

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF in Olin 103**

Plan for Lecture 37

Review of topics covered in this course

- 1. Mathematical methods**
- 2. Classical mechanics concepts**

31	Mon, 11/07/2022	Chap. 9	Linear sound waves	#24	11/09/2022
32	Wed, 11/09/2022	Chap. 9	Scattering of sound and non-linear effects	#25	11/11/2022
33	Fri, 11/11/2022	Chap. 10	Surface waves in fluids	#26	11/16/2022
34	Mon, 11/14/2022	Chap. 10	Surface waves in fluids; soliton solutions		
35	Wed, 11/16/2022	Chap. 11	Heat conduction		
36	Fri, 11/18/2022	Chap. 12	Viscous effects on hydrodynamics		
37	Mon, 11/21/2022	Chap 1-12	Review		
	Wed, 11/23/2022		Thanksgiving Holiday		
	Fri, 11/25/2022		Thanksgiving Holiday		
	Mon, 11/28/2022		Presentations I		
	Wed, 11/30/2022		Presentations II		
	Fri, 12/02/2022		Presentations III		



Comments on take-home final –
Similar to mid-term in form
Available: Fri. Dec. 2, 2022
Due: Mon. Dec. 12, 2022??
(before 11 AM)

November 2022

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S	M	T	W	T	F	S
30	31	1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	1	2	3
4	5	6	7	8	9	10

- Grading based on correct answer and reasoning.
- Make sure that your work is legible
- Include Mathematica, Maple, Wolfram, Matlab work if appropriate

Review of mathematical methods

Some useful identities for vectors and vector operators

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla \psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

Vector relations for spherical polar coordinates

$$\nabla\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\begin{aligned}\nabla\times\mathbf{A} &= \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] \\ &+ \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\end{aligned}$$

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$$

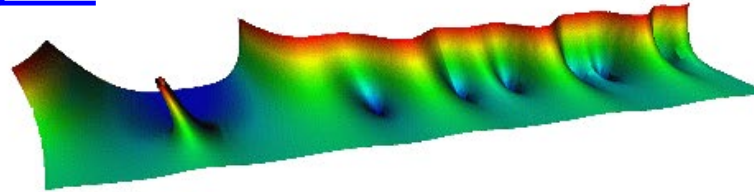
$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$$

$$\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$$

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \cos\theta\cos\phi\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \cos\theta\sin\phi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{\partial}{\partial\theta}$$



NIST Digital Library of Mathematical Functions

Project News

2018-09-15 [DLMF Update; Version 1.0.20](#)
2018-06-22 [DLMF Update; Version 1.0.19](#)
2018-06-22 [Philip J. Davis, A&S Author, dies at age 95](#)
2018-03-27 [DLMF Update; Version 1.0.18](#)
[More news](#)

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Example – special functions

10 Bessel Functions Bessel and Hankel Functions

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§10.2 Definitions

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- §10.2(ii) [Standard Solutions](#)
- §10.2(iii) [Numerically Satisfactory Pairs of Solutions](#)

§10.2(i) Bessel's Equation

10.2.1

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = 0.$$

This differential equation has a regular singularity at $z = 0$ with indices $\pm\nu$, and an irregular singularity at $z = \infty$ of rank 1; compare §§[2.7\(i\)](#) and [2.7\(ii\)](#).

§10.2(ii) Standard Solutions

Bessel Function of the First Kind

10.2.2

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{k! \Gamma(\nu + k + 1)}.$$

This solution of [\(10.2.1\)](#) is an analytic function of $z \in \mathbb{C}$, except for a branch point at $z = 0$ when ν is not an integer. The *principal branch* of $J_\nu(z)$ corresponds to the principal value of $\left(\frac{1}{2}z\right)^\nu$ ([§4.2\(iv\)](#)) and is analytic in the z -plane cut along the interval $(-\infty, 0]$.

Complex numbers

$$i \equiv \sqrt{-1} \quad i^2 = -1$$

$$\text{Define } z = x + iy$$

$$|z|^2 = zz^* = (x + iy)(x - iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos \phi + i \sin \phi) = \rho e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x, y) + iv(x, y)$$

Derivatives: Cauchy-Riemann equations

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i \frac{\partial v(z)}{\partial x} \quad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i \frac{\partial v(z)}{i\partial y} = \frac{\partial v(z)}{\partial y} - i \frac{\partial u(z)}{\partial y}$$

$$\text{Argue that } \frac{df}{dz} = \frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{i\partial y} \Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y} \quad \text{and} \quad \frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$$

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

→ A closed integral of an analytic function is zero.

