PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Notes on Lecture 35: Chap. 11 in F&W

Heat conduction

1. Basic equations
2. Boundary value problems

30	Fri, 11/04/2022	Chap. 9	Linearized hydrodynamics equations	#23	11/07/2022
31	Mon, 11/07/2022	Chap. 9	Linear sound waves	#24	11/09/2022
32	Wed, 11/09/2022	Chap. 9	Scattering of sound and non-linear effects	#25	11/11/2022
33	Fri, 11/11/2022	Chap. 10	Surface waves in fluids	#26	11/16/2022
34	Mon, 11/14/2022	Chap. 10	Surface waves in fluids; soliton solutions		
35	Wed, 11/16/2022	Chap. 11	Heat conduction		
36	Fri, 11/18/2022	Chap. 12	Viscous effects on hydrodynamics		
37	Mon, 11/21/2022	Chap 1-12	Review		
	Wed, 11/23/2022		Thanksgiving Holiday		
	Fri, 11/25/2022		Thanksgiving Holiday		
	Mon, 11/28/2022		Presentations I		
	Wed, 11/30/2022		Presentations II		
	Fri, 12/02/2022		Presentations III		

PHY 711 Presentation Schedule for Fall 2022

Monday, November 28, 2022

	Name	Title/Topic
10:00-10:15	Lee Pryor	Foucault Pend. on a spinning torus
10:17-10:32	Samuel Griffith	Normal Modes of Oscillation
10:35-10:50	Moti Mirhosseini	Fourier Transform

Wednesday, November 30,, 2022

	Name	Title/Topic
10:00-10:15	Katie Koch	2D Wave Equation
10:17-10:32	Banasree Sarkar Mou	Moment of inertia tensor of rigid body and the dynamics of spinning top
10:35-10:50	Arezoo Nameny	Foucault pendulum

Friday, December 2, 2022

	Name	Title/Topic
10:00-10:15	Zezhong Zhang	Acoustic Tweezer
10:17-10:32	Athul Prem	Green's function methods
10:35-10:50	Evan Kumar	Laplace Transforms

Physics Colloquium

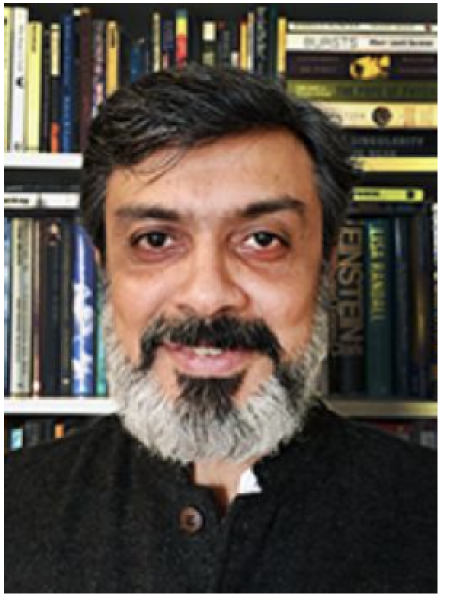
Thursday

·

November 17, 2022

Strong light-matter interaction in 2D materials

Strong exciton-photon interaction results in the formation of half-light half-matter quasiparticles called exciton-polaritons (EPs) that take on the properties of both its constituents. In this talk, I will first introduce polariton formation in 2D semiconductors 1 followed by a discussion of Rydberg excitons 2 and dipolar excitons 3 to realize highly nonlinear interactions to achieve single photon nonlinearity. Following this, I will discuss the use of strain to control exciton flow and nonlinear response 4 . Finally, I will present some recent results on EPs in correlated van der Waals materials and their potential to realize hybridization between excitons, photons and magnons 5 .



Vinod Menon

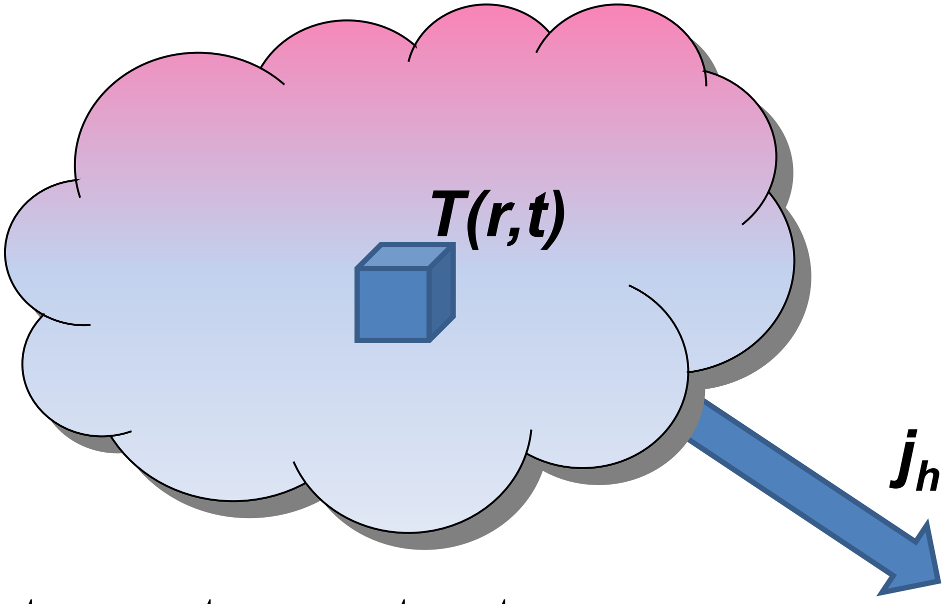
Dept. of Physics
City College of New York CUNY

4:00 pm - Olin 101*

*Link provided for those unable to attend in person.

