PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 5

Review of classical mechanical scattering theory – Chap 1 F&W

1. Overview
2. Some numerical considerations
3. Discussion questions
4. Review

Physics Colloquium Thursday, September 1, 2022
4 PM in Olin 101
Refreshments at 3:30 PM in Olin Lobby

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/22/2022		Introduction	#1	8/26/2022
2	Wed, 8/24/2022	Chap. 1	Scattering theory		
3	Fri, 8/26/2022	Chap. 1	Scattering theory	#2	8/29/2022
4	Mon, 8/29/2022	Chap. 1	Scattering theory	#3	8/31/2022
5	Wed, 8/31/2022	Chap. 1	Summary of scattering theory	#4	9/02/2022
6	Fri, 9/02/2022	Chap. 2	Non-inertial coordinate systems		

PHY 711 – Assignment #4

08/31/2022

1. Equation 5.28 in Fetter and Walecka represents the differential equation evaluated in the center of mass parameters of an alpha particle ($z = 2$) having center mass energy E acting on a gold particle ($Z = 79$) assumed to be initially at rest. In the lab frame, the initial kinetic energy of the alpha particle is 5 mega electron volts (MeV). Use reliable sources for the fundamental constants such as those from the NIST website `https://physics.nist.gov/cuu/Constants/index.html` in order to perform numerical evaluations.

   (a) Evaluate in MeV units the energy associated with the center of mass motion of the two particle system (alpha particle and gold nucleus). (Is it significant or neglibible?)

   (b) Fetter and Walecka use cgs Gaussian units for the Coulomb interaction. Using SI units for the Coulomb interaction, rewrite the equation for the differential cross section.

   (c) Evaluate the differential equation (in units of m^2) at various center of mass scattering angles θ at least for $\theta = 45, 90$, and 135 degrees.

Your questions –

From Sam -- If one particle in a 2 particle pair is stationary, doesn't the kinetic energy related to the center of mass have to be the same as the energy of particle one in the lab frame? I looked ahead to Friday's hw and was slightly confused.

From Lee -- Could you please remind me of what K (kappa) represents in the Rutherford scattering equations (near the end of your presentation)?

From Zezong -- The notes say there are relatively few forms of (1/u) for which the integral has an analytic result, does it mean in real life it is relatively easy to find numerical results in most cases?

Comments on numerical evaluations

- Fetter and Walecka like many older textbooks use cgs Gaussian units (centimeters, grams, seconds plus miscellaneous factors of 4π) while "modern" texts use SI units (meters, kilograms, seconds plus other miscellaneous factors of $4\pi\varepsilon_0$)

Coulomb force between *ze* and *Ze* at separation **r**:

cgs Gaussian units

$$\mathbf{F}_{Coulomb} = \frac{zZe^2\mathbf{r}}{r^3}$$

SI units

$$\mathbf{F}_{Coulomb} = \frac{zZe^2\mathbf{r}}{4\pi\epsilon_0 r^3}$$

Comments on numerical evaluations -- continued

- MeV is a convenient unit of energy related to SI unit of Joules

$$1\ \text{eV}=1.602176634\text{x}10^{-19}\ \text{J}$$
$$1\ \text{MeV}= 10^{6}\ \text{eV}=1.602176634\text{x}10^{-13}\ \text{J}$$

- More generally a reliable source for fundamental constants is available at the NIST website https://physics.nist.gov/cuu/Constants/index.html

The NIST Reference on Constants, Units, and Uncertainty

Information at the foundation of modern science and technology from the Physical Measurement Laboratory of NIST

CODATA Internationally recommended 2018 values of the Fundamental Physical Constants

Constants Topics:
- Values
- Energy Equivalents
- Searchable Bibliography
- Background

Constants Bibliography

Constants, Units & Uncertainty home page

Version history and disclaimer

(e.g., **electron mass**, most misspellings okay)

Search by name [ ] Search

Display ◉ alphabetical list, ○ table (image), or ○ table (pdf)

by clicking a category below

Universal	Defined constants	Frequently used constants
Electromagnetic	Non-SI units	Extensive listings
Atomic and nuclear	Conversion factors for energy equivalents	All values (ascii)
Physico-chemical	X-ray values	

Find the correlation coefficient between any pair of constants

What constants will you need to know?

What units will your answer have?

Reviewing – scattering in the laboratory

Figure from F&W:

