PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes on Lecture 36: Chap. 12 in F & W

Viscous fluids

1. Viscous stress tensor
2. Navier-Stokes equation
3. Example for incompressible fluid – Stokes "law"
4. Effects on linearize sound waves

31	Mon, 11/07/2022	Chap. 9	Linear sound waves	#24	11/09/2022
32	Wed, 11/09/2022	Chap. 9	Scattering of sound and non-linear effects	#25	11/11/2022
33	Fri, 11/11/2022	Chap. 10	Surface waves in fluids	#26	11/16/2022
34	Mon, 11/14/2022	Chap. 10	Surface waves in fluids; soliton solutions		
35	Wed, 11/16/2022	Chap. 11	Heat conduction		
36	Fri, 11/18/2022	Chap. 12	Viscous effects on hydrodynamics		
37	Mon, 11/21/2022	Chap 1-12	Review		
	Wed, 11/23/2022		Thanksgiving Holiday		
	Fri, 11/25/2022		Thanksgiving Holiday		
	Mon, 11/28/2022		Presentations I		
	Wed, 11/30/2022		Presentations II		
	Fri, 12/02/2022		Presentations III		

Equations for motion of non-viscous fluid

Newton-Euler equation of motion:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} = \rho \mathbf{f}_{applied} - \nabla p$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) = 0 \qquad \Rightarrow \quad \mathbf{v} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) \right) = 0$$

Add the two equations:

$$\underbrace{\rho \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \rho}{\partial t} \mathbf{v}}_{\frac{\partial \left(\rho \mathbf{v} \right)}{\partial t}} + \underbrace{\rho \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} + \mathbf{v} \nabla \cdot \left(\rho \mathbf{v} \right)}_{\sum_{j=1}^{3} \frac{\partial \left(\rho v_j \mathbf{v} \right)}{\partial x_j}} = \rho \mathbf{f}_{applied} - \nabla p$$

Equations for motion of non-viscous fluid -- continued

Modified Newton-Euler equation in terms of fluid momentum:

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j\mathbf{v})}{\partial x_j}=\rho\mathbf{f}_{applied}-\nabla p$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\sum_{j=1}^{3}\frac{\partial(\rho v_j\mathbf{v})}{\partial x_j}+\nabla p=\rho\mathbf{f}_{applied}$$

Fluid momentum: $\rho\mathbf{v}$

Stress tensor: $T_{ij}\equiv\rho v_i v_j+p\delta_{ij}$

i^{th} component of Newton-Euler equation:

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{j=1}^{3}\frac{\partial T_{ij}}{\partial x_j}=\rho f_i$$

Now consider the effects of viscosity

In terms of stress tensor:

$$T_{ij} = T_{ij}^{\text{ideal}} + T_{ij}^{\text{viscous}}$$

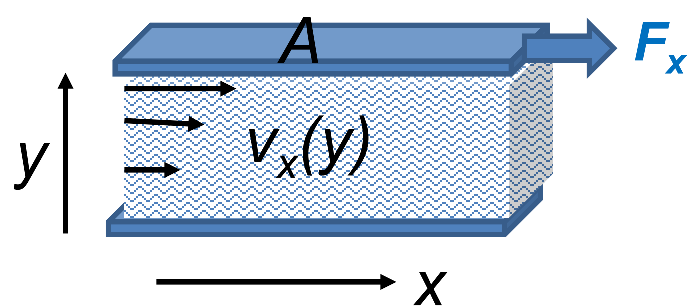
$$T_{ij}^{\text{ideal}} = \rho v_i v_j + p\delta_{ij} = T_{ji}^{\text{ideal}}$$

As an example of a viscous effect, consider --

Newton's "law" of viscosity

$$\frac{F_x}{A} = \eta \frac{\partial v_x}{\partial y}$$

material dependent parameter



Effects of viscosity

Argue that viscosity is due to shear forces in a fluid of the form:

$$\frac{F_{drag}}{A} = \eta \frac{\partial v_x}{\partial y}$$

Formulate viscosity stress tensor with traceless and diagonal terms:

$$T_{kl}^{\text{viscous}} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

viscosity

bulk viscosity

Total stress tensor: $T_{kl} = T_{kl}^{ideal} + T_{kl}^{\text{viscous}}$

$$T_{kl}^{\text{ideal}} = \rho v_k v_l + p\delta_{kl}$$

$$T_{kl}^{\text{viscous}} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

Effects of viscosity -- continued

Incorporating generalized stress tensor into Newton-Euler equations

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_{i=1}^{3} \frac{\partial T_{ij}}{\partial x_j} = \rho f_i$$

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_{j=1}^{3} \frac{\partial(\rho v_i v_j)}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} + \eta \sum_{j=1}^{3} \frac{\partial^2 v_i}{\partial x_j^2} + \left(\zeta + \frac{1}{3}\eta\right) \sum_{j=1}^{3} \frac{\partial^2 v_j}{\partial x_i \partial x_j}$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial(\rho v_j)}{\partial x_j} = 0$$

Vector form (Navier-Stokes equation)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{v}) = 0$$