Note: For additional information on the seminar or to obtain the video conference link, contact

Conduction of heat



Enthalpy of a system at constant pressure p

non uniform temperature $T(\mathbf{r},t)$

mass density ρ and heat capacity c_p

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Note that in this treatment we are considering a system at constant pressure *p*

Notation: Heat added to system -- $dQ = TdS$

External work done on system -- $dW = -pdV$

Internal energy -- $dE = dQ + dW = TdS - pdV$

Entropy -- dS

Enthalpy -- $dH = d(E + pV) = TdS + Vdp$

Heat capacity at constant pressure:

$$C_p \equiv \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial H}{\partial T}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$

$$C_p = \int \rho c_p d^3 r$$

More generally, note that c_p can depend on *T*; we are assuming that dependence to be trivial.

Conduction of heat -- continued

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Time rate of change of enthapy:

$$\frac{dH}{dt} = \int_V \rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} d^3r = -\int_A \mathbf{j}_h \cdot d\mathbf{A} + \int_V \rho \dot{q} d^3r$$

heat flux

heat source

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Conduction of heat -- continued

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Empirically: $\mathbf{j}_h = -k_{th} \nabla T(\mathbf{r},t)$

$$\Rightarrow \frac{\partial T(\mathbf{r},t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r},t) + \frac{\dot{q}}{c_p}$$

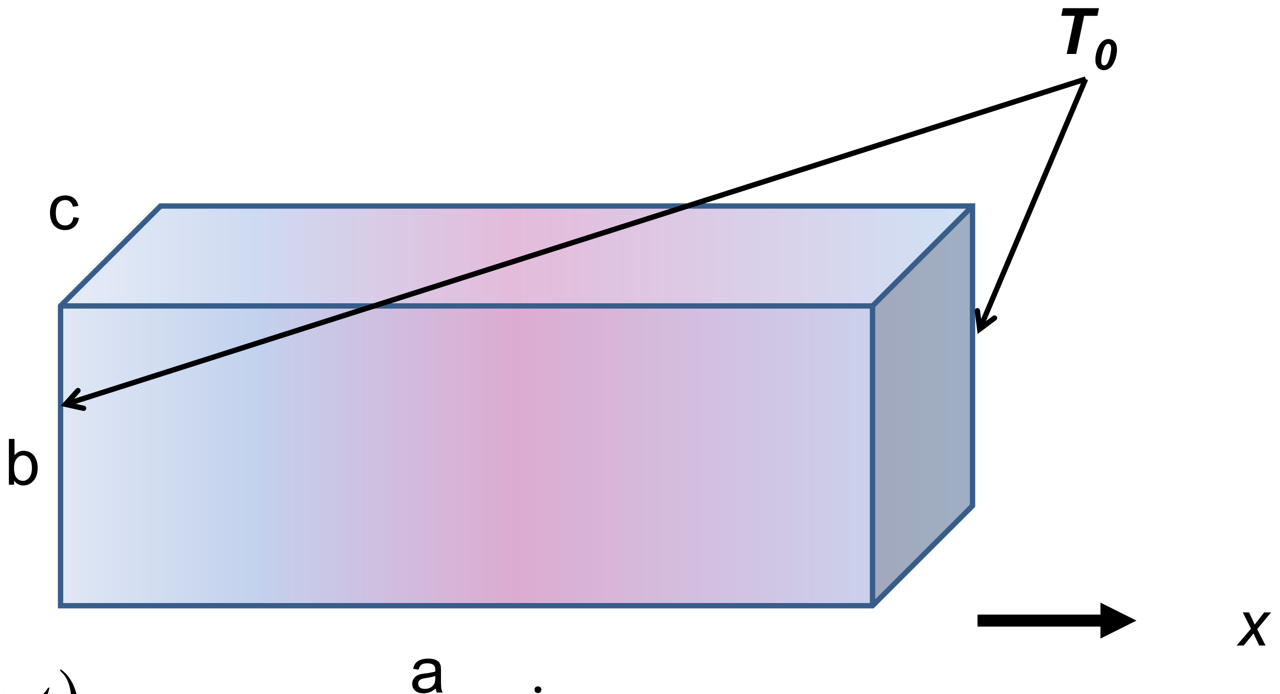
$\kappa \equiv \dfrac{k_{th}}{\rho c_p}$ thermal diffusivity

https://www.engineersedge.com/heat_transfer/thermal_diffusivity_table_13953.htm

Typical values (m^2/s)

Air	2×10^{-5}
Water	1×10^{-7}
Copper	1×10^{-4}

Boundary value problems for heat conduction



$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = \frac{\dot{q}}{c_p}$$

Without source term: $$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

Example with boundary values: $T(0,y,z,t) = T(a,y,z,t) = T_0$

Have you ever encountered the following equation in other contexts and if so where?