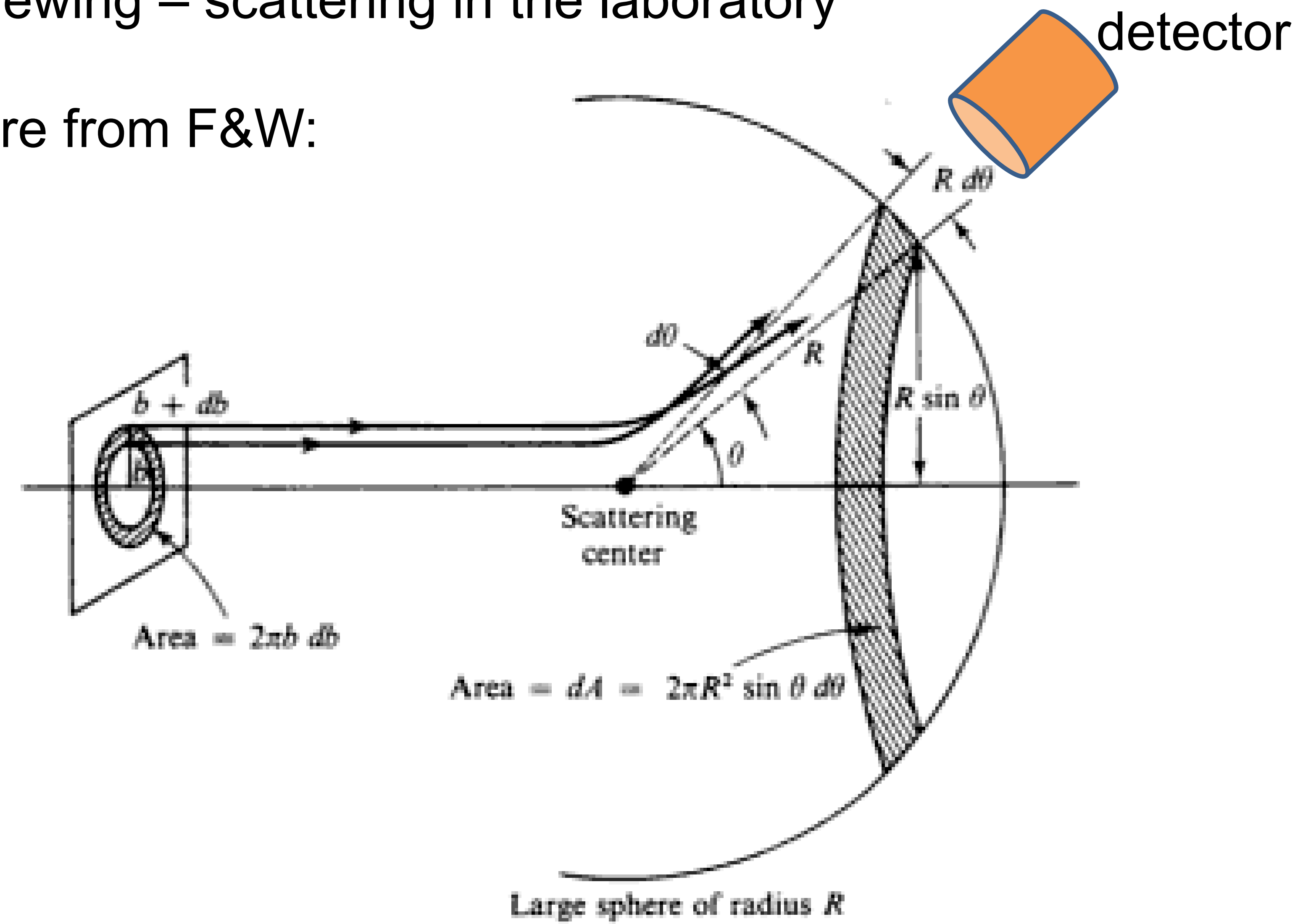


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Comment about angularly dependent interaction potentials *V(**r**)*

Comments -- These certainly occur in nature and are important. However, the equations we have used here have to be modified. The experimental set up will be the same, but the differential cross section will depend both on θ and ϕ.