However:

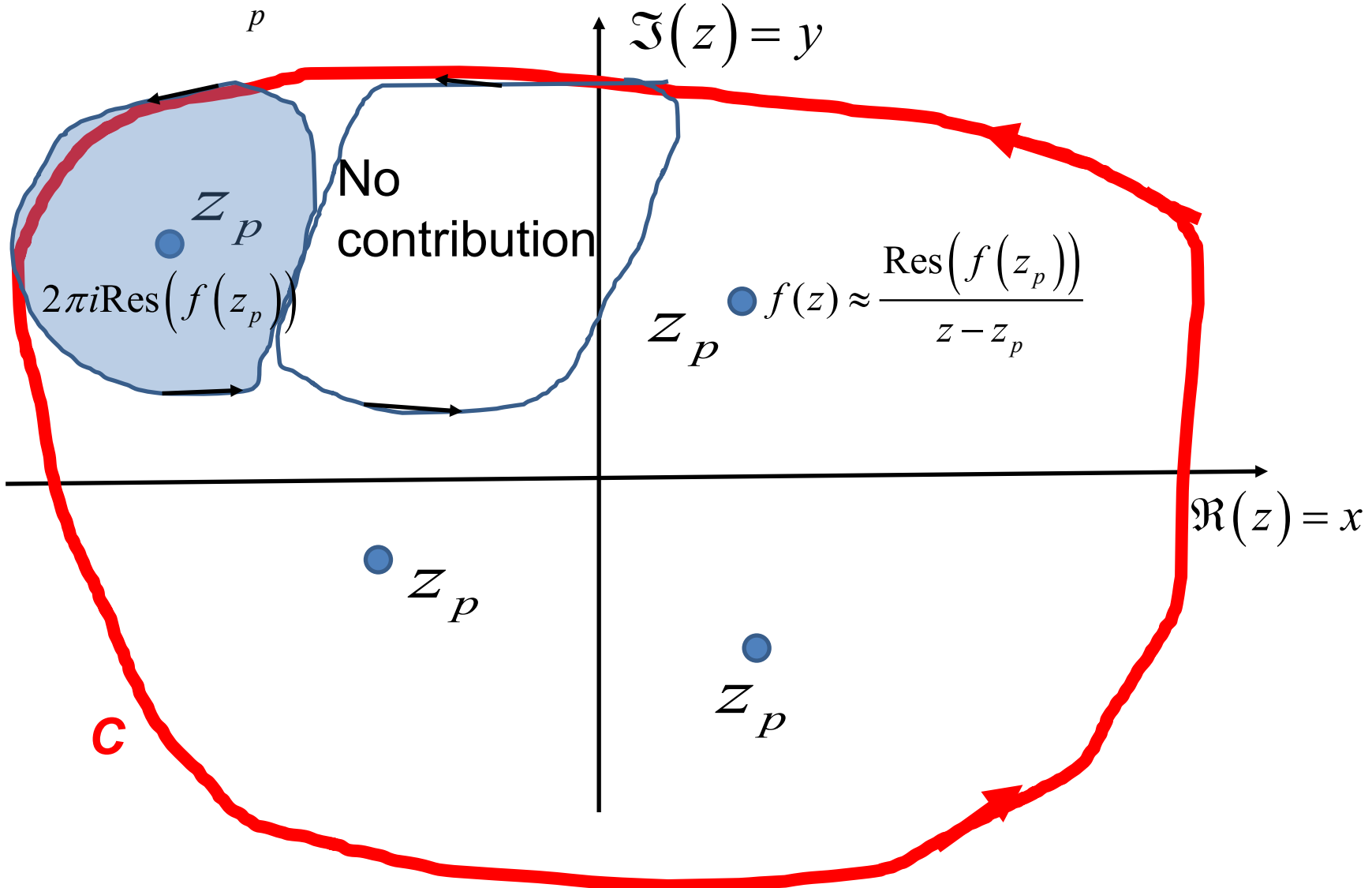
Behavior of $f(z) = \frac{1}{z^n}$ about the point $z = 0$:

For an integer n , consider

$$\oint \frac{1}{z^n} dz = \int_0^{2\pi} \frac{\rho e^{i\phi} i d\phi}{\rho^n e^{in\phi}} = \rho^{1-n} \int_0^{2\pi} e^{i(1-n)\phi} i d\phi = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

Contour integration methods --

$$\oint_C f(z) dz = 2\pi i \sum_p \text{Res}(f(z_p))$$



General formula for determining residue:

Suppose that in the neighborhood of z_p , $f(z) \approx \frac{g(z)}{(z - z_p)^m} \stackrel{z \rightarrow z_p}{\equiv} \frac{\text{Res}(f(z_p))}{z - z_p}$