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

Boundary value problems for heat conduction

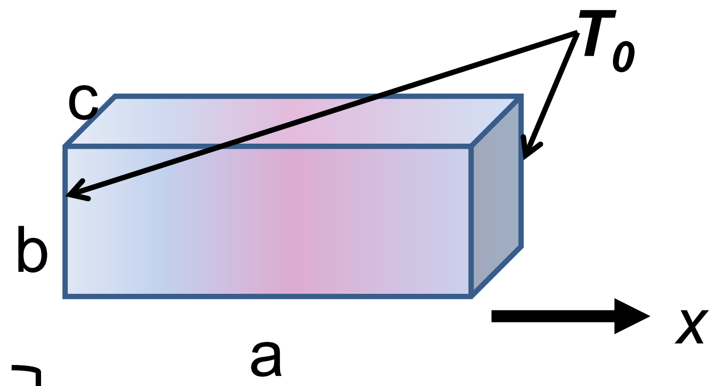
$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$$

$$T(0,y,z,t) = T(a,y,z,t) = T_0$$

$$\frac{\partial T(x,0,z,t)}{\partial y} = \frac{\partial T(x,b,z,t)}{\partial y} = 0$$

$$\frac{\partial T(x,y,0,t)}{\partial z} = \frac{\partial T(x,y,c,t)}{\partial z} = 0$$

Assuming thermally insulated boundaries

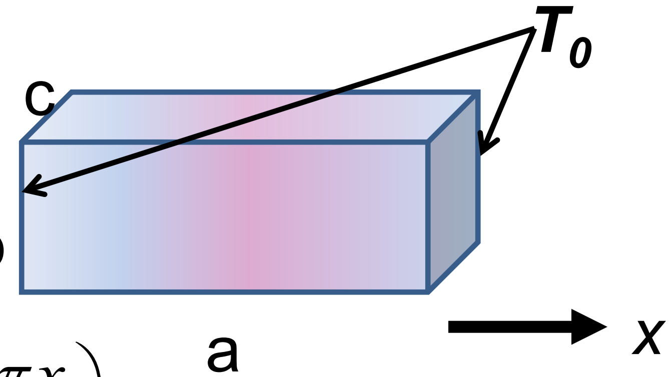


$$\text{Separation of varables: } T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$$

$$\text{Let } \frac{d^2X}{dx^2} = -\alpha^2 X \quad \frac{d^2Y}{dy^2} = -\beta^2 Y \quad \frac{d^2Z}{dz^2} = -\gamma^2 Z$$

$$\Rightarrow -\lambda + \kappa(\alpha^2 + \beta^2 + \gamma^2) = 0$$

Boundary value problems for heat conduction



$$T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$$

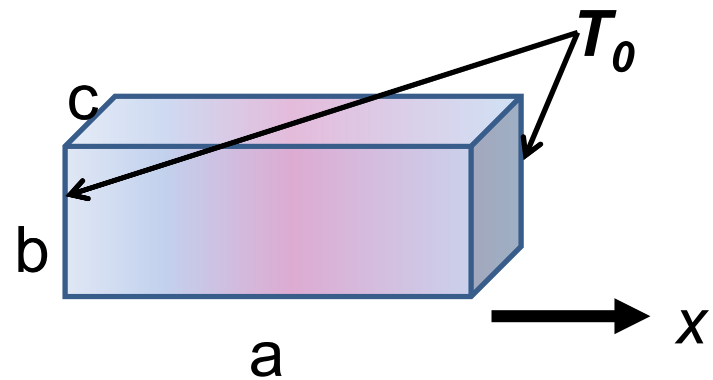
$$X(0) = X(a) = 0 \qquad \Rightarrow X(x) = \sin\left(\frac{m\pi x}{a}\right)$$

$$\frac{dY(0)}{dy} = \frac{dY(b)}{dy} = 0 \qquad \Rightarrow Y(y) = \cos\left(\frac{n\pi y}{b}\right)$$

$$\frac{dZ(0)}{dz} = \frac{dZ(c)}{dz} = 0 \qquad \Rightarrow Z(z) = \cos\left(\frac{p\pi z}{c}\right)$$

$$-\lambda_{nmp} + \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right) = 0$$

Boundary value problems for heat conduction



Full solution:

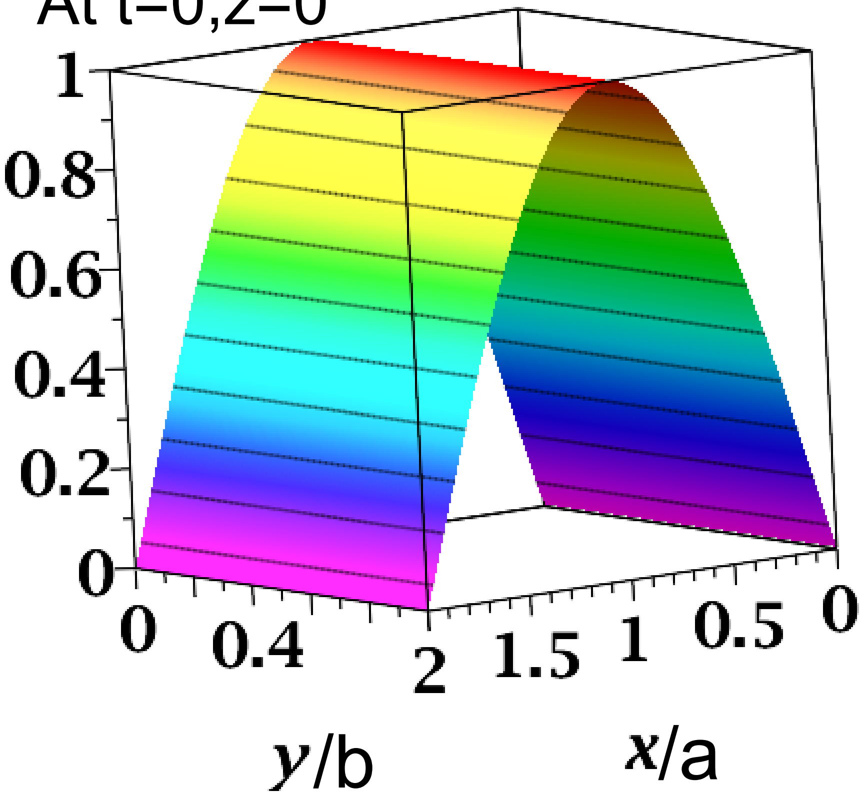
$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

$$\lambda_{nmp} = \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right)$$

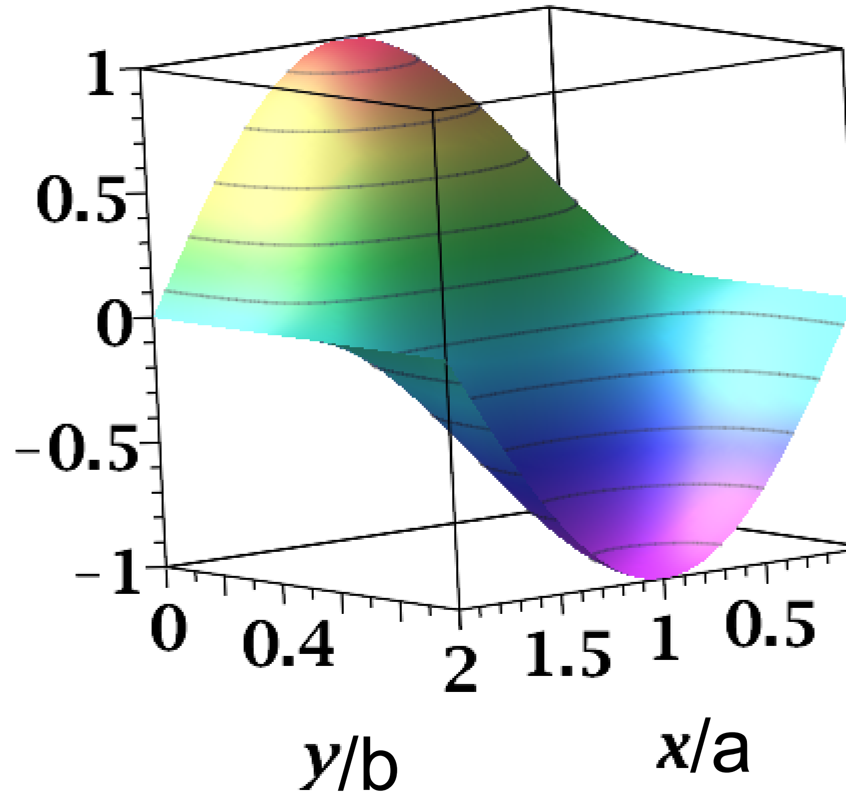
Full solution:

$$T\left(x,y,z,t\right) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

At t=0,z=0



$m=1, n=0, p=0$



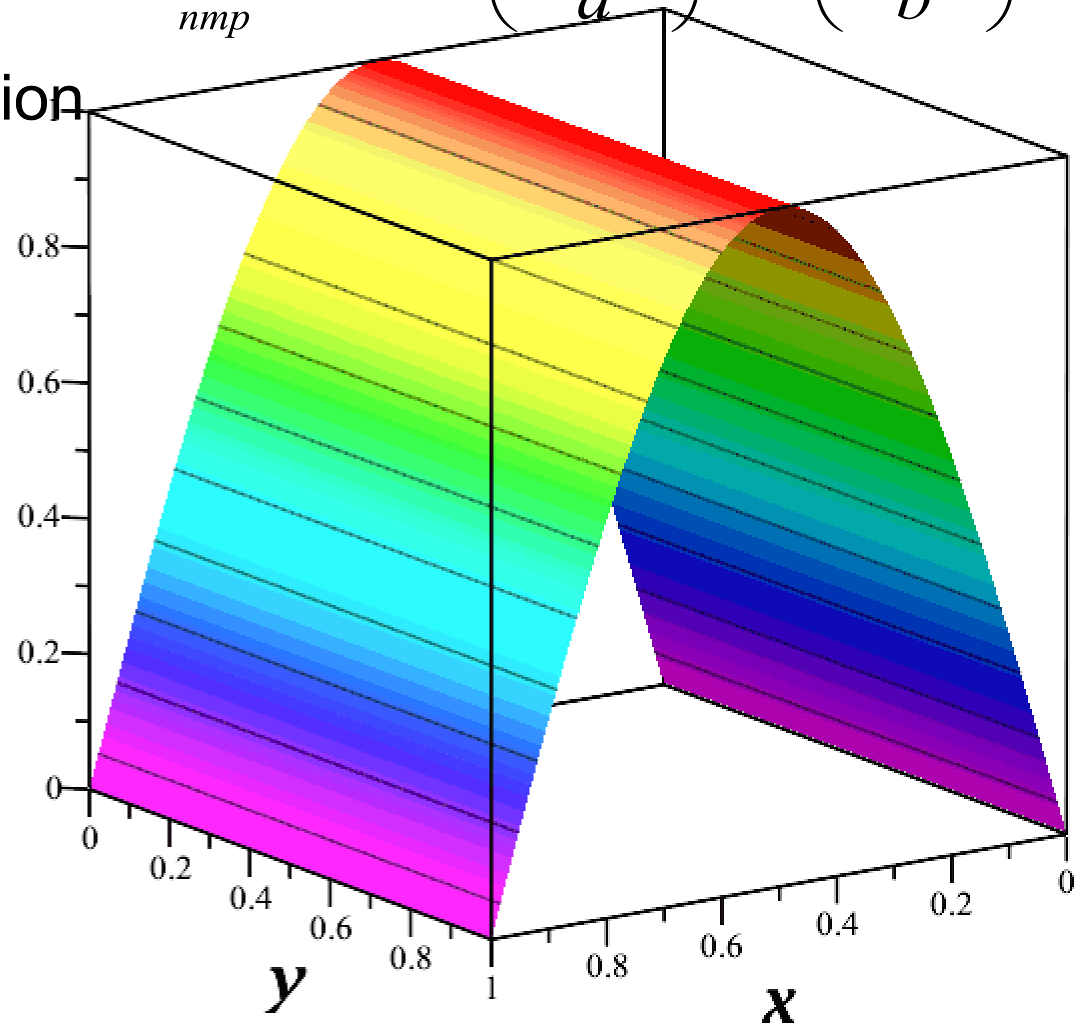
$m=1, n=1, p=0$

Full solution:

$t=0.$

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

Time evolution
nmp=100
at *z=0*



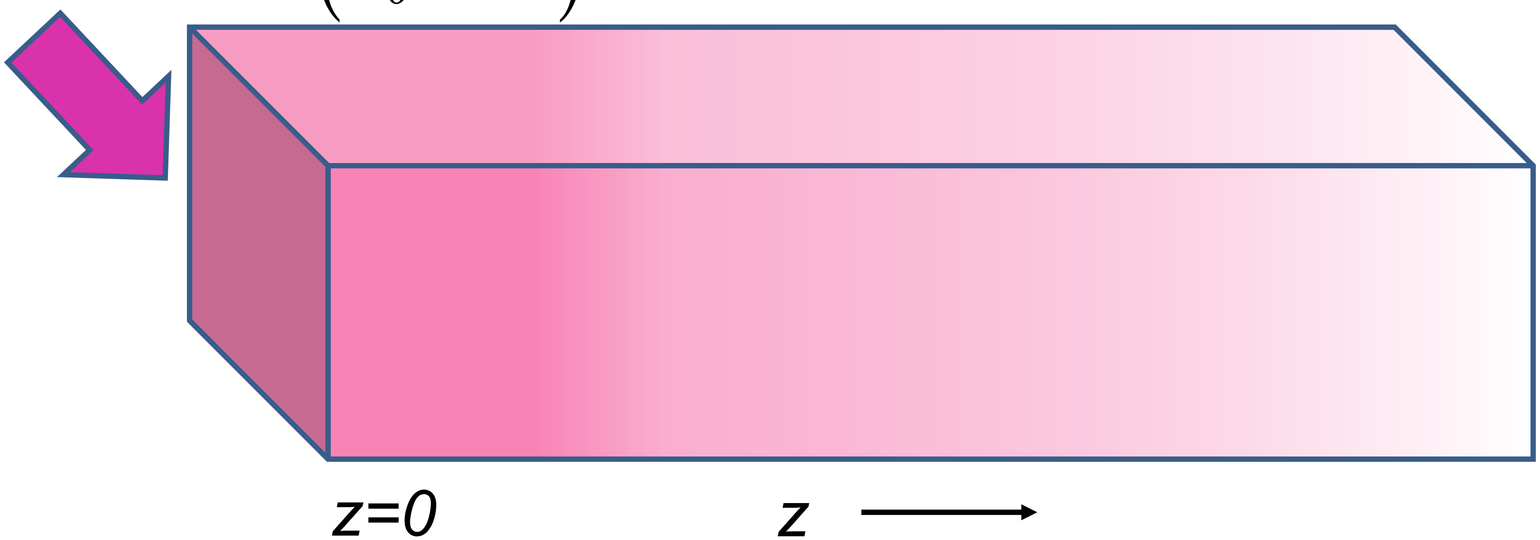
What real system could have such a temperature distribution?

Comment – While one can imagine that the boundary conditions can be readily realized, the single normal mode patterns are much harder. On the other hand, we see that the smallest values of lambda have the longest time constants.