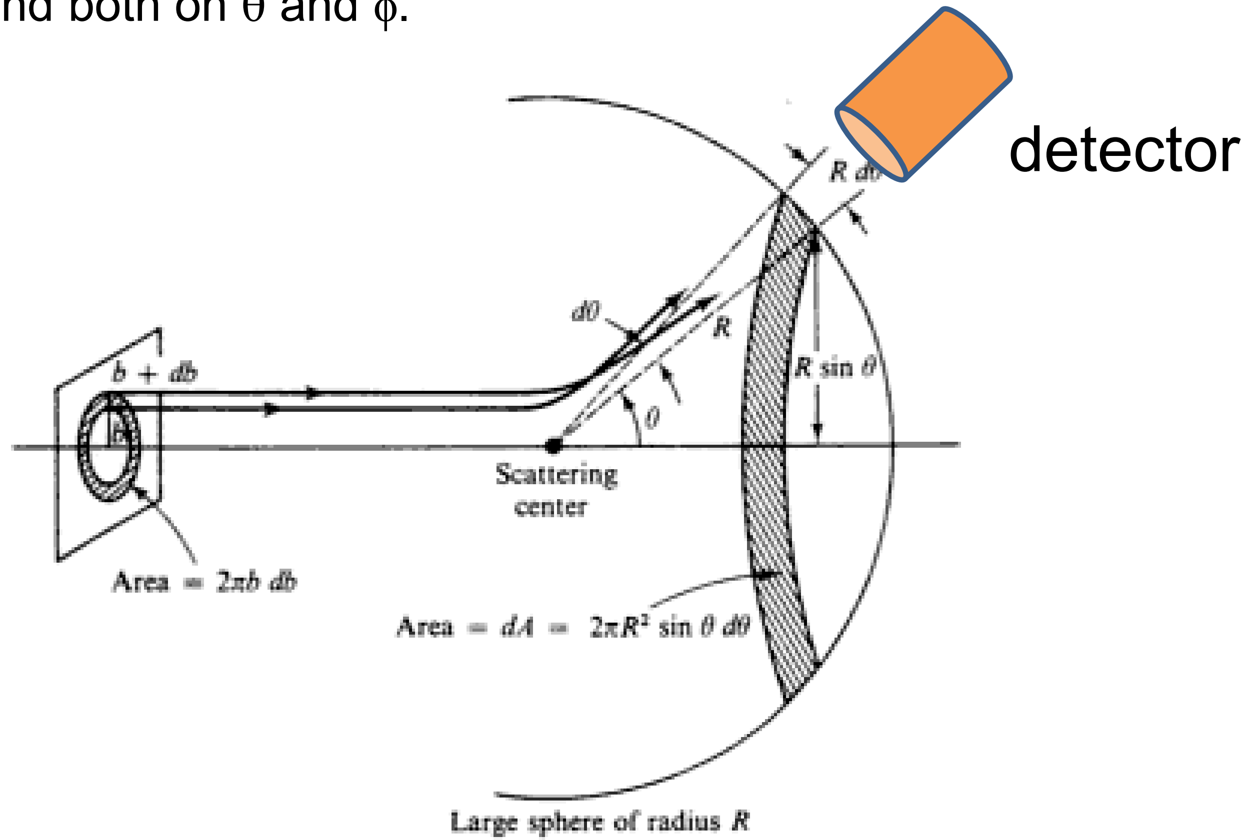


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

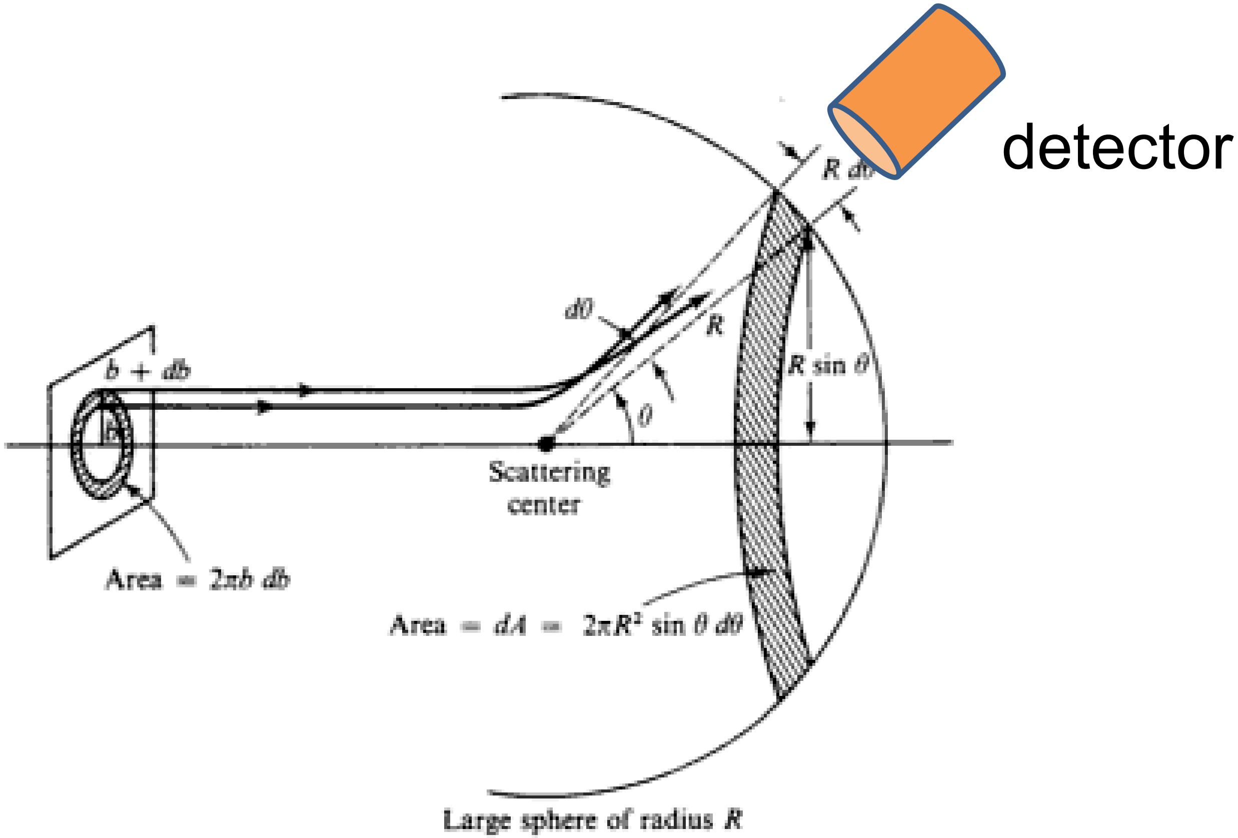
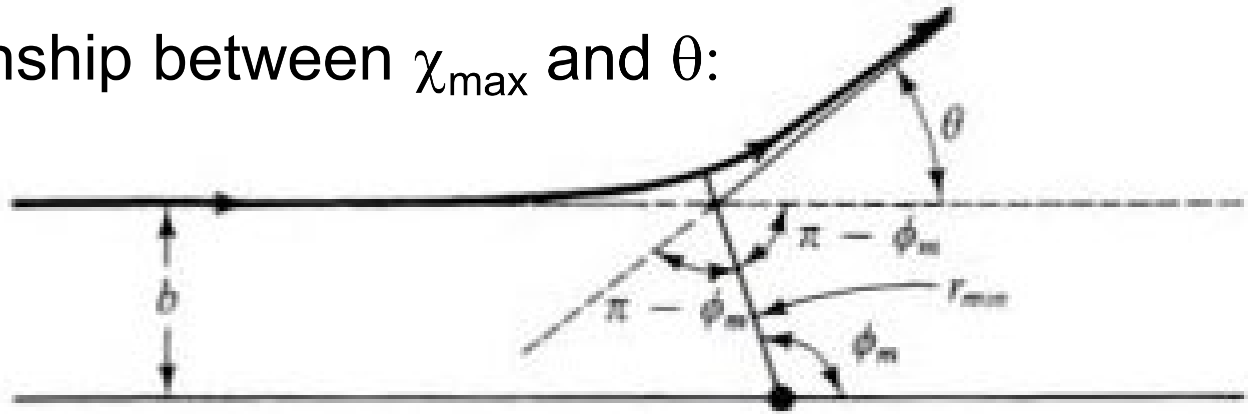


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

Note that this diagram implies a repulsive interaction.
How would it look if the interaction was attractive?

