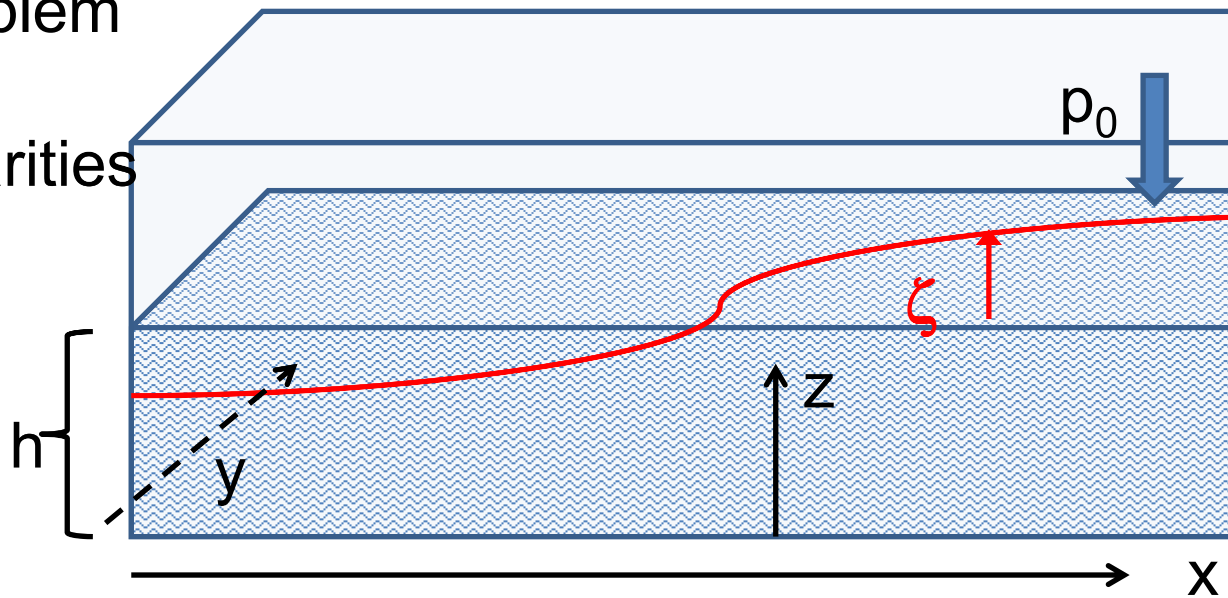
$$A\cosh(k\left(h+\zeta\right))\cos\left(k(x-ct)\right)\left(k^2c^2 - gk\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))}\right) = 0$$

$$\Rightarrow c^2 = \frac{g}{k}\frac{\sinh(k\left(h+\zeta\right))}{\cosh(k\left(h+\zeta\right))} \approx \frac{g}{k}\tanh(kh)$$

Note that this solution represents a pure plane wave. More likely, there would be a linear combination of wavevectors *k*. Additionally, your text considers the effects of surface tension. In this lecture, we will focus on the effects of the non-linear effects of Euler and continuity equations.

Surface waves in an incompressible fluid

General problem including non-linearities



Within fluid: $0 \le z \le h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}v^2 + g\left(z-h\right) = \text{constant} \qquad \Phi = \Phi(x,y,z,t)$$

$$-\nabla^2 \Phi = 0 \qquad \mathbf{v} = \mathbf{v}(x,y,z,t) = -\nabla\Phi(x,y,z,t)$$

At surface: $z = h + \zeta$ with $\zeta = \zeta\left(x,y,t\right)$

$$v_z(h+\zeta) = \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x\frac{\partial \zeta}{\partial x} + v_y\frac{\partial \zeta}{\partial y} = -\left.\frac{\partial \Phi(x,y,z,t)}{\partial z}\right|_{z=h+\zeta} \quad \text{where } v_{x,y} = v_{x,y}\left(x,y,h+\zeta,t\right)$$