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Typical viscosities at 20° C and 1 atm:

Fluid	η/ρ (m^2/s)	η (Pa s)
Water	1.00×10^{-6}	1×10^{-3}
Air	14.9×10^{-6}	0.018×10^{-3}
Ethyl alcohol	1.52×10^{-6}	1.2×10^{-3}
Glycerine	1183×10^{-6}	1490×10^{-3}

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R*

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v} \cdot \nabla\right)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla \cdot \mathbf{v}\right)$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v}\right) = 0$$

Note that $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$

Incompressible fluid $\Rightarrow \; \nabla \cdot \mathbf{v} = 0$

Steady flow $\Rightarrow \; \frac{\partial \mathbf{v}}{\partial t} = 0$

Irrotational flow $\Rightarrow \; \nabla \times \mathbf{v} = 0$

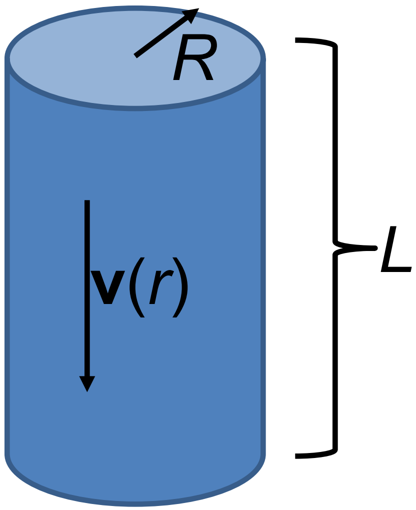
No applied force $\Rightarrow \; \mathbf{f} = 0$

Neglect non-linear terms $\Rightarrow \nabla\left(v^2\right) = 0$

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

Navier-Stokes equation becomes:

$$0 = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v}$$



Assume that $\mathbf{v}(\mathbf{r},t) = v_z(r)\hat{\mathbf{z}}$

$$\frac{\partial p}{\partial z} = \eta \nabla^2 v_z(r) \qquad \text{(independent of } z\text{)}$$

Suppose that $\frac{\partial p}{\partial z} = -\frac{\Delta p}{L}$ (uniform pressure gradient)

$$\Rightarrow \nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

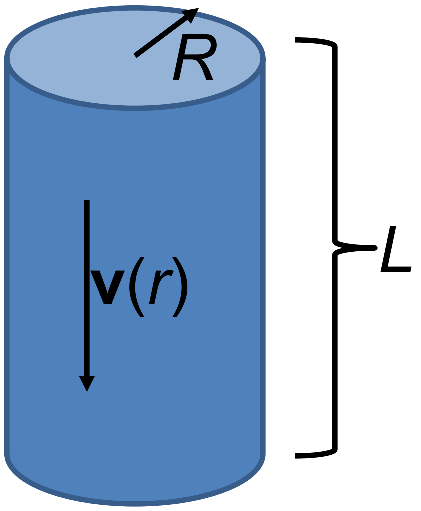
Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r}\frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + \mathrm{C}_2$$

$$\Rightarrow C_1 = 0 \qquad v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + \mathrm{C}_2$$