Close up of repulsive interaction

Relationship between χ_{max} and θ:



$$2(\pi - \chi_{\max}) + \theta = \pi$$

$$\Rightarrow \chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2}$$

Using the diagram from your text, θ represents the scattering angle in the center of mass frame and ϕ is used instead of χ.

More details

Total energy of system:

$$E = \frac{1}{2}(m_1 + m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r) \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$E = \quad E_{\text{Center of mass}} \quad + \quad E_{rel}$$

Recall that $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Focus on relative motion: $E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$

Since **r**(t) represents motion in a plane, we will analyze the system in that plane and use polar coordinates.

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$$

$$x(t) = r(t)\cos(\chi(t))$$

$$y(t) = r(t)\sin(\chi(t))$$

Note that $\left|\dot{\mathbf{r}}(t)\right|^2 = \dot{x}^2(t) + \dot{y}^2(t)$

$$= \dot{r}^2(t) + r^2(t)\dot{\chi}^2(t)$$

Some details

Total energy of system:

$$E_{\text{total}} = \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V\left(r\right)$$

$$E_{\text{total}} = \quad E_{\text{Center of mass}} \quad + \quad E_{rel} \quad = \quad \text{constant}$$

$$E_{\text{Center of mass}} = \frac{1}{2}\left(m_1 + m_2\right)V_{CM}^2 \quad = \quad \text{constant}$$

$$\mathbf{V}_{CM} \equiv \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} = \frac{m_1\mathbf{v}_1(t=0)}{m_1 + m_2}$$

$$E_{\text{Center of mass}} = \frac{1}{2}\frac{m_1^2 v_1^2(t=0)}{m_1 + m_2} = \frac{1}{2}m_1 v_1^2(t=0)\left(\frac{m_1}{m_1 + m_2}\right)$$

Note that $E_{rel} = \frac{1}{2}m_1 v_1^2(t=0)\left(\frac{m_2}{m_1 + m_2}\right)$

Clarification –

$$E_{total} = E_{\text{Center of mass}} + E_{rel}$$

Energy of the center mass motion

Energy within the center of mass reference frame

In the following slides E_{rel} is written E

Also note that the relative angular momentum of the system is a constant

$\ell=\mu r^2\dot{\chi}$

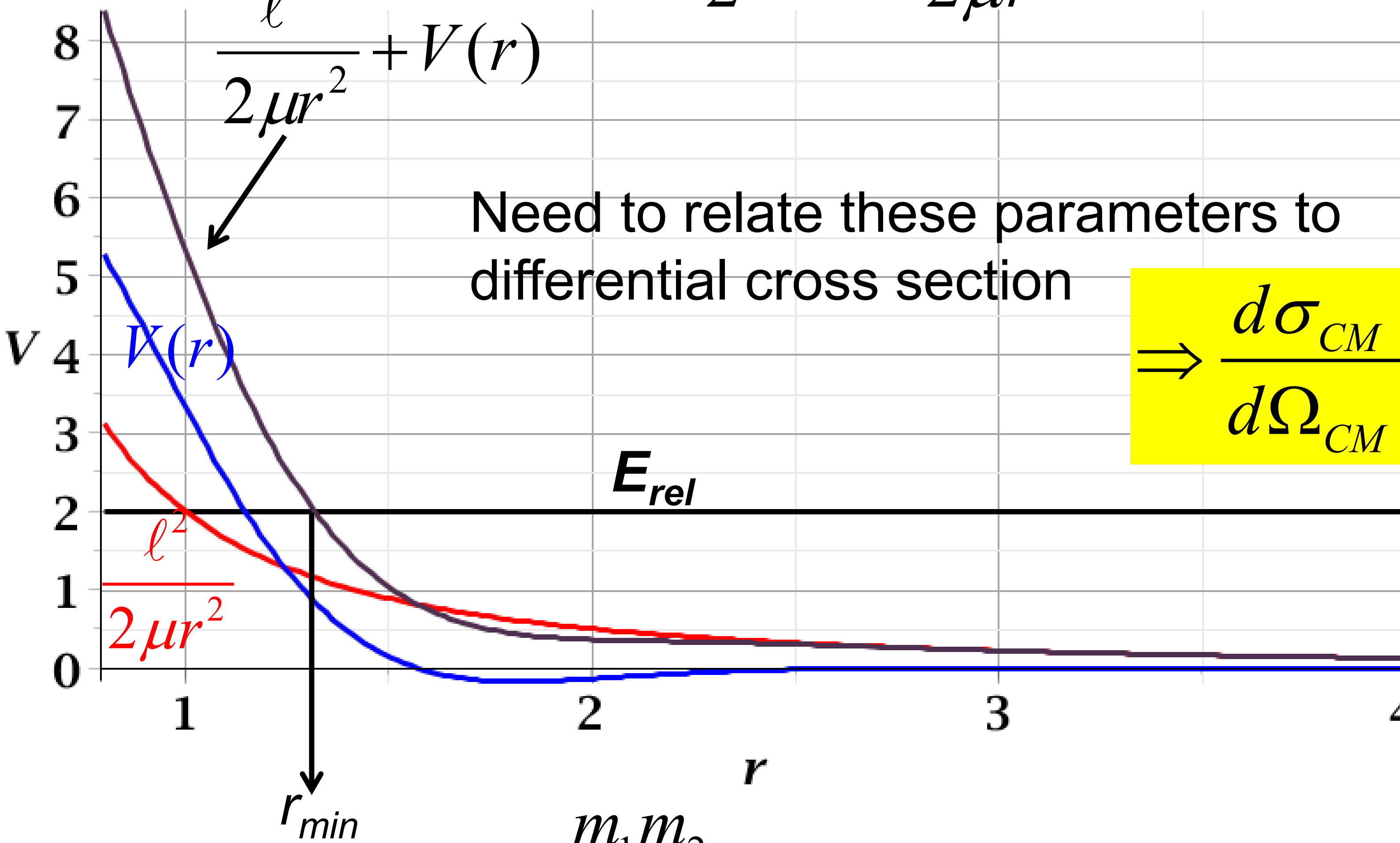
So that $\frac{1}{2}\mu\ \left|\dot{\mathbf{r}}(t)\right|^2 = \frac{1}{2}\mu\left(\dot{r}^2(t)+r^2(t)\dot{\chi}^2(t)\right)$