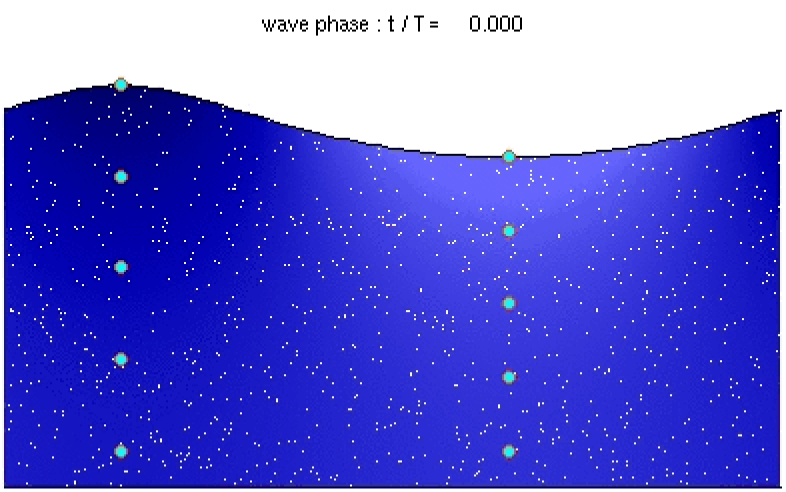
Some relationships at surface --

At surface: $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

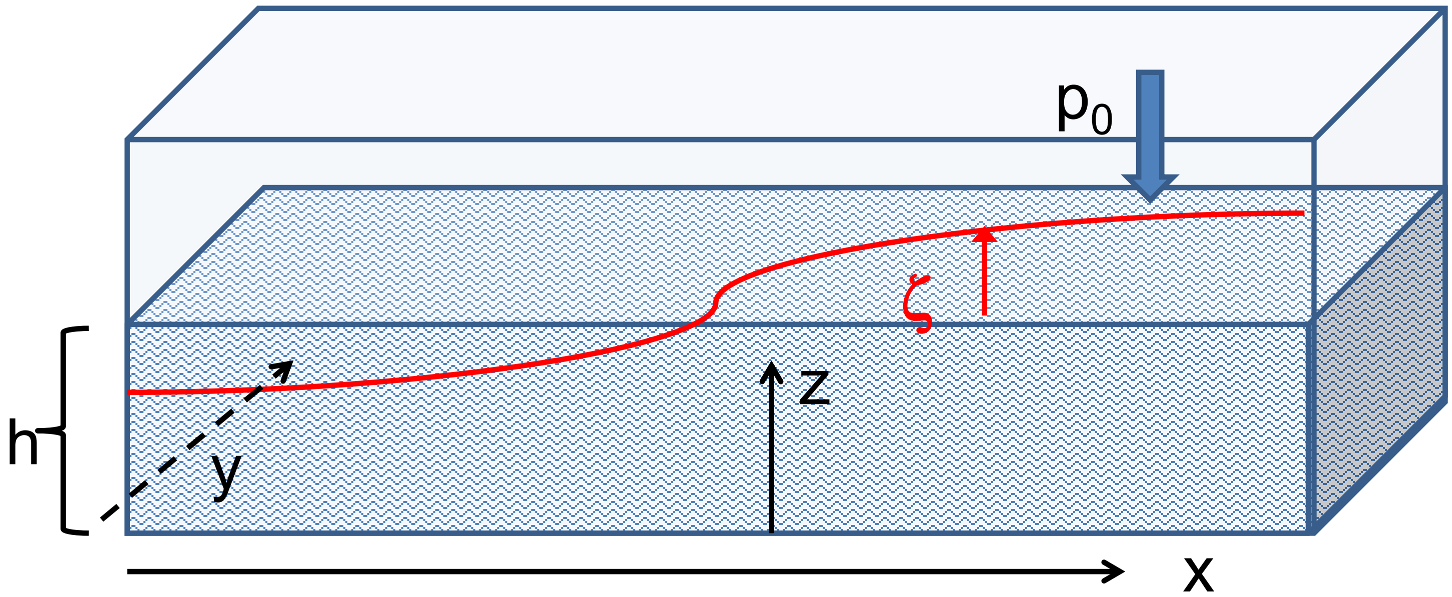
$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} = -\left.\frac{\partial \Phi(x,y,z,t)}{\partial z}\right|_{z=h+\zeta}$$