Since $g(z)$ is analytic near z_p , we can make a Taylor expansion about z_p :

$$g(z) \approx g(z_p) + (z - z_p) \frac{dg(z_p)}{dz} + \dots + \frac{(z - z_p)^{m-1}}{(m-1)!} \frac{d^{m-1}g(z_p)}{dz^{m-1}} + \dots$$

$$\Rightarrow \text{Res}(f(z_p)) = \lim_{z \rightarrow z_p} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1} \left((z - z_p)^m f(z) \right)}{dz^{m-1}} \right\}$$

Fourier transforms --

Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega F(\omega) e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dt' f(t') e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt' f(t') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt' f(t') \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt (f(t))^* f(t) = 2\pi \int_{-\infty}^{\infty} d\omega (F(\omega))^* F(\omega)$$

Check:

$$\begin{aligned} \int_{-\infty}^{\infty} dt (f(t))^* f(t) &= \int_{-\infty}^{\infty} dt \left(\left(\int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t} \right)^* \int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t} \right) \\ &= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega') \int_{-\infty}^{\infty} dt e^{i(\omega' - \omega)t} \\ &= \int_{-\infty}^{\infty} d\omega F^*(\omega) \int_{-\infty}^{\infty} d\omega' F(\omega') 2\pi \delta(\omega' - \omega) \\ &= 2\pi \int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega) \end{aligned}$$

Doubly discrete Fourier Transforms

Doubly periodic functions

$$\omega \rightarrow \frac{2\pi\nu}{T} \quad t \rightarrow \frac{\mu T}{2N+1} \quad (N, \nu, \text{ and } \mu \text{ integers})$$

$$\tilde{f}_\mu = \frac{1}{2N+1} \sum_{\nu=-N}^N \tilde{F}_\nu e^{-i2\pi\nu\mu/(2N+1)}$$

$$\tilde{F}_\nu = \sum_{\mu=-N}^N \tilde{f}_\mu e^{i2\pi\nu\mu/(2N+1)}$$

➔ Fast Fourier Transforms (FFT)

Notions of eigenvalues and eigenvectors

In the context of linear algebra --

Eigenvalue properties of matrices

$$\mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$$

Hermitian matrix: $\mathbf{H}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$

$$H_{ij} = H_{ji}^*$$

Theorem for Hermitian matrices:

$$\lambda_\alpha \text{ have real values and } \mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$$

Unitary matrix: $\mathbf{U}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$ $\mathbf{U}^H \mathbf{U} = \mathbf{I}$

$$|\lambda_\alpha| = 1 \quad \text{and} \quad \mathbf{y}_\alpha^H \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$$

In the context of Sturm-Liouville differential equations --

Notions of eigenvalues and eigenvectors -- continued

Sturm Liouville differential equations, in terms of given functions $\tau(x)$, $\nu(x)$, and $\sigma(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + \nu(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x)$$

Orthogonality of eigenfunctions: $\int_a^b \sigma(x) f_n(x) f_m(x) dx = \delta_{nm} N_n$,

$$\text{where } N_n \equiv \int_a^b \sigma(x) (f_n(x))^2 dx.$$

Calculus of variation – a method to find a function $(y(x))$ which optimizes a particular integral relationship.

$$\text{For } f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right),$$

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx :$

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x, y} \right] = 0$$



Euler-Lagrange equation

Lagrangian in the presence of electromagnetic forces

Lagrangian: (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad U = q\Phi(\mathbf{r}, t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\text{where } \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\Phi(\mathbf{r}, t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$
2. Compute generalized momenta : $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function : $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Note that the equations of motion should yield equivalent trajectories for the Lagrangian and Hamiltonian formulations.

Mechanics topics

- Scattering theory
- Lagrangian mechanics
- Hamiltonian mechanics
- Liouville theorem
- Rigid body motion
- Normal modes of oscillation about equilibrium
- Wave motion
- Fluid mechanics (ideal or including viscosity; linear and nonlinear)
- Heat conduction
- Effects of viscosity on fluid mechanics

Note: The following review slides are necessarily brief. Please refer to the original “Extra” lecture slides for details. Please email: natalie@wfu.edu with any corrections/suggestions

Scattering theory

Note: The notion of cross section is common to many areas of physics including classical mechanics, quantum mechanics, optics, etc. Only in the **classical mechanics** can we calculate it from a knowledge of the particle trajectory as it relates to the scattering geometry.