$$= \frac{1}{2}\mu\dot{r}^2(t)+\frac{\ell^2}{2\mu r^2}$$

➔ $E_{rel}=\frac{1}{2}\mu\dot{r}^2+\frac{\ell^2}{2\mu r^2}+V\left(r\right)$

For a continuous potential interaction in center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



Need to relate these parameters to differential cross section

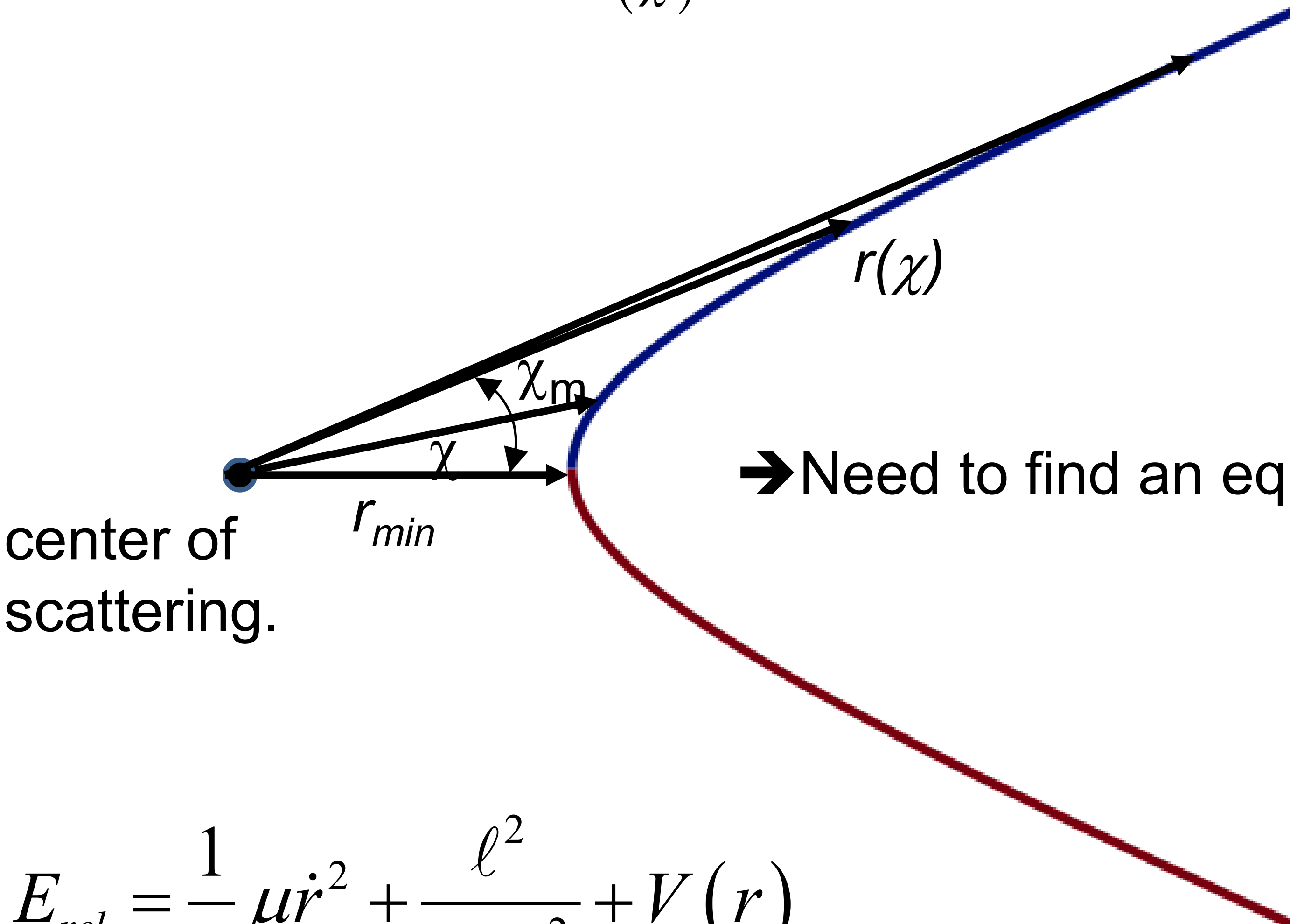
$$\Rightarrow \frac{d\sigma_{CM}}{d\Omega_{CM}}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