where $v_{x,y} = v_{x,y}(x, y, h+\zeta, t)$

Note that $v_z(x, y, h+\zeta, t) = \frac{d\zeta}{dt}$



From wikipedia



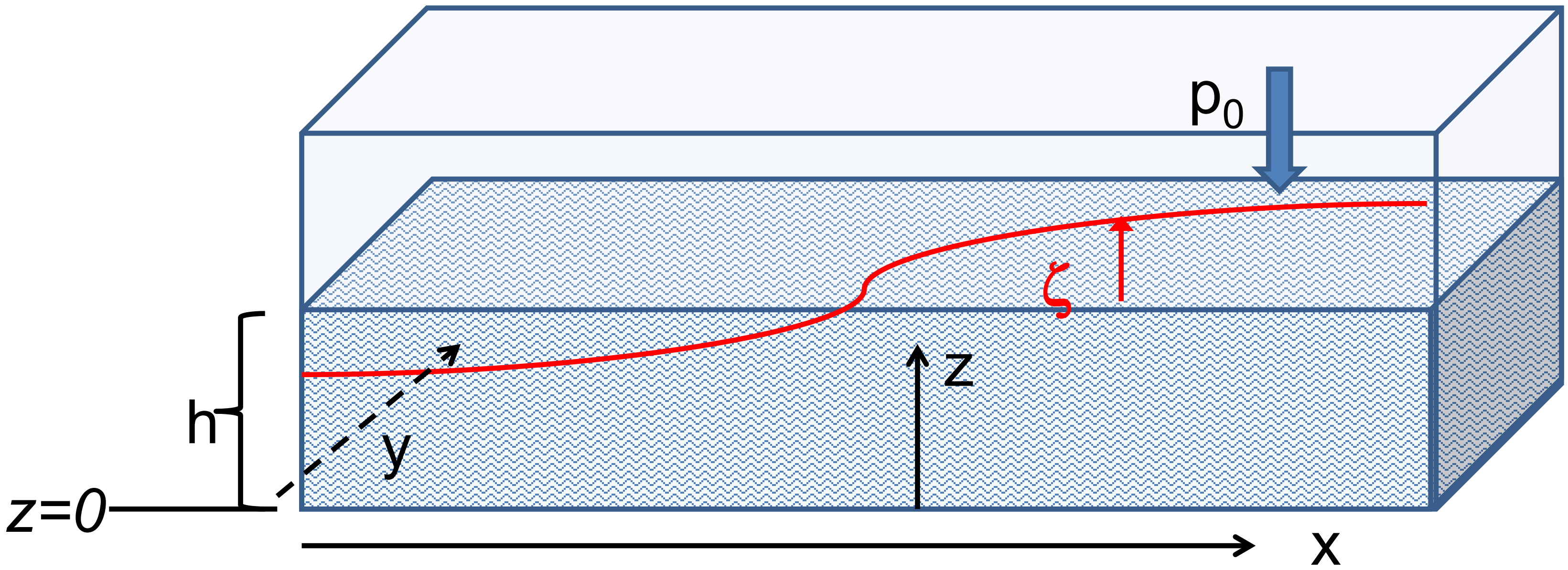
Further simplifications; assume trivial y - dependence

$$\Phi = \Phi(x, z, t) \qquad\qquad \zeta = \zeta(x, t)$$

Within fluid : $\qquad 0 \leq z \leq h + \zeta$

At surface : $\qquad v_z(x, z = h + \zeta, t) = -\frac{\partial \Phi}{\partial z} = \frac{d\zeta}{dt}$

Non-linear effects in surface waves:



Dominant non-linear effects $\Rightarrow$ soliton solutions

$$\zeta(x,t) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right) \qquad \eta_0 = \text{constant}$$

$$\text{where } c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right)$$

Detailed analysis of non-linear surface waves

[Note that these derivations follow Alexander L. Fetter and John Dirk Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw Hill, 1980), Chapt. 10.]

We assume that we have an incompressible fluid: $\rho = \text{constant}$

Velocity potential: $\Phi(x,z,t)$; $\mathbf{v}(x,z,t) = -\nabla\Phi(x,z,t)$

The surface of the fluid is described by $z=h+\zeta(x,t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$.

Defining equations for $\Phi(x,z,t)$ and $\zeta(x,t)$

where $0 \leq z \leq h + \zeta(x,t)$

Continuity equation:

$$\nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Bernoulli equation (assuming irrotational flow) and gravitation potential energy

$$-\frac{\partial \Phi(x,z,t)}{\partial t} + \frac{1}{2}\left[\underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial x}\right)^2}_{v_x^2} + \underbrace{\left(\frac{\partial \Phi(x,z,t)}{\partial z}\right)^2}_{v_z^2}\right] + g(z-h) = 0.$$

Boundary conditions on functions –

Zero velocity at bottom of tank:

$$\frac{\partial \Phi(x,0,t)}{\partial z} = 0.$$

Consistent vertical velocity at water surface

$$v_z(x,z,t)\Big|_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v}\cdot\nabla\zeta + \frac{\partial \zeta}{\partial t}$$

$$= v_x \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t}$$

$$\Rightarrow -\frac{\partial \Phi(x,z,t)}{\partial z} + \frac{\partial \Phi(x,z,t)}{\partial x}\frac{\partial \zeta(x,t)}{\partial x} - \frac{\partial \zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta} = 0$$

Analysis assuming water height *z* is small relative to variations in the direction of wave motion *(x)*

Taylor's expansion about $z = 0$:

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

Note that the zero vertical velocity at the bottom suggest that to a good approximation, that all odd derivatives $\frac{\partial^n \Phi}{\partial z^n}(x,0,t)$ vanish from the Taylor expansion. In addition, the Laplace equation allows us to convert all even derivatives with respect to *z* to derivatives with respect to *x*.