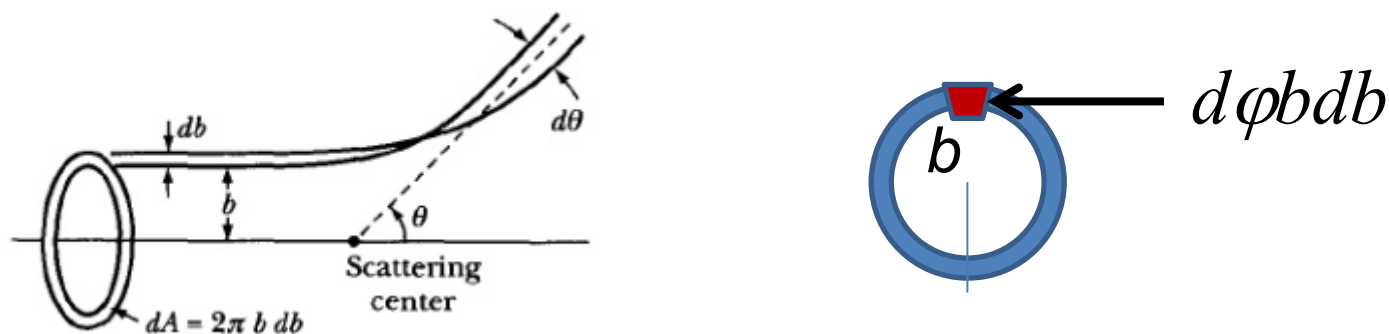


Figure from Marion & Thorton, Classical Dynamics

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{d\phi b db}{d\phi \sin\theta d\theta} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$$

Note: We are assuming that the process is isotropic in ϕ

Lagrangian mechanics

Given the Lagrangian function: $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$,

The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

Are those which minimize the action: $S = \int L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$

Euler-Lagrange equations:

$$\sum_\sigma \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) \delta q_\sigma = 0 \quad \Rightarrow \text{for each } \sigma : \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

For the case that there both mechanical and

electromagnetic contributions in terms of electric and magnetic fields:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$$

$$L = T - U_{\text{mech}} - q\Phi(\mathbf{r}, t) + \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function: $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$

2. Compute generalized momenta: $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$

3. Construct Hamiltonian expression: $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$

4. Form Hamiltonian function: $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

5. Analyze canonical equations of motion:

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Question – When can you bypass the 5 step derivation process and directly write the Hamiltonian of the system as

$$H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t) = \sum_\sigma \frac{p_\sigma^2}{2m_\sigma} + V(\{q_\sigma\})$$

1. Only when Natalie Holzwarth is not looking
2. When you have a simple system that has no explicit velocity and/or time dependence
3. Usually

Important tool for analyzing Lagrangian and/or Hamiltonian systems -- finding constants of the motion

In Lagrangian formulation --

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t) \quad \Rightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \dot{q}_\sigma} = (\text{constant})$

Additionally:
$$\frac{d}{dt} \left(L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma \right) = \frac{\partial L}{\partial t}$$

For $\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_\sigma} \dot{q}_\sigma = -E \quad (\text{constant})$

Constants of the motion in the Hamiltonian formulation

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \sum_\sigma (-\dot{p}_\sigma \dot{q}_\sigma + \dot{q}_\sigma \dot{p}_\sigma) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

$$\Rightarrow \text{constant } H \text{ if } \frac{\partial H}{\partial t} = 0$$

Question – Why use this fancy formalism when simple conservation of energy or momentum intuitively apply?

- a. You should use your intuition whenever possible.
- b. You should never trust your intuition.
- c. The equations should be consistent with correct intuitive solutions and also reveal additional solutions (perhaps beyond intuition)

Liouville's Theorem (1838)

The density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion.

Denote the density of particles in phase space: $D = D(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dD}{dt} = \sum_{\sigma} \left(\frac{\partial D}{\partial q_{\sigma}} \dot{q}_{\sigma} + \frac{\partial D}{\partial p_{\sigma}} \dot{p}_{\sigma} \right) + \frac{\partial D}{\partial t}$$

According to Liouville's theorem: $\frac{dD}{dt} = 0$

Rigid body motion

Moment of inertia tensor :

$$\tilde{\mathbf{I}} \equiv \sum_p m_p (\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p) \quad (\text{dyad notation})$$