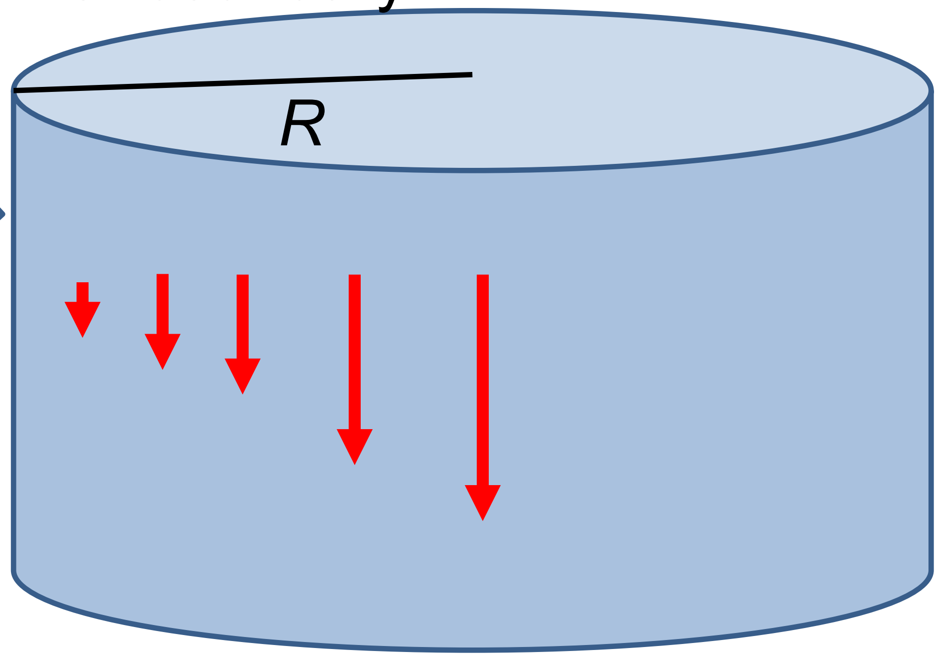


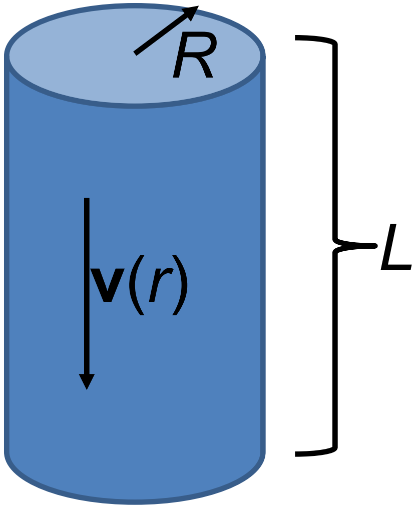
$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

Comment on boundary condition

$$v_z(R) = 0$$

Fluid approximately stationary at boundary



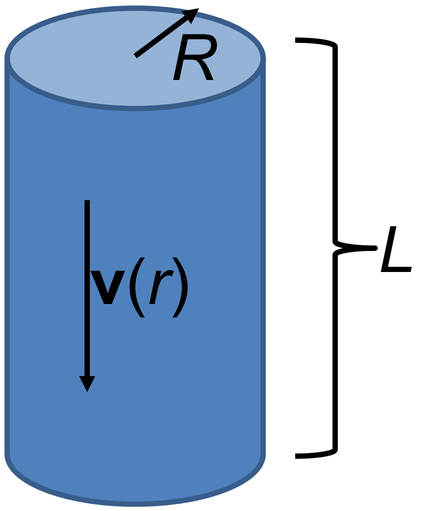


Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

Mass flow rate through the pipe:

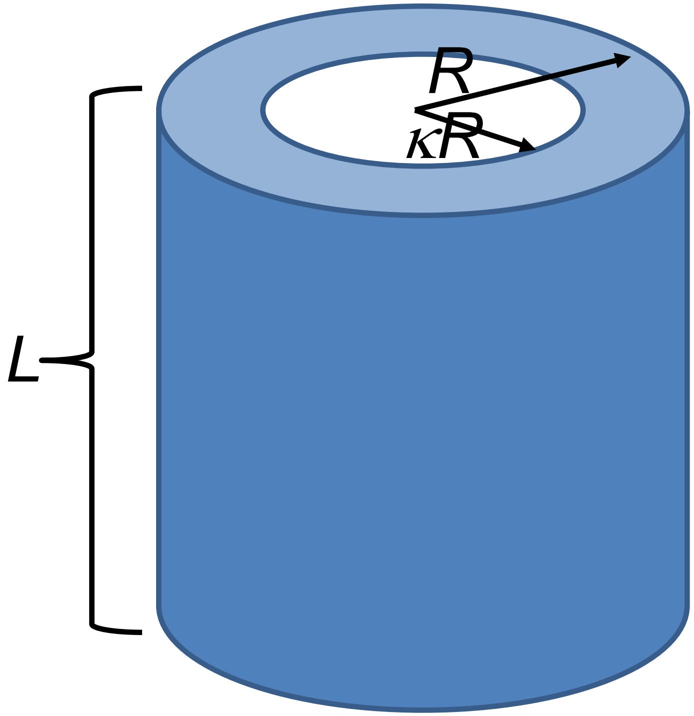
$$\frac{dM}{dt} = 2\pi\rho\int_0^R r\,dr\,v_z(r) = \frac{\Delta p\,\rho\pi R^4}{8\eta L}$$



Poiseuille formula;

➔Method for measuring η

Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius *R* and inner radius κR



$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

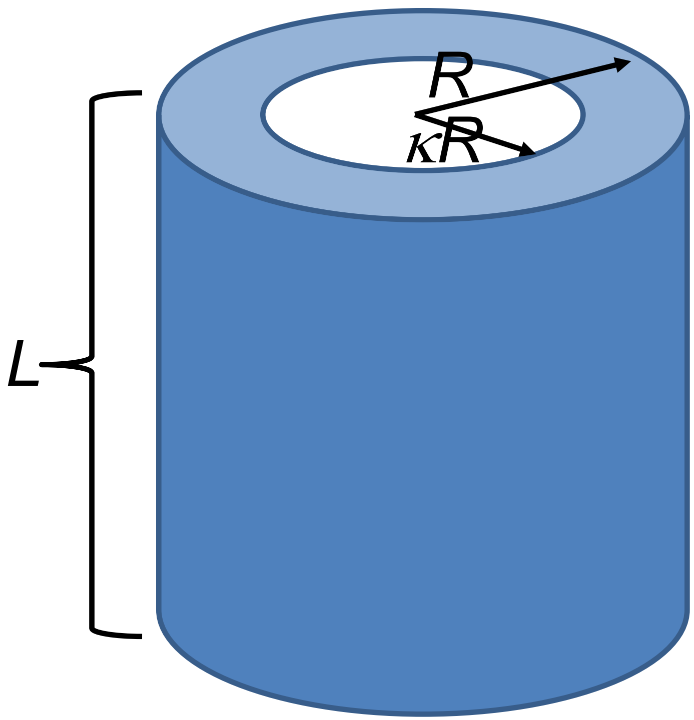
$$\frac{1}{r}\frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + \mathrm{C}_2$$

$$v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_1 \ln(R) + \mathrm{C}_2$$

$$v_z(\kappa R) = 0 = -\frac{\Delta p \kappa^2 R^2}{4\eta L} + C_1 \ln(\kappa R) + \mathrm{C}_2$$

Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius *R* and inner radius κR -- continued



Solving for C_1 and C_2 :

$$v_z(r) = \frac{\Delta p R^2}{4\eta L}\left(1-\left(\frac{r}{R}\right)^2 - \frac{1-\kappa^2}{\ln\kappa}\ln\left(\frac{r}{R}\right)\right)$$

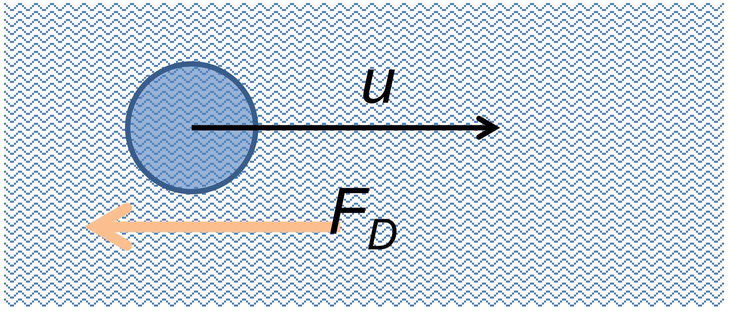
Mass flow rate through the pipe:

$$\frac{dM}{dt} = 2\pi\rho\int_{\kappa R}^{R} r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}\left(1-\kappa^4 + \frac{\left(1-\kappa^2\right)^2}{\ln\kappa}\right)$$

More discussion of viscous effects in incompressible fluids

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η:

$$F_D = -\eta\left(6\pi R u\right)$$



Plan:

1. Consider the general effects of viscosity on fluid equations
2. Consider the solution to the linearized equations for the case of steady-state flow of a sphere of radius R
3. Infer the drag force needed to maintain the steady-state flow

Have you ever encountered Stokes law in previous contexts?

a. Milliken oil drop experiment
b. A sphere falling due to gravity in a viscous fluid, reaching a terminal velocity
c. Other?

Newton-Euler equation for incompressible fluid, modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \underbrace{\frac{\eta}{\rho}}_{\upsilon} \nabla^2 \mathbf{v}$$

υ Kinematic viscosity

Typical kinematic viscosities at 20º C and 1 atm:

Fluid	ν (m²/s)
Water	1.00 x 10^{-6}
Air	14.9 x 10^{-6}
Ethyl alcohol	1.52 x 10^{-6}
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Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η :

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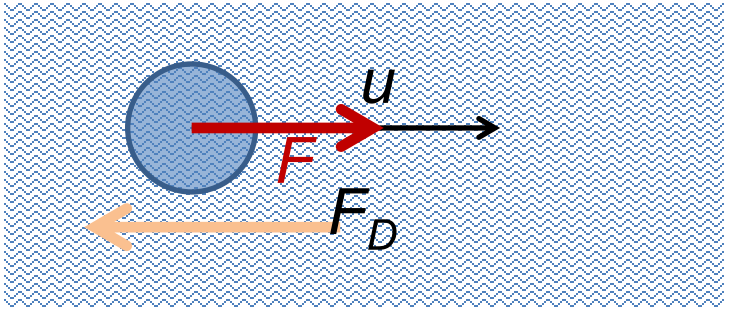

Effects of drag force on motion of particle of mass m with constant force F :

$$F - 6\pi R\eta u = m\frac{du}{dt} \qquad \text{with } u(0) = 0$$

$$\Rightarrow u(t) = \frac{F}{6\pi R\eta}\left(1 - e^{-\frac{6\pi R\eta}{m}t}\right)$$

Effects of drag force on motion of particle of mass m with constant force F :