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

$$\Rightarrow \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Modified Taylor's expansion: $\Phi(x,z,t) \approx \Phi(x,0,t) - \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial x^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial x^4}(x,0,t)\cdots$

Some details --

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z\frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

At bottom: $z = 0$ and $v_z(x,0,t) = 0 \quad \Rightarrow \frac{\partial \Phi}{\partial z}(x,0,t) = 0$

Further, you textbook argues that using Fourier transforms,

$$\Phi(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \cosh(kz) e^{ikx} \tilde{f}(k,t) \approx \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left(1 + \frac{(kz)^2}{2!} + \frac{(kz)^4}{4!} + \ldots\right) e^{ikx} \tilde{f}(k,t)$$

$$\Phi(x,z,t) \approx \Phi(x,0,t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4 \Phi}{\partial z^4}(x,0,t)\cdots$$

Check linearized equations and their solutions:

Bernoulli equations --

Bernoulli equation evaluated at $z = h + \zeta(x,t)$

$$-\frac{\partial \Phi(x,h,t)}{\partial t} + g\zeta(x,t) = 0$$

Consistent vertical velocity at $z = h + \zeta(x,t)$

$$-\frac{\partial \Phi(x,z,t)}{\partial z} - \frac{\partial \zeta(x,t)}{\partial t}\bigg|_{z=h+\zeta} = 0$$

Using Taylor's expansion results to lowest order

$$-\frac{\partial \Phi(x,h,t)}{\partial z} \approx h\frac{\partial^2 \Phi(x,0,t)}{\partial x^2} = -\frac{\partial \zeta(x,t)}{\partial t} \qquad -\frac{\partial \Phi(x,h,t)}{\partial t} \approx -\frac{\partial \Phi(x,0,t)}{\partial t} = -g\zeta(x,t)$$

Decoupled equations: $$\frac{\partial^2 \Phi(x,0,t)}{\partial t^2} = gh\frac{\partial^2 \Phi(x,0,t)}{\partial x^2}.$$

➔linear wave equation with $c^2=gh$

Analysis of non-linear equations --

Bernoulli equation evaluated at surface:

$$-\frac{\partial\Phi(x,z,t)}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi(x,z,t)}{\partial x}\right)^2+\left(\frac{\partial\Phi(x,z,t)}{\partial z}\right)^2\right]\Bigg|_{z=h+\zeta}+g\zeta(x,t)=0.$$

Consistency of surface velocity

$$-\frac{\partial\Phi(x,z,t)}{\partial z}+\frac{\partial\Phi(x,z,t)}{\partial x}\frac{\partial\zeta(x,t)}{\partial x}-\frac{\partial\zeta(x,t)}{\partial t}\Bigg|_{z=h+\zeta}=0$$

Representation of velocity potential from Taylor′s expansion:

$$\Phi(x,z,t)\approx\Phi(x,0,t)-\frac{z^2}{2}\frac{\partial^2\Phi}{\partial x^2}(x,0,t)+\frac{z^4}{4!}\frac{\partial^4\Phi}{\partial x^4}(x,0,t)\cdots$$

Analysis of non-linear equations -- keeping the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms. Let $\phi(x,t) \equiv \Phi(x,0,t)$

Approximate form of Bernoulli equation evaluated at surface: $z = h + \zeta$

$$-\frac{\partial \phi}{\partial t} + \frac{(h+\zeta)^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left((h+\zeta)\frac{\partial^2 \phi}{\partial x^2}\right)^2\right] + g\zeta = 0$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g\zeta = 0.$$

Approximate form of surface velocity expression :

$$\Rightarrow \frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

These equations represent non-linear coupling of $\phi(x,t)$ and $\zeta(x,t)$.