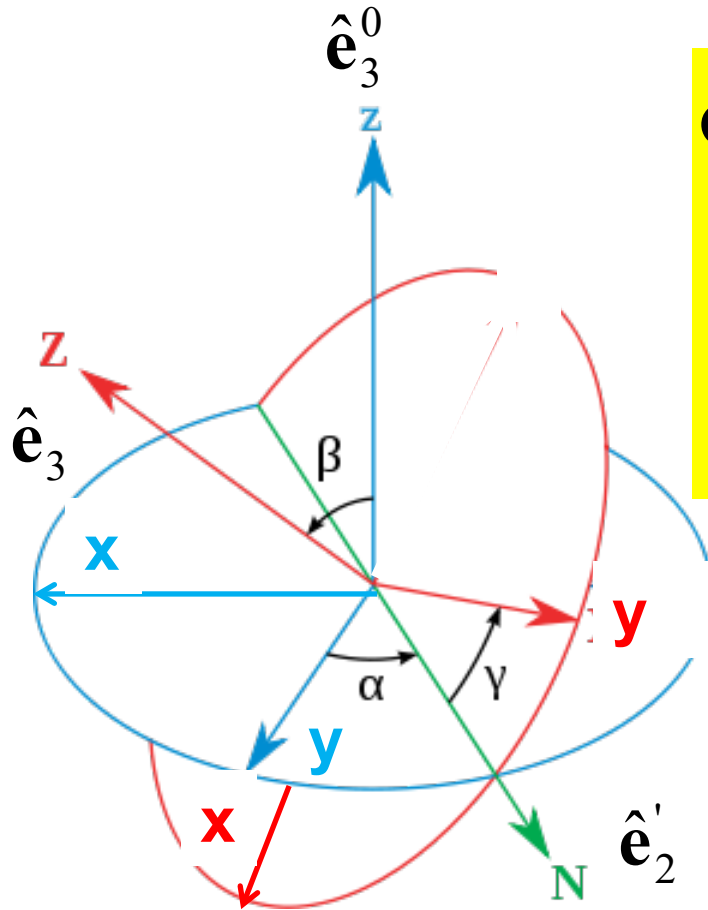
In a reference frame attached to the object, there are 3 moments of inertia and 3 distinct principal axes

Representation of rotational kinetic energy:

$$\begin{aligned} T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \frac{1}{2} I_1 \tilde{\omega}_1^2 + \frac{1}{2} I_2 \tilde{\omega}_2^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2 \\ &= \frac{1}{2} I_1 \left[\dot{\alpha} (-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma \right]^2 \\ &\quad + \frac{1}{2} I_2 \left[\dot{\alpha} (\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma \right]^2 \\ &\quad + \frac{1}{2} I_3 \left[\dot{\alpha} \cos \beta + \dot{\gamma} \right]^2 \end{aligned}$$

Euler's transformation between body fixed and inertial reference frames

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha} \hat{\mathbf{e}}_3^0 + \dot{\beta} \hat{\mathbf{e}}_2' + \dot{\gamma} \hat{\mathbf{e}}_3$$