ℓ=angular momentum

Trajectory of relative vector in center of mass frame

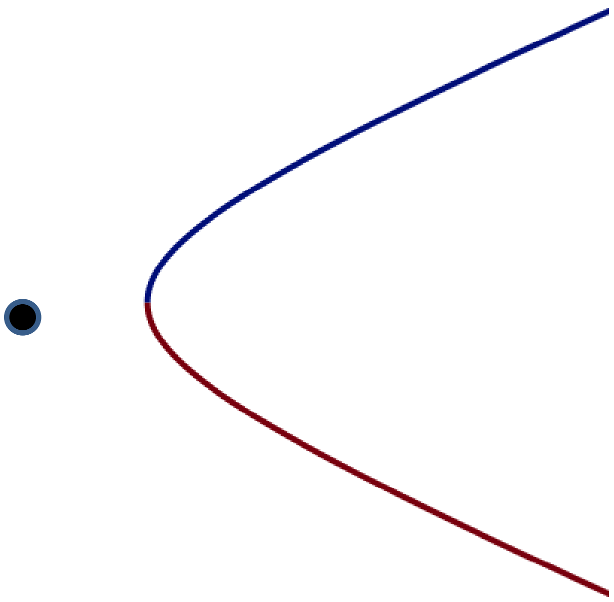
$r(\chi)$



➔Need to find an equation for $r(\chi)$

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

How is this related to scattering?



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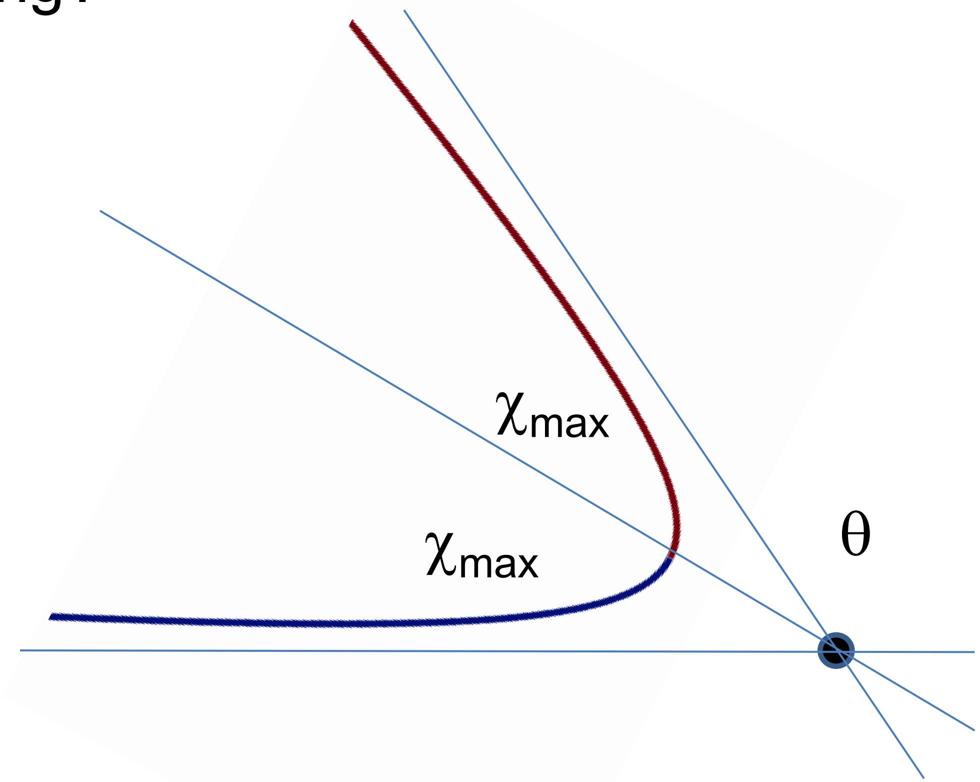


Note that here θ is used for the scattering angle

Note that we have use ψ to denote the scattering angle in the center of mass frame, but your textbook uses θ (which we had used to denote the scattering angle in the lab frame). In this lecture our analysis is entirely in the center of mass frame and some of the equations use θ to denote the scattering angle.

Questions:

1. How can we find $r(\chi)$?
2. If we find $r(\chi)$, how can we relate χ to ψ? (Here ψ is CM scattering angle.)
3. How can we find $b(\psi)$?