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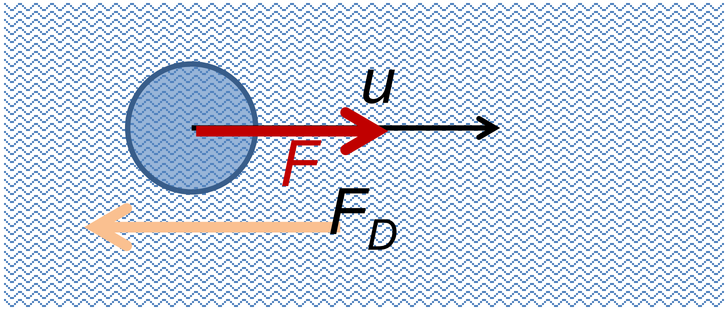
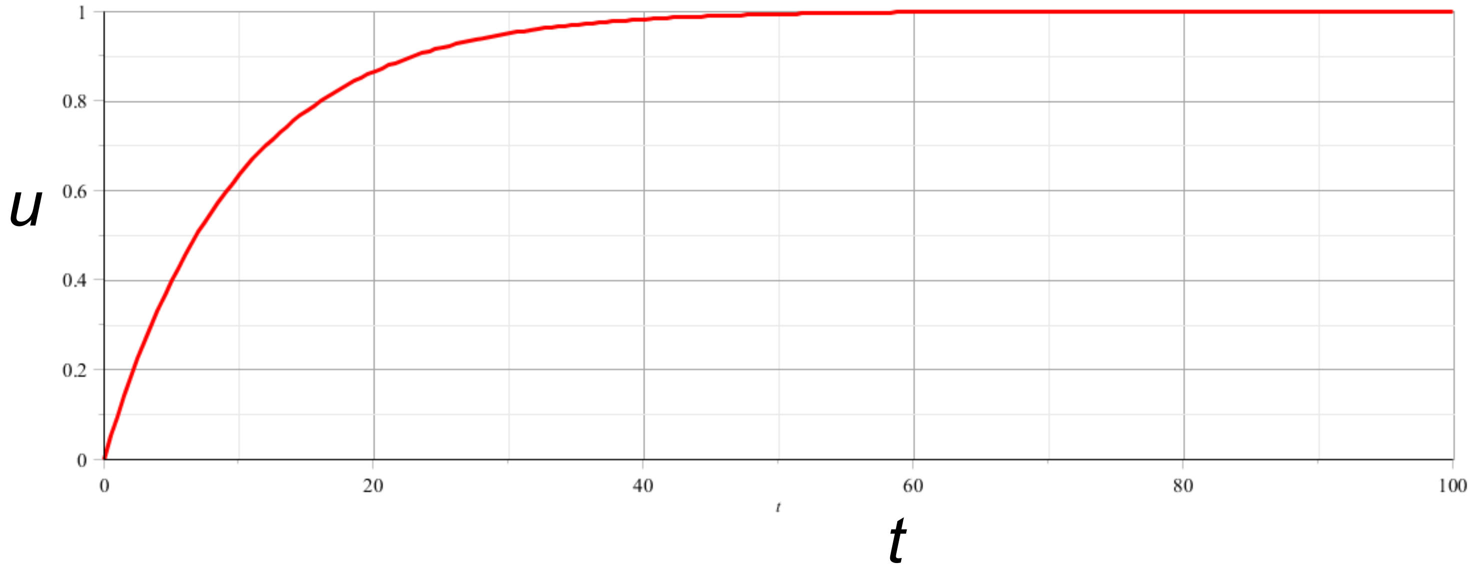


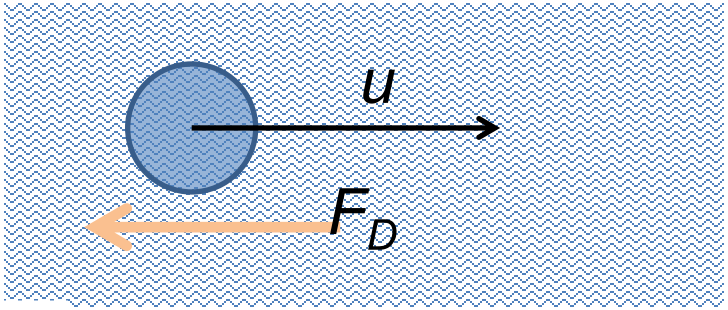


Effects of drag force on motion of particle of mass m

with an initial velocity with $u(0) = U_0$ and no external force

$$-6\pi R\eta u = m\frac{du}{dt}$$

$$\Rightarrow u(t) = U_0 e^{-\frac{6\pi R\eta}{m}t}$$

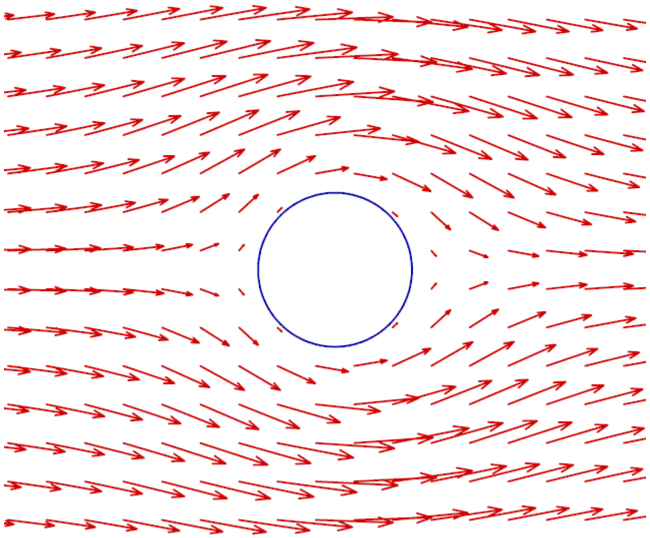


Recall: PHY 711 -- Assignment #22 Nov. 02, 2022

Determine the form of the velocity potential for an incompressible fluid representing uniform velocity in the **z** direction at large distances from a spherical obstruction of radius *a*. Find the form of the velocity potential and the velocity field for all *r* > *a*. Assume that for *r* = *a*, the velocity in the radial direction is 0 but the velocity in the azimuthal direction is not necessarily 0.

$$\nabla^2\Phi = 0$$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$



In the present viscous case, we will assume that **v**(*a*)=0.