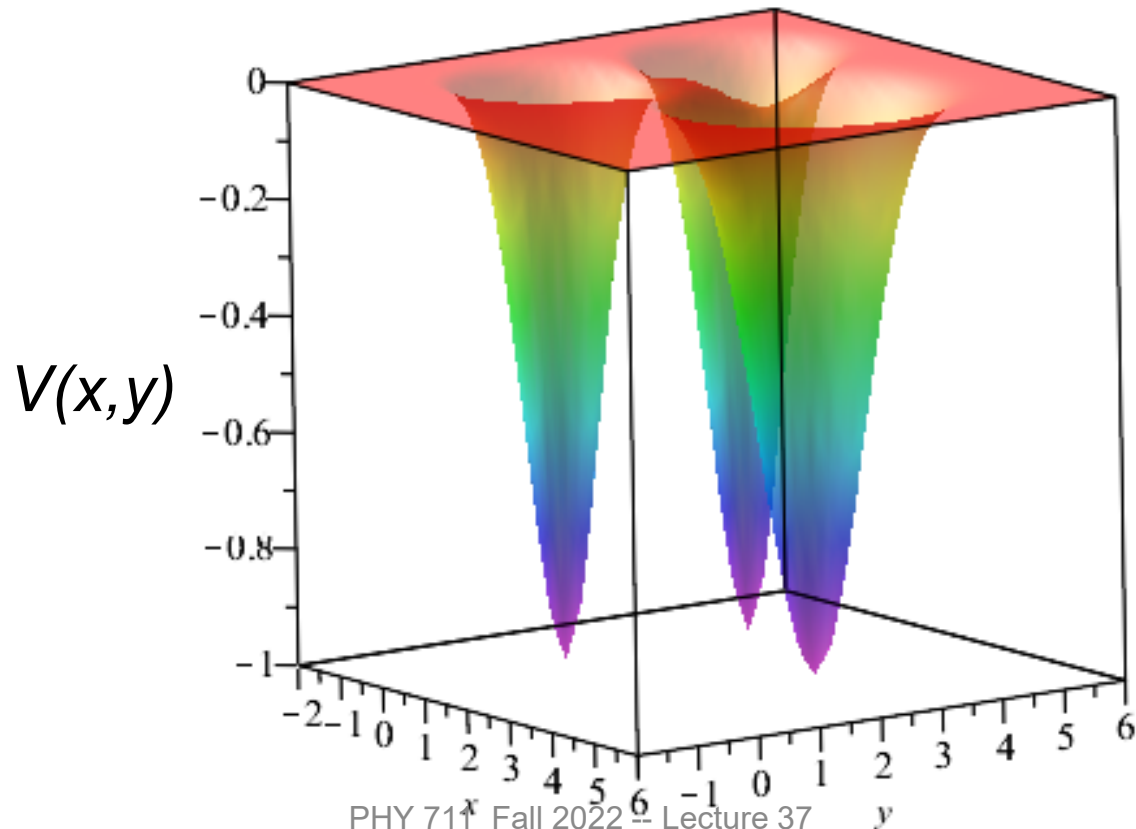


$$\begin{aligned} \tilde{\boldsymbol{\omega}} = & \left[\dot{\alpha} (-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma \right] \hat{\mathbf{e}}_1 \\ & + \left[\dot{\alpha} (\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma \right] \hat{\mathbf{e}}_2 \\ & + \left[\dot{\alpha} \cos \beta + \dot{\gamma} \right] \hat{\mathbf{e}}_3 \end{aligned}$$

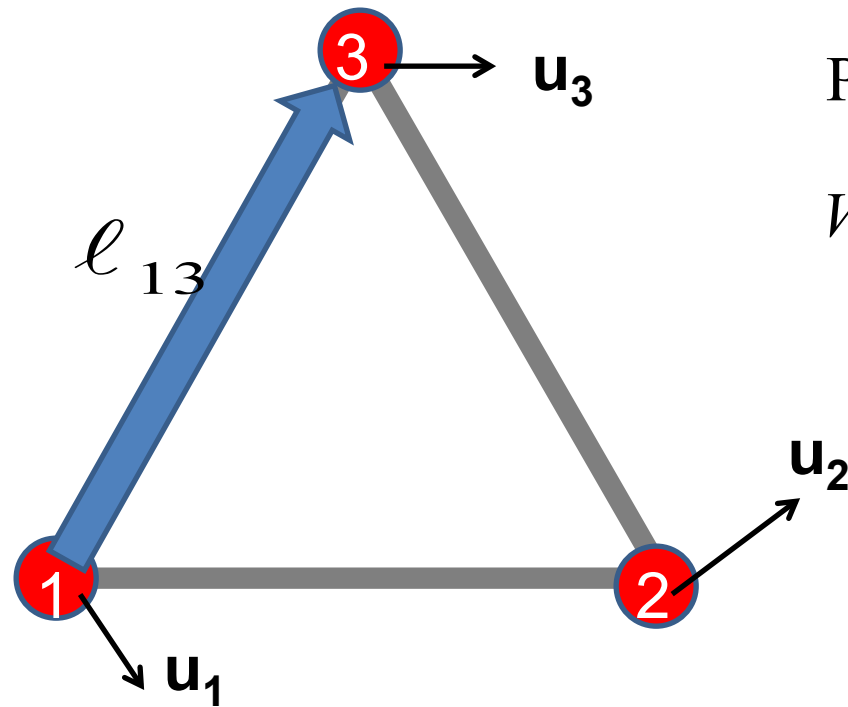
Normal modes of vibration -- potential in 2 and more dimensions

$$V(x, y) \approx V(x_{eq}, y_{eq}) + \frac{1}{2} (x - x_{eq})^2 \left. \frac{\partial^2 V}{\partial x^2} \right|_{x_{eq}, y_{eq}}$$

$$+ \frac{1}{2} (y - y_{eq})^2 \left. \frac{\partial^2 V}{\partial y^2} \right|_{x_{eq}, y_{eq}} + (x - x_{eq})(y - y_{eq}) \left. \frac{\partial^2 V}{\partial x \partial y} \right|_{x_{eq}, y_{eq}}$$



Example – normal modes of a system with the symmetry of an equilateral triangle -- continued



Potential contribution for spring 13:

$$V_{13} = \frac{1}{2}k \left(|\ell_{13} + \mathbf{u}_3 - \mathbf{u}_1| - |\ell_{13}| \right)^2$$

$$\approx \frac{1}{2}k \left(\frac{\ell_{13} \cdot (\mathbf{u}_3 - \mathbf{u}_1)}{|\ell_{13}|} \right)^2$$

$$\approx \frac{1}{2}k \left(\frac{1}{2}(u_{x3} - u_{x1}) + \frac{\sqrt{3}}{2}(u_{y3} - u_{y1}) \right)^2$$

$$\ell_{13} = |\ell_{13}| \left(\frac{1}{2} \hat{\mathbf{x}} + \frac{\sqrt{3}}{2} \hat{\mathbf{y}} \right)$$