Coupled equations: $$-\frac{\partial \phi}{\partial t}+\frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2}+\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2+g\zeta=0.$$

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \phi}{\partial x}\right)-\frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4}-\frac{\partial \zeta}{\partial t}=0.$$

Traveling wave solutions with new notation:

$u \equiv x-ct \qquad \phi(x,t)\equiv \chi(u) \qquad$ and $\quad \zeta(x,t)\equiv \eta(u)$

Note that the wave “speed” c will be consistently determined

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$

Integrating and re-arranging coupled equations

$$c\frac{d\chi(u)}{du}-\frac{ch^2}{2}\frac{d^3\chi(u)}{du^3}+\frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2+g\eta(u)=0.$$

$$\chi'=-\frac{g}{c}\eta+\frac{h^2}{2}\chi'''-\frac{1}{2c}(\chi')^2\approx-\frac{g}{c}\eta-\frac{h^2g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right)-\frac{h^3}{6}\frac{d^4\chi(u)}{du^4}+c\frac{d\eta(u)}{du}=0.$$

$$\Rightarrow(h+\eta)\frac{d\chi(u)}{du}-\frac{h^3}{6}\frac{d^3\chi(u)}{du^3}+c\eta(u)=0$$

Now we can express $\frac{d\chi(u)}{du}=\chi'$ in terms of η:

$$\chi'\approx-\frac{g}{c}\eta-\frac{h^2g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2$$

Integrating and re-arranging coupled equations – continued -- Expressing modified surface velocity equation in terms of $\eta(u)$:

$$(h+\eta)\left(-\frac{g}{c}\eta-\frac{h^2 g}{2c}\eta''-\frac{g^2}{2c^3}\eta^2\right)+\frac{h^3 g}{6c}\eta''+c\eta=0$$

$$\Rightarrow\left(1-\frac{gh}{c^2}\right)\eta-\frac{gh^3}{3c^2}\eta''-\frac{g}{c^2}\left(1+\frac{gh}{2c^2}\right)\eta^2=0$$

$$\Rightarrow\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}[\eta(u)]^2=0.$$

Note: $c^2 = gh + \ldots$

Solution of the famous Korteweg-de Vries equation

Modified surface amplitude equation in terms of η

$$\Rightarrow \left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}[\eta(u)]^2=0.$$

Soliton solution

$$\zeta(x,t)=\eta(x-ct)=\eta_0 \ \text{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\ \frac{x-ct}{2h}\right)$$

$$c=\sqrt{\frac{gh}{1-\eta_0/h}}\approx\sqrt{gh}\left(1+\frac{\eta_0}{2h}\right) \quad \text{where } \eta_0 \text{ is a constant}$$

Steps to solution

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Let $1-\frac{hg}{c^2}\equiv\frac{\eta_0}{h}$ $\quad\Rightarrow \frac{\eta_0}{h}\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$

Multiply equation by $\eta'(u)$ $\quad\Rightarrow \frac{d}{du}\left(\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)\right)=0$

Integrate wrt u and assume solution vanishes for $u\to\infty$

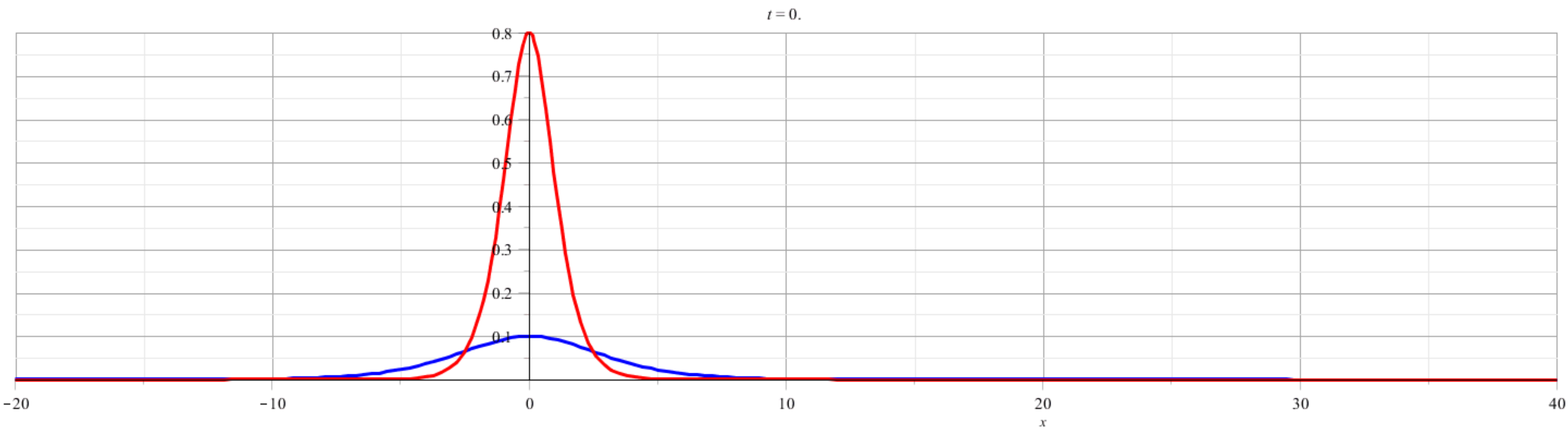
$$\frac{\eta_0}{2h}\eta^2(u)-\frac{h^2}{6}\eta'^2(u)-\frac{1}{2h}\eta^3(u)=0$$