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{b}{\sin\psi}\left|\frac{db}{d\psi}\right|$$

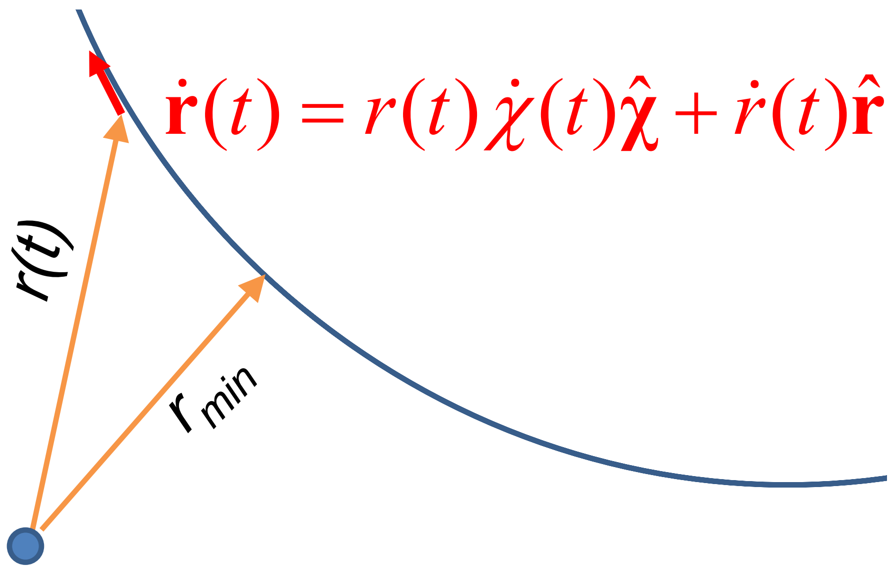
Evaluation of constants far from scattering center --

$$\ell = \mathbf{r} \times \left(\mu \dot{\mathbf{r}} \right) = r \mu r \frac{d\chi}{dt} = \mu r^2 \frac{d\chi}{dt}$$

$$\text{also:} \quad \ell = b \mu \dot{r}(t = -\infty)$$

$$E_{rel} = \frac{1}{2} \mu \left(\dot{r}(t = -\infty) \right)^2$$

$$\Rightarrow \quad \ell = b\sqrt{2 \mu E_{rel}}$$



Conservation of energy in the center of mass frame:

$$E_{rel} \equiv E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\chi)$$

$$\frac{dr}{dt} = \frac{dr}{d\chi}\frac{d\chi}{dt} = \frac{dr}{d\chi}\frac{\ell}{\mu r^2}$$

Here, constant angular momentum is: $\ell = \mu r^2\left(\frac{d\chi}{dt}\right)$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{d\chi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\chi) \Leftrightarrow \chi(r)$:

From: $$E = \frac{1}{2}\mu\left(\frac{dr}{d\chi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$\left(\frac{dr}{d\chi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\chi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

When the dust clears:

$$d\chi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\chi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

$$\Rightarrow \chi_{\max}(b, E) = \chi\left(r \to \infty\right) - \chi\left(r = r_{\min}\right)$$

$$\int_0^{\chi_{max}} d\chi = \int_{r_{min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{min}^2} - \frac{V(r_{min})}{E} = 0$$

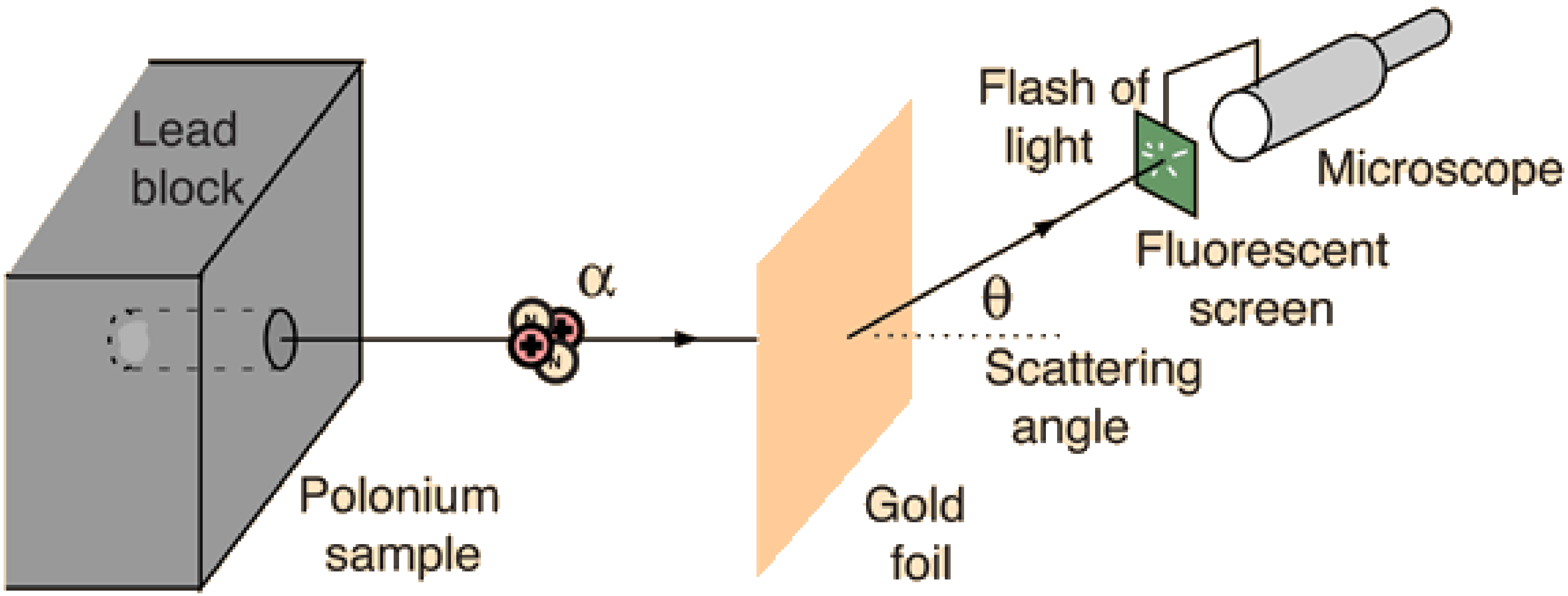
General equations for central potential V(r)

$$\chi_{\max} = \frac{\pi}{2} + \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1 / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = -\pi + 2b \int_{0}^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