Example – normal modes of a system with the symmetry of an equilateral triangle -- continued

Potential contributions: $V = V_{12} + V_{13} + V_{23}$

$$\approx \frac{1}{2}k \left(\frac{\ell_{12} \cdot (\mathbf{u}_2 - \mathbf{u}_1)}{|\ell_{12}|} \right)^2 + \frac{1}{2}k \left(\frac{\ell_{13} \cdot (\mathbf{u}_3 - \mathbf{u}_1)}{|\ell_{13}|} \right)^2$$

$$+ \frac{1}{2}k \left(\frac{\ell_{23} \cdot (\mathbf{u}_3 - \mathbf{u}_2)}{|\ell_{23}|} \right)^2$$

$$\approx \frac{1}{2}k (u_{x2} - u_{x1})^2$$

$$+ \frac{1}{2}k \left(\frac{1}{2}(u_{x3} - u_{x1}) + \frac{\sqrt{3}}{2}(u_{y3} - u_{y1}) \right)^2$$

$$+ \frac{1}{2}k \left(\frac{1}{2}(u_{x2} - u_{x3}) - \frac{\sqrt{3}}{2}(u_{y2} - u_{y3}) \right)^2$$

Example – normal modes of a system with the symmetry of an equilateral triangle -- continued

$$\frac{k}{m} \begin{bmatrix} \frac{5}{4} & -1 & -\frac{1}{4} & \frac{1}{4}\sqrt{3} & 0 & -\frac{1}{4}\sqrt{3} \\ -1 & \frac{5}{4} & -\frac{1}{4} & 0 & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 \\ \frac{1}{4}\sqrt{3} & 0 & -\frac{1}{4}\sqrt{3} & \frac{3}{4} & 0 & -\frac{3}{4} \\ 0 & -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 & \frac{3}{4} & -\frac{3}{4} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4}\sqrt{3} & 0 & -\frac{3}{4} & -\frac{3}{4} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{y1} \\ u_{y2} \\ u_{y3} \end{bmatrix} = \omega^2 \begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{y1} \\ u_{y2} \\ u_{y3} \end{bmatrix}$$

Discrete particle interactions \rightarrow continuous media \rightarrow
The wave equation

Initial value solutions $\mu(x, t)$ to the wave equation;
attributed to D'Alembert:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x, 0) = \varphi(x) \text{ and } \frac{\partial \mu}{\partial t}(x, 0) = \psi(x)$$

$$\Rightarrow \mu(x, t) = \frac{1}{2} (\varphi(x - ct) + \varphi(x + ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

Mechanical motion of fluids

Newton's equations for fluids

Use Euler formulation; following “particles” of fluid

Variables : Density $\rho(x,y,z,t)$

Pressure $p(x,y,z,t)$

Velocity $\mathbf{v}(x,y,z,t)$

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f} - \frac{1}{\rho} \nabla p + \underbrace{\frac{\eta}{\rho} \nabla^2 \mathbf{v} + \frac{1}{\rho} \left(\zeta + \frac{1}{3} \eta \right) \nabla (\nabla \cdot \mathbf{v})}_{\text{Viscosity contributions}}$$