Oscillatory thermal behavior

Here we assume that the spatial variation is along z

$$T(z=0,t) = \Re\left(T_0 e^{-i\omega t}\right)$$



$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2}$$

Assume: $T(z,t) = \Re\left(f(z) e^{-i\omega t}\right)$

$$(-i\omega) f = \kappa \frac{d^2 f}{dz^2}$$

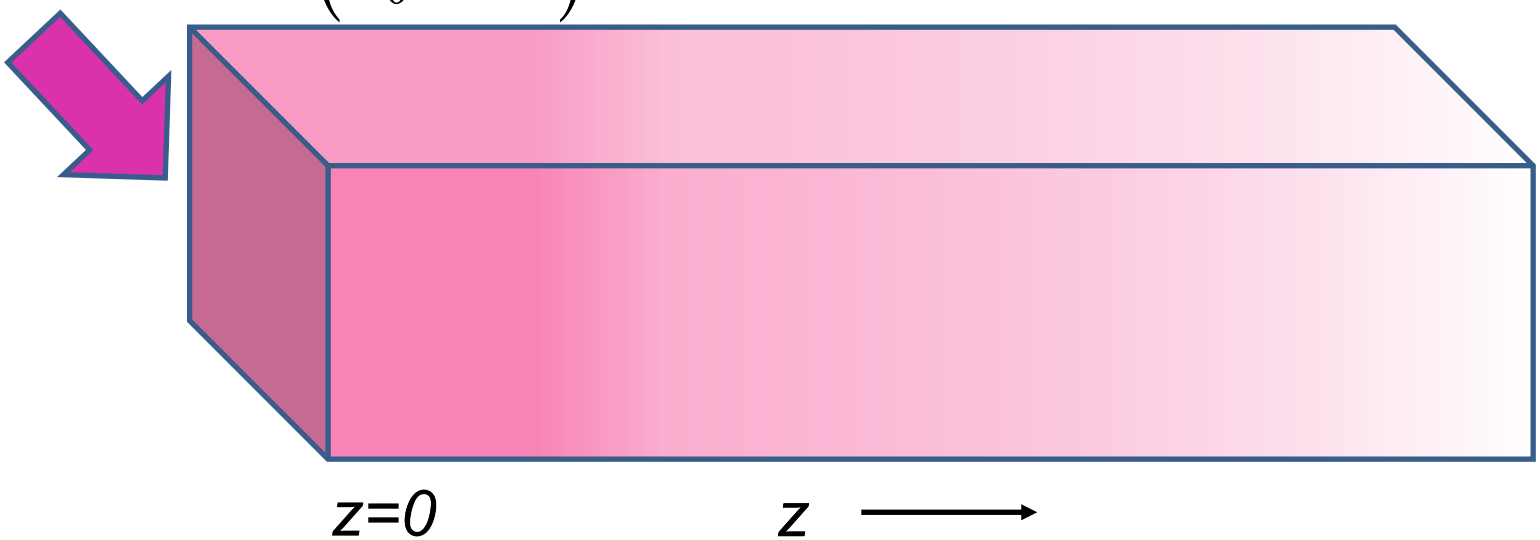
Let $\quad f(z) = A e^{\alpha z}$

$$\alpha^2 = -\frac{i\omega}{\kappa} = e^{3i\pi/2} \; \frac{\omega}{\kappa}$$

$$\alpha = \pm(1-i)\sqrt{\frac{\omega}{2\kappa}}$$

Oscillatory thermal behavior -- continued

$$T(z=0,t) = \Re\left(T_0 e^{-i\omega t}\right)$$

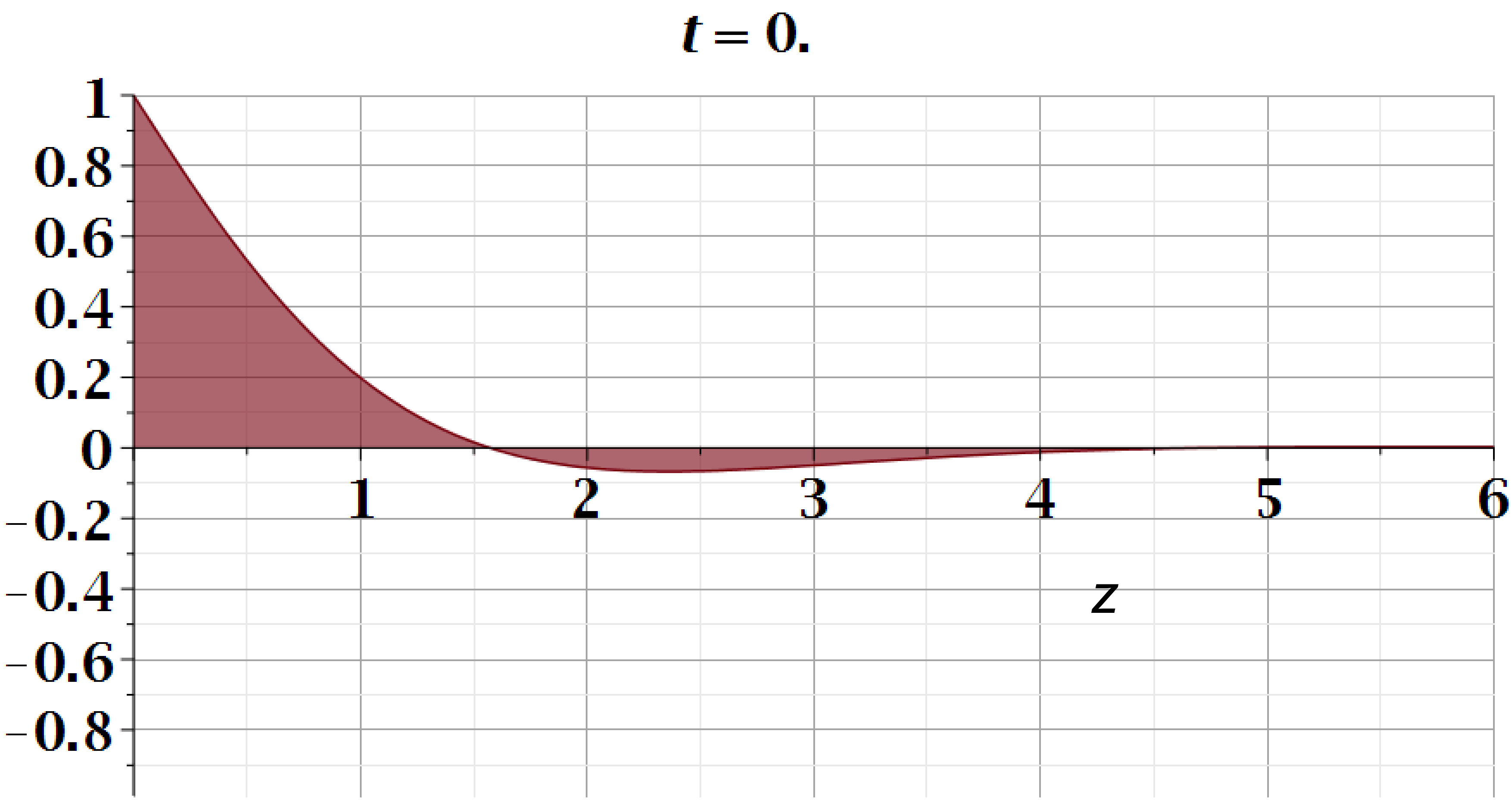


$$T(z,t) = \Re\left(A e^{\pm(1-i)z/\delta} e^{-i\omega t}\right)$$

$$\text{where} \quad \delta \equiv \sqrt{\frac{2\kappa}{\omega}}$$

$$\text{Physical solution:} \qquad T(z,t) = T_0 e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t\right)$$

$$T(z,t) = T_0 e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t\right)$$



Does this expression say the temperature transmits along the z axis?

Comment – In this case, our setup approximates trivial variation in the x-y plane so that all variation is along z. Given the oscillating boundary condition at z=0, the spatial form along z is a result of the form of the heat equation.

Initial value problem in an infinite domain; Fourier transform

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},0) = f(\mathbf{r})$$

$$\text{Let:} \quad \tilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t)$$

$$\tilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$\Rightarrow \tilde{T}(\mathbf{q},0) = \tilde{f}(\mathbf{q})$$

$$\Rightarrow \frac{\partial \tilde{T}(\mathbf{q},t)}{\partial t} = -\kappa q^2 \tilde{T}(\mathbf{q},t)$$