Example: Diagram of Rutherford scattering experiment
http://hyperphysics.phy-astr.gsu.edu/hbase/rutsca.html



Scattering angle equation:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad 1 - \frac{b^2}{r_{\min}^2} - \frac{\kappa}{r_{\min}} = 0$$

$$\frac{1}{r_{\min}} = \frac{1}{b}\left(-\frac{\kappa}{2\mathrm{b}} + \sqrt{\left(\frac{\kappa}{2\mathrm{b}}\right)^2 + 1} \right)$$

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

Some details –

Rutherford scattering example:

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r}$$

SI units

$$V(r) = \frac{zZe^2}{4\pi\epsilon_0 r}$$ where e represents the elementary charge in Coulombs

r represents the particle separation in meters

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad \Rightarrow \kappa = \frac{zZe^2}{4\pi\epsilon_0 E}$$

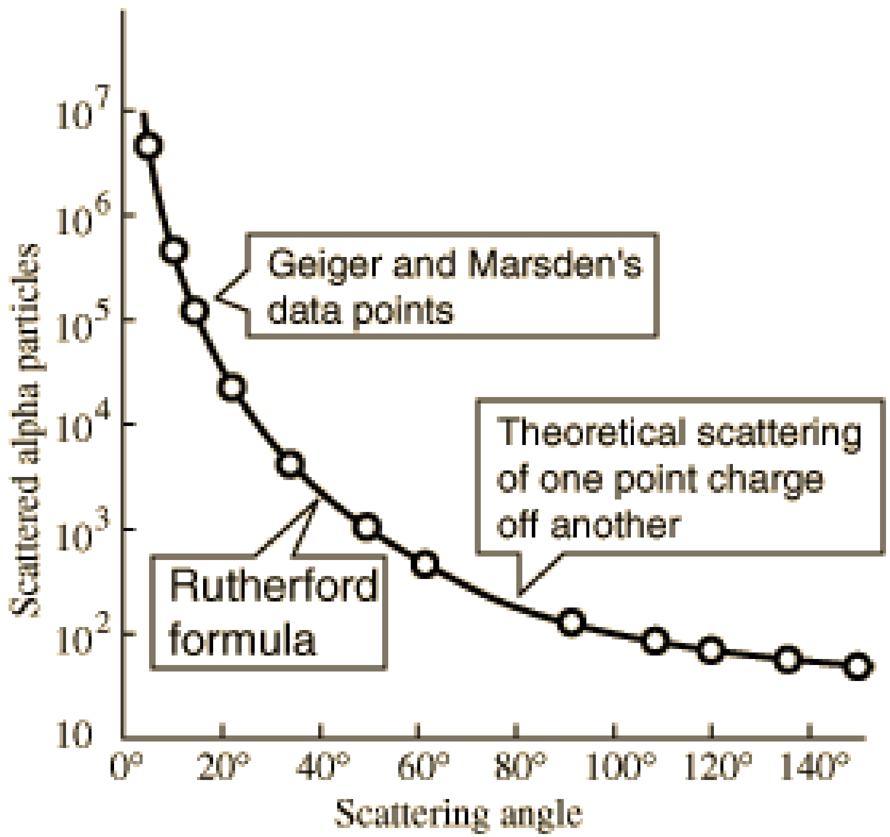
Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2+1}}\right)$$

$$\frac{2b}{\kappa} = \left|\frac{\cos(\theta/2)}{\sin(\theta/2)}\right|$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|=\frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$



What happens as $\theta \rightarrow 0$?

From webpage: http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/rutsca2.html#c3

Original experiment performed with α particles on gold

$$\frac{\kappa}{4} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{8\pi\epsilon_0 \mu v_\infty^2} = \frac{Z_\alpha Z_{\mathrm{Au}} e^2}{16\pi\epsilon_0 E_{rel}}$$

(in SI units)

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$

General formula relating b and θ:

$$\theta = -\pi + 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

where:

$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

$\Rightarrow$ There are relatively few forms of $V(1/u)$ for which the integral has an analytic result.

See Problem 1.16 in F&W for an example:

$$V(r) = \frac{\gamma}{r^2} \text{ where } \frac{d\sigma}{d\Omega} = \frac{\gamma\pi^2}{E\sin\theta} \frac{(\pi - \theta)}{\theta^2 (2\pi - \theta)^2}$$

More generally, it is possible to use numerical integration methods (with care) to evaluate *b(θ).*

Transformation between lab and center of mass results:
Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\left|\frac{d\cos\psi}{d\cos\theta}\right|$$

$$\left(\frac{d\sigma_{LAB}(\theta)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\psi)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\psi+(m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\psi+1}$$

For elastic scattering

where: $$\tan\theta = \frac{\sin\psi}{\cos\psi + m_1/m_2}$$