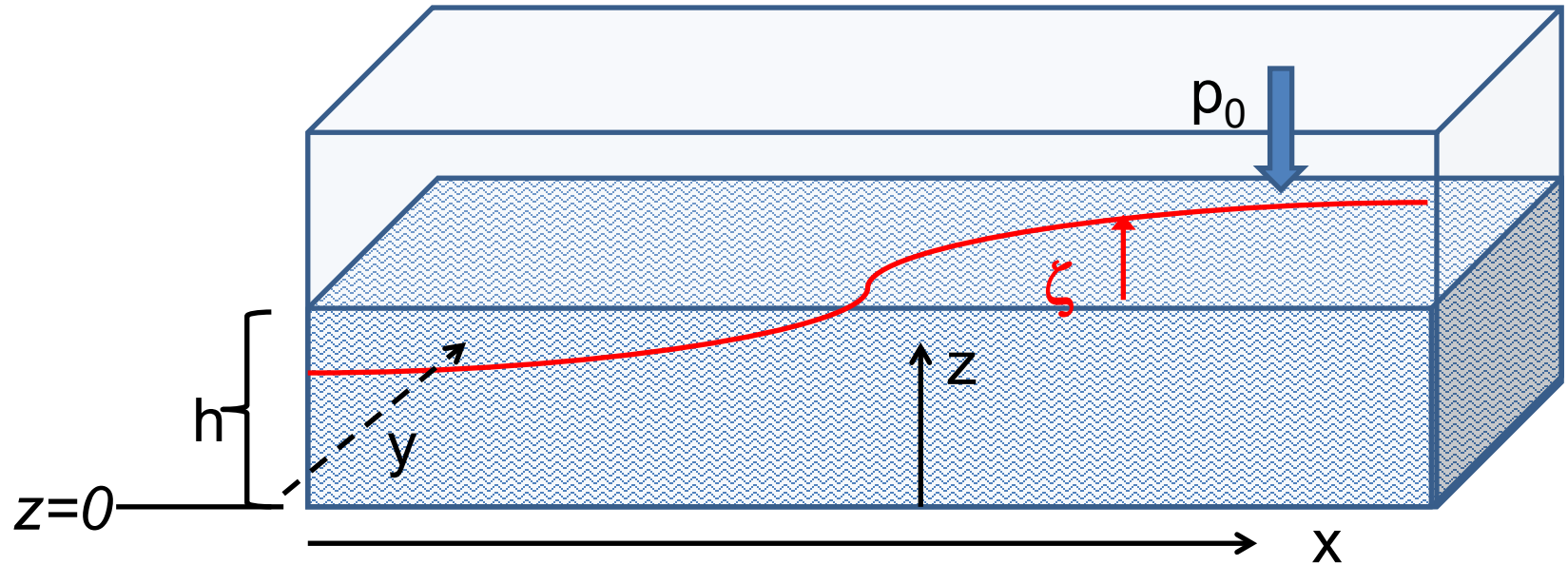
Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Viscosity contributions

Fluid mechanics of incompressible fluid plus surface

Non-linear effects in surface waves:



Dominant non-linear effects \Rightarrow soliton solutions

$$\zeta(x, t) = \eta_0 \operatorname{sech}^2 \left(\sqrt{\frac{3\eta_0}{h}} \frac{x - ct}{2h} \right) \quad \eta_0 = \text{constant}$$

$$\text{where } c = \sqrt{\frac{gh}{1 - \eta_0/h}} \approx \sqrt{gh} \left(1 + \frac{\eta_0}{2h} \right)$$

Example of solving coupled equations

Lagrangian equations of motion for a Lorentz force

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$\frac{d}{dt} \left(m\dot{x} - \frac{q}{2c} B_0 y \right) - \frac{q}{2c} B_0 \dot{y} = 0 \quad \Rightarrow \quad m\ddot{x} - \frac{q}{c} B_0 \dot{y} = 0$$

$$\frac{d}{dt} \left(m\dot{y} + \frac{q}{2c} B_0 x \right) + \frac{q}{2c} B_0 \dot{x} = 0 \quad \Rightarrow \quad m\ddot{y} + \frac{q}{c} B_0 \dot{x} = 0$$

$$\frac{d}{dt} m\dot{z} = 0 \quad \Rightarrow \quad m\ddot{z} = 0$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c}B_0(-\dot{x}y + \dot{y}x)$$

$$m\ddot{x} = +\frac{q}{c}B_0\dot{y}$$

$$m\ddot{y} = -\frac{q}{c}B_0\dot{x}$$

$$m\ddot{z} = 0$$

Need to find $z(t), x(t), y(t)$.

In this case, the initial conditions are

$$z(0) = 0, x(0) = 0, y(0) = 0 \quad \dot{z}(0) = 0, \dot{x}(0) = U_0, \dot{y}(0) = 0$$

Comment on linear equations vs non-linear equations
 Example of the hydrodynamic and continuity equations for one spatial dimension in terms of $\rho(x,t)$, $p(x,t)$, $v(x,t)$:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$$

In the linear approximation:

$$\rho(x,t) = \rho_0 + \delta\rho(x,t)$$

$$p(x,t) = p_0 + \delta p(x,t)$$

$$v(x,t) = v_0 + \delta v(x,t)$$

$$\frac{\partial \delta v}{\partial t} = - \frac{1}{\rho_0} \frac{\partial \delta p}{\partial x}$$

$$\frac{\partial \delta \rho}{\partial t} + \rho_0 \frac{\partial \delta v}{\partial x} = 0$$

Properties of linear equations –

1. Superposition of solutions
2. Many systematic solution methodologies
3. For the case of Sturm-Liouville equations
 - a. Completeness
 - b. Green's functions

Properties of non-linear equations –

1. Specialized solutions for each case
2. Sometimes, beautiful analytic forms (solitons)
3. Sometimes solutions lead to unphysical effects (shock waves)
4. Within the context of Fourier analysis, non-linearities often lead to frequency doubling or other frequency modifications