$$\tilde{T}(\mathbf{q},t) = \tilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$\widetilde{T}(\mathbf{q},t)=\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}T(\mathbf{r},t) \qquad \Rightarrow T(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int d^3q\, e^{i\mathbf{q}\cdot\mathbf{r}}\widetilde{T}(\mathbf{q},t)$$

$$\widetilde{T}(\mathbf{q},t)=\widetilde{T}(\mathbf{q},0)e^{-\kappa q^2 t}$$

$$T(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int d^3q\, e^{i\mathbf{q}\cdot\mathbf{r}}\widetilde{T}(\mathbf{q},0)e^{-\kappa q^2 t}$$

$$\widetilde{T}(\mathbf{q},0)=\widetilde{f}(\mathbf{q})=\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}f(\mathbf{r})$$

$$T(\mathbf{r},t)=\int d^3r'\, G(\mathbf{r}-\mathbf{r}',t)T(\mathbf{r}',0)$$

$$\text{with } G(\mathbf{r}-\mathbf{r}',t)\equiv\frac{1}{(2\pi)^3}\int d^3q\, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$T(\mathbf{r},t) = \int d^3r'\, G(\mathbf{r}-\mathbf{r}',t)\, T(\mathbf{r}',0)$$

$$\text{with}\quad G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3}\int d^3q\, e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

$$G(\mathbf{r}-\mathbf{r}',t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-|\mathbf{r}-\mathbf{r}'|^2/(4\kappa t)}$$

Heat equation in half-space

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},t) \Rightarrow T(z,t) \quad \text{with initial and boundary values:}$$

$$T(z,t) \equiv 0 \quad \text{for } z < 0$$

$$T(z,0) = 0 \quad \text{for } z > 0$$

$$T(0,t) = T_0 \quad \text{for } t \geq 0$$

$$\text{Solution:} \quad T = T_0 \, \text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\text{where } \text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