$$\eta'^2(u)=\frac{3}{h^3}\eta^2(u)\left(\eta_0-\eta(u)\right)$$

$$\frac{d\eta}{\eta\left(\eta_0-\eta\right)^{1/2}}=\sqrt{\frac{3}{h^3}}du \qquad \Rightarrow \eta(u)=\frac{\eta_0}{\cosh^2\left(\sqrt{\frac{3\eta_0}{4h^3}}u\right)}$$

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \ \mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\ \frac{x-ct}{2h}\right)$$

Two soliton solutions with different amplitudes --



Relationship to “standard” form of Korteweg-de Vries equation

New variables:

$$\beta = 2\eta_0, \qquad \bar{x} = \sqrt{\frac{3}{2h}}\frac{x}{h}, \qquad \text{and} \qquad \bar{t} = \sqrt{\frac{3}{2h}}\frac{ct}{2\eta_0 h}.$$

Standard Korteweg-de Vries equation

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta\frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

Soliton solution:

$$\eta(\bar{x},\bar{t}) = \frac{\beta}{2}\operatorname{sech}^2\left[\frac{\sqrt{\beta}}{2}(\bar{x} - \beta\bar{t})\right].$$

More details

Modified surface amplitude equation in terms of η :

$$\left(1-\frac{hg}{c^2}\right)\eta(u)-\frac{h^2}{3}\eta''(u)-\frac{3}{2h}\left[\eta(u)\right]^2=0.$$

Some identities: $\frac{\eta_0}{h}=1-\frac{gh}{c^2}; \quad \frac{\partial\eta}{\partial t}=-c\frac{d\eta}{du}; \quad \frac{\partial\eta}{\partial x}=\frac{d\eta}{du}.$

Derivative of surface amplitude equation:

$$\frac{\eta_0}{h}\eta'-\frac{h^2}{3}\eta'''-\frac{3}{h}\eta\eta'=0.$$

Expression in terms of x and t:

$$-\frac{\eta_0}{ch}\frac{\partial\eta}{\partial t}-\frac{h^2}{3}\frac{\partial^3\eta}{\partial x^3}-\frac{3}{h}\eta\frac{\partial\eta}{\partial x}=0.$$

Expression in terms of $\overline{x}$ and $\overline{t}$:

$$\frac{\partial\eta}{\partial\overline{t}}+6\eta\frac{\partial\eta}{\partial\overline{x}}+\frac{\partial^3\eta}{\partial\overline{x}^3}=0.$$

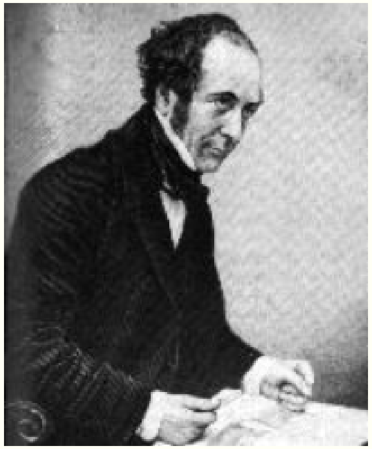
Summary

Soliton solution

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \ \mathrm{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right)$$

$$c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1+\frac{\eta_0}{2h}\right) \quad \text{where } \eta_0 \text{ is a constant}$$

John Scott Russell and the solitary wave



Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. As he described it in his "Report on Waves": (Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).

https://www.macs.hw.ac.uk/~chris/scott_russell.html

``I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".
(Cet passage en francais)

This event took place on the Union Canal at Hermiston, very close to the Riccarton campus of Heriot-Watt University, Edinburgh.

Photo of canal soliton http://www.ma.hw.ac.uk/solitons/
(link no longer active)