Newton-Euler equation for incompressible fluid, modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \frac{\eta}{\rho}\nabla^2\mathbf{v}$$

Continuity equation: $\nabla \cdot \mathbf{v} = 0$

Assume steady state: $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} = 0$

Assume non-linear effects small

Initially set $\mathbf{f}_{applied} = 0$;

$$\Rightarrow \nabla p = \eta \nabla^2 \mathbf{v}$$

$$\nabla p = \eta \nabla^2 \mathbf{v}$$

Take curl of both sides of equation:

$$\nabla \times (\nabla p) = 0 = \eta \nabla^2 (\nabla \times \mathbf{v})$$

Assume (with a little insight from Landau):

$$\mathbf{v} = \nabla \times (\nabla \times f(r)\mathbf{u}) + \mathbf{u}$$

where $f(r) \xrightarrow[r \to \infty]{} 0$

Note that:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Digression

Some comment on assumption: $\mathbf{v} = \nabla \times \left(\nabla \times f(r)\mathbf{u}\right) + \mathbf{u}$

$$\nabla \times \left(\nabla \times \mathbf{A}\right) = \nabla\left(\nabla \cdot \mathbf{A}\right) - \nabla^2 \mathbf{A}$$

Here $\mathbf{A} = f(r)\mathbf{u}$

$$\nabla \times \mathbf{v} = \nabla \times \left(\nabla \times \left(\nabla \times \mathbf{A}\right)\right) = -\nabla \times \left(\nabla^2 \mathbf{A}\right)$$

Also note: $\nabla p = \eta \nabla^2 \mathbf{v}$

$$\Rightarrow \nabla \times \nabla p = 0 = \nabla \times \eta \nabla^2 \mathbf{v} \qquad \text{or} \qquad \nabla^2\left(\nabla \times \mathbf{v}\right) = 0$$

$$\nabla^2\left(\nabla \times \nabla^2 \mathbf{A}\right) = \nabla^4\left(\nabla \times \mathbf{A}\right) = 0$$

$$\mathbf{v} = \nabla \times \left(\nabla \times f(r)\mathbf{u}\right) + \mathbf{u}$$

$$\mathbf{u} = u\hat{\mathbf{z}}$$

$$\nabla \times \left(\nabla \times f(r)\hat{\mathbf{z}}\right) = \nabla\left(\nabla \cdot f(r)\hat{\mathbf{z}}\right) - \nabla^2 f(r)\hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \nabla^2\left(\nabla \times \mathbf{v}\right) = 0$$

$$\nabla^4\left(\nabla \times f(r)\hat{\mathbf{z}}\right) = 0 \quad \Rightarrow \nabla^4\left(\nabla f(r) \times \hat{\mathbf{z}}\right) = 0 \quad \Rightarrow \nabla^4 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$v_r = u\cos\theta\left(1 - \frac{2}{r}\frac{df}{dr}\right) = u\cos\theta\left(1 - 4C_1 - \frac{2C_2}{r} + \frac{2C_4}{r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1 - \frac{d^2 f}{dr^2} - \frac{1}{r}\frac{df}{dr}\right) = -u\sin\theta\left(1 - 4C_1 - \frac{C_2}{r} - \frac{C_4}{r^3}\right)$$

Some details:

$$\nabla^4 f(r) = 0 \qquad \Rightarrow \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \right)^2 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$\mathbf{v} = u\left(\nabla \times \left(\nabla \times f(r)\hat{\mathbf{z}} \right) + \hat{\mathbf{z}} \right)$$

$$= u\left(\nabla\left(\nabla \cdot \left(f(r)\hat{\mathbf{z}} \right) \right) - \nabla^2 f(r)\hat{\mathbf{z}} + \hat{\mathbf{z}} \right)$$

Note that: $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$

$$\mathbf{v} = u\left(\nabla\left(\frac{df}{dr}\cos\theta \right) - \left(\nabla^2 \left(f(r) \right) - 1 \right)\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \right) \right)$$

$$v_r = u\cos\theta\left(1-\frac{2}{r}\frac{df}{dr}\right) = u\cos\theta\left(1-4C_1-\frac{2C_2}{r}+\frac{2C_4}{r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1-\frac{d^2 f}{dr^2}-\frac{1}{r}\frac{df}{dr}\right) = -u\sin\theta\left(1-4C_1-\frac{C_2}{r}-\frac{C_4}{r^3}\right)$$