More details about the error function -- https://dlmf.nist.gov/7

§7.2(i) Error Functions

7.2.1 $$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt,$$

7.2.2 $$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt = 1 - \operatorname{erf} z,$$

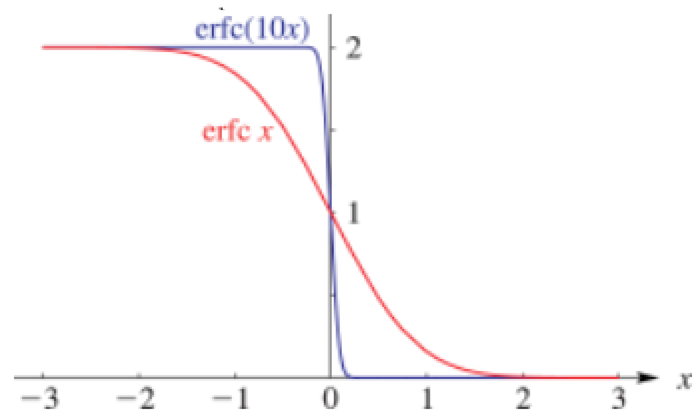


Figure 7.3.1: Complementary error functions $\operatorname{erfc} x$ and $\operatorname{erfc}(10x)$, $-3 \le x \le 3$.

Heat equation in half-space -- continued

$$\frac{\partial T(z,t)}{\partial t} - \kappa \frac{\partial^2 T(z,t)}{\partial z^2} = 0$$

$$\text{Solution :} \quad T = T_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

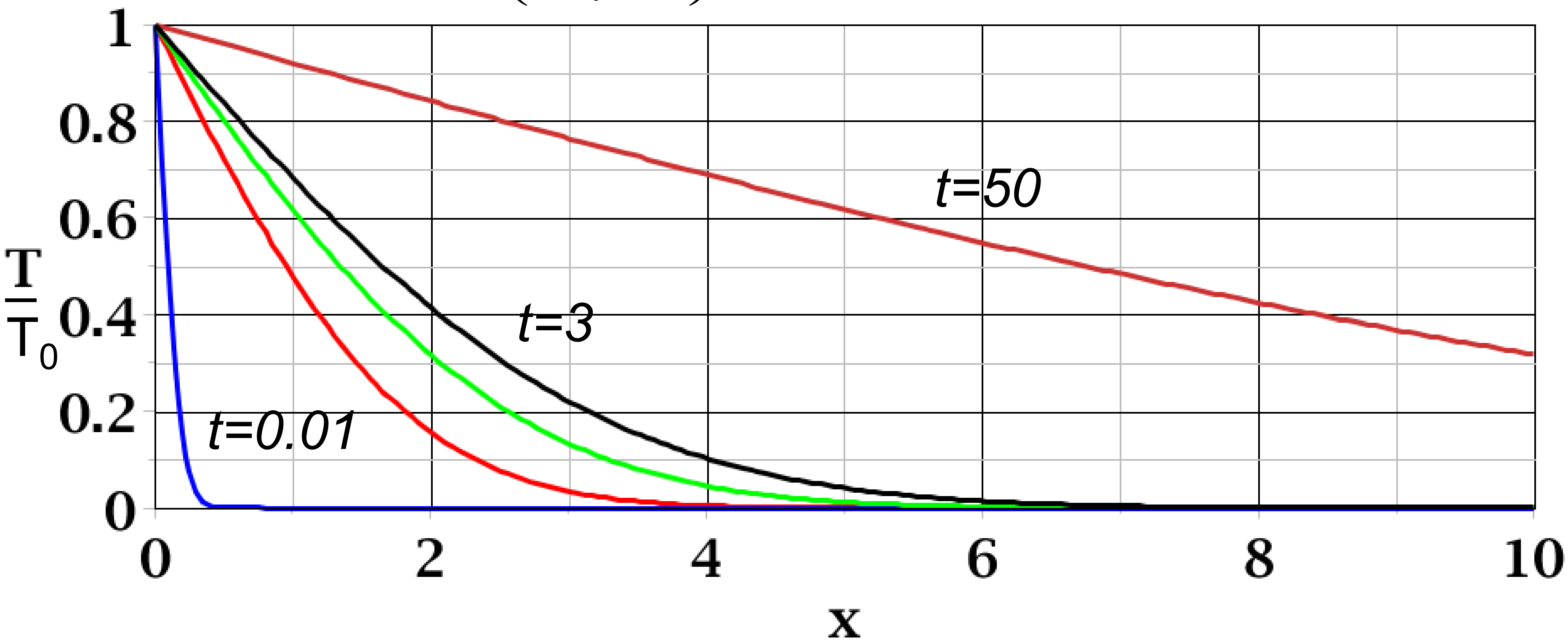
$$\text{where } \operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du$$

$$\text{Note that} \quad \frac{d \operatorname{erfc}(x)}{dx} = \frac{d}{dx} \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du = -\frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$\frac{\partial}{\partial t} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\sqrt{\kappa t^3}}\right)$$

$$\frac{\partial^2}{\partial z^2} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\kappa\sqrt{\kappa t^3}}\right)$$

$$T = T_0 \ \mathrm{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$



Temperature profile