To satisfy $\mathbf{v}(r\rightarrow\infty)=\mathbf{u}$: $\Rightarrow C_1 = 0$

To satisfy $\mathbf{v}(R)=0$ solve for C_2, C_4

$$v_r = u\cos\theta\left(1-\frac{3R}{2r}+\frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1-\frac{3R}{4r}-\frac{R^3}{4r^3}\right)$$

$$v_r = u\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)$$

Determining pressure:

$$\nabla p = \eta\nabla^2\mathbf{v} = -\eta\nabla\left(u\cos\theta\left(\frac{3R}{2r^2}\right)\right)$$

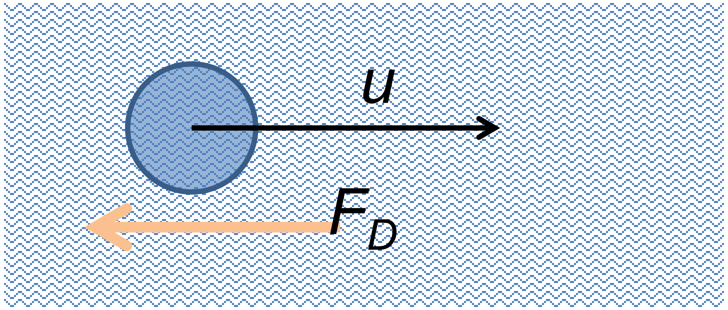
$$\Rightarrow p(r) = p_0 - \eta u\cos\theta\left(\frac{3R}{2r^2}\right)$$

$$p(r) = p_0 - \eta u \cos\theta \left(\frac{3R}{2r^2} \right)$$

Corresponds to:

$$F_D \cos\theta = \left(p(R) - p_0 \right) 4\pi R^2 = -\eta u \cos\theta (6\pi R)$$

$$\Rightarrow \mathrm{F}_D = -\eta u \left(6\pi R \right)$$



Additional effects of viscosity – allowing for changes in entropy

$$p(\rho, s) = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s$$

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Newton-Euler equations for viscous fluids – effects on sound

Without viscosity terms:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p \qquad\qquad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v}$ $\qquad \mathbf{f}=0 \qquad \rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv p_0 + c^2\delta\rho$$

Linearized equations: $\dfrac{\partial \delta\mathbf{v}}{\partial t} = -\dfrac{c^2}{\rho_0}\nabla\delta\rho \qquad \dfrac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot(\delta\mathbf{v}) = 0$

Let $\delta\mathbf{v} \equiv \delta\mathbf{v}_0\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \delta\rho \equiv \delta\rho_0\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

Sound waves without viscosity -- continued

Linearized equations: $\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v}\right) = 0$

Let $\quad \delta \mathbf{v} \equiv \delta \mathbf{v}_0 \; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \delta\rho \equiv \delta\rho_0 \; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho \qquad \Rightarrow \omega \delta \mathbf{v}_0 = \frac{c^2 \delta\rho_0}{\rho_0}\mathbf{k}$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v}\right) = 0 \qquad \Rightarrow -\omega\delta\rho_0 + \rho_0 \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2} \qquad \frac{\delta\rho_0}{\rho_0} = \frac{\hat{\mathbf{k}}\cdot\delta\mathbf{v}_0}{c}$$

➔Pure longitudinal harmonic wave solutions

Newton-Euler equations for viscous fluids – effects on sound

Recall full equations:

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v}$ $\quad\mathbf{f}=0\quad$ $\rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + c^2\delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s$$

where $c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$

viscosity causes heat transfer

Newton-Euler equations for viscous fluids – effects on sound

Note that pressure now depends both on density and entropy so that entropy must be coupled into the equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v})$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad\qquad \rho T \frac{\partial s}{\partial t} = k_{th} \nabla^2 T$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v}$ $\quad \mathbf{f}=0$ $\quad \rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + c^2 \delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s \quad \text{where } c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$$

$$T = T_0 + \delta T = T_0 + \left(\frac{\partial T}{\partial \rho}\right)_s \delta\rho + \left(\frac{\partial T}{\partial s}\right)_\rho \delta s$$

$$s = s_0 + \delta s$$

Newton-Euler equations for viscous fluids – linearized equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

$$\Rightarrow \frac{\partial \delta\mathbf{v}}{\partial t} = \underbrace{-\frac{1}{\rho_0}\nabla\delta p}_{} + \frac{\eta}{\rho}\nabla^2\delta\mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\delta\mathbf{v})$$

$$-\frac{1}{\rho_0}\left\{\left(\frac{\partial p}{\partial\rho}\right)_s \nabla\delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \nabla\delta s\right\} = -\frac{c^2}{\rho_0}\nabla\delta\rho - \rho_0\left(\frac{\partial T}{\partial\rho}\right)_s \nabla\delta s$$

Digression -- from the first law of thermodynamics:

$$d\epsilon = Tds + \frac{p}{\rho^2}d\rho$$

$$\left(\frac{\partial}{\partial\rho}\left(\frac{\partial\epsilon}{\partial s}\right)_\rho\right)_s = \left(\frac{\partial T}{\partial\rho}\right)_s \quad\Leftrightarrow\quad \left(\frac{\partial}{\partial s}\left(\frac{\partial\epsilon}{\partial\rho}\right)_s\right)_\rho = \left(\frac{\partial p/\rho^2}{\partial s}\right)_\rho \approx \frac{1}{\rho_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho$$

Newton-Euler equations for viscous fluids – linearized equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\Rightarrow \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot (\delta \mathbf{v}) = 0$$

$$\rho T \frac{\partial s}{\partial t} = k_{th} \nabla^2 T$$

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \frac{k_{th}}{\rho_0 T_0}\left(\left(\frac{\partial T}{\partial s}\right)_\rho \nabla^2 \delta s + \left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho \right)$$

Further relationships:

$$\left(\frac{\partial T}{\partial s}\right)_\rho \approx \frac{T_0}{c_v} \qquad \kappa = \frac{k_{th}}{\rho c_p}$$

heat capacity at constant volume

Newton-Euler equations for viscous fluids – linearized equations

$$\Rightarrow \frac{\partial \delta s}{\partial t} = \left(\gamma\kappa\nabla^2 \delta s + \frac{c_p \kappa}{T_0} \left(\frac{\partial T}{\partial \rho} \right)_s \nabla^2 \delta\rho \right) \quad \text{where } \gamma \equiv \frac{c_p}{c_v}$$

Newton-Euler equations for viscous fluids – effects on sound

Linearized equations (with the help of various thermodynamic relationships):

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho - \rho_0 \left(\frac{\partial T}{\partial \rho}\right)_s \nabla \delta s + \frac{\eta}{\rho_0}\nabla^2 \delta \mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla \cdot \delta \mathbf{v}\right)$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v}\right) = 0$$

$$\frac{\partial \delta s}{\partial t} = \gamma\kappa\nabla^2 \delta s + \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho$$

Here: $\gamma = \frac{c_p}{c_v}$ $\quad\kappa = \frac{k_{th}}{c_p \rho_0}$

Linearized hydrodynamic equations

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho - \rho_0 \left(\frac{\partial T}{\partial \rho}\right)_s \nabla \delta s + \frac{\eta}{\rho_0}\nabla^2 \delta \mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla \cdot \delta \mathbf{v}\right)$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v}\right) = 0$$

$$\frac{\partial \delta s}{\partial t} = \gamma\kappa\nabla^2 \delta s + \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho$$

It can be shown that

$$\left(\frac{\partial T}{\partial \rho}\right)_s = \frac{Tc^2\beta}{\rho c_p} \qquad \text{where} \qquad \beta \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \qquad \text{(thermal expansion)}$$

Let $\quad \delta \mathbf{v} \equiv \delta \mathbf{v}_0 \; e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \delta\rho \equiv \delta\rho_0 \; e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \delta s \equiv \delta s_0 \; e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$

Linearized hydrodynamic equations; plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

In the absense of thermal expansion, $\beta = 0$

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0$$

➔Entropy and mechanical modes are independent

Linearized hydrodynamic equations; full plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

Longitudinal solutions: $(\delta\mathbf{v}\cdot\mathbf{k} \neq 0)$:

$$\left(\omega^2 - c^2k^2 + i\frac{\omega k^2}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\right)\delta\rho_0 - \frac{\rho_0 T_0\beta c^2k^2}{c_p}\delta s_0 = 0$$

$$\frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

Linearized hydrodynamic equations; full plane wave solutions:

Longitudinal solutions: $(\delta\mathbf{v}\cdot\mathbf{k}\neq 0)$:

$$\left(\omega^2 - c^2k^2 + i\frac{\omega k^2}{\rho_0}\left(\frac{4}{3}\eta+\zeta\right)\right)\delta\rho_0 - \frac{\rho_0 T_0 \beta c^2 k^2}{c_p}\delta s_0 = 0$$

$$\frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

Approximate solution: $\quad k = \dfrac{\omega}{c} + i\alpha$

where $\quad \alpha \approx \dfrac{\omega^2}{2c^3\rho_0}\left(\dfrac{4}{3}\eta+\zeta\right) + \dfrac{\kappa T_0 \beta^2\omega^2}{2c_p c}$

$$\delta\rho = \delta\rho_0 e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{r}} e^{i\frac{\omega}{c}\left(\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}$$

Linearized hydrodynamic equations; full plane wave solutions:

$$\omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} + \frac{T_0\beta c^2}{c_p}\delta s_0\mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta\mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot\delta\mathbf{v}_0\right)$$

$$\omega\delta\rho_0 - \rho_0\ \mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\omega\delta s_0 = -i\gamma\kappa k^2\delta s_0 - \frac{i\kappa\beta c^2}{\rho_0}k^2\delta\rho_0$$

Transverse modes $(\delta\mathbf{v}\cdot\mathbf{k} = 0)$:

$$\delta\rho_0 = 0 \qquad \delta s_0 = 0$$

$$\left(\omega + \frac{i\eta k^2}{\rho_0}\right)\left(\delta\mathbf{v}\times\ \mathbf{k}\right) = 0 \qquad k = \pm\left(\frac{i\omega\rho_0}{\eta}\right)^{1/2}$$