PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin 103

Notes for Lecture 6

Physics analyzed in accelerated coordinate frames – Chap 2 F&W

1. Angular acceleration
2. Linear and angular acceleration
3. Foucault pendulum

Your questions –

From Katie -- Could you clarify the difference between the fixed coordinate frame versus the moving coordinate frame? Mainly just explaining the equations on slide 4 and where they are coming from. Also, I am having trouble visualizing where the cross products are coming from when comparing the diagram and the equations. Could you try and draw connections between these two things when going over the slides in class (specifically with the Foucault Pendulum example)?

From Zezong -- I want to know more details about how to get the approximations in slide 17 and 18.

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/22/2022		Introduction	#1	8/26/2022
2	Wed, 8/24/2022	Chap. 1	Scattering theory		
3	Fri, 8/26/2022	Chap. 1	Scattering theory	#2	8/29/2022
4	Mon, 8/29/2022	Chap. 1	Scattering theory	#3	8/31/2022
5	Wed, 8/31/2022	Chap. 1	Summary of scattering theory	#4	9/02/2022
6	Fri, 9/02/2022	Chap. 2	Non-inertial coordinate systems	#5	9/05/2022
7	Mon, 9/05/2022	Chap. 3	Calculus of Variation		

Class on Labor Day!!!

PHY 711 -- Assignment #5

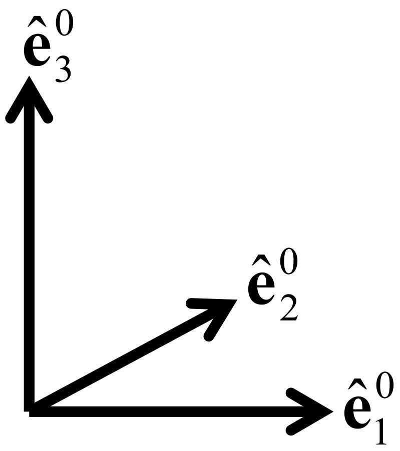
Sept. 2, 2022

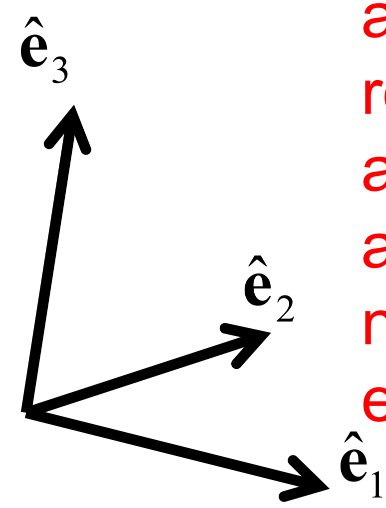
Read Chapter 2 in **Fetter & Walecka**.

1. Suppose that you would like to install a Foucault Pendulum at a location of your choice. Find the latitude of your location and determine the period of the pendulum to make a complete circle of the direction of its swing.

Physical laws as described in non-inertial coordinate systems

- Newton's laws are formulated in an inertial frame of reference $\left\{ \hat{\mathbf{e}}_i^0 \right\}$
- For some problems, it is convenient to transform the the equations into a non-inertial coordinate system $\left\{ \hat{\mathbf{e}}_i(t) \right\}$





Note that in addition to rotation, linear acceleration can also contribute to non-inertial effects.

Comparison of analysis in "inertial frame" versus "non-inertial frame"

Denote by $\hat{\mathbf{e}}_i^0$ an fixed coordinate system in 3 orthogonal directions

Denote by $\hat{\mathbf{e}}_i$ a moving coordinate system in 3 orthogonal directions

$$\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{\mathbf{e}}_i^0 = \sum_{i=1}^{3} V_i \hat{\mathbf{e}}_i$$

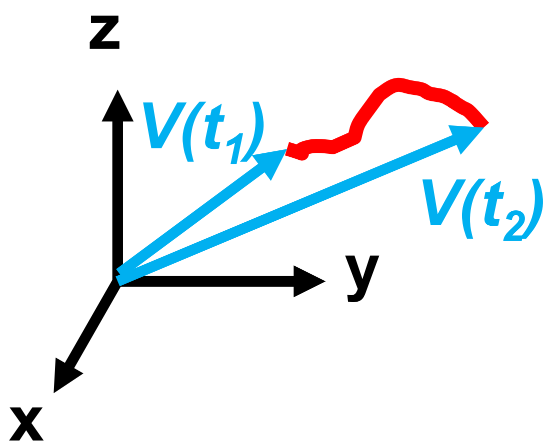
$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dV_i^0}{dt}\hat{\mathbf{e}}_i^0 = \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{\mathbf{e}}_i + \sum_{i=1}^{3} V_i \frac{d\hat{\mathbf{e}}_i}{dt}$$

Define: $$\left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{\mathbf{e}}_i$$

This represents the time rate of change of V measured within the e frame.

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{\mathbf{e}}_i}{dt}$$

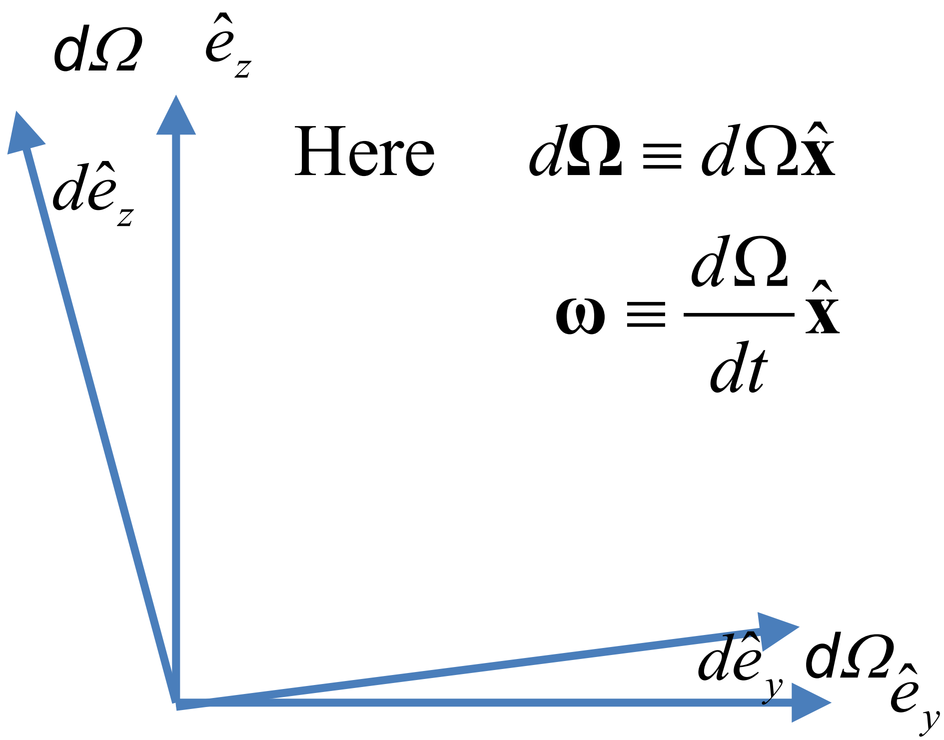
e^0 example



$$\frac{d\mathbf{V}(t)}{dt} = \frac{dV_x^0(t)}{dt}\hat{\mathbf{x}} + \frac{dV_y^0(t)}{dt}\hat{\mathbf{y}} + \frac{dV_z^0(t)}{dt}\hat{\mathbf{z}}$$

e example – same motion described in moving coordinate system.

Properties of the frame motion (rotation only):



Here $d\boldsymbol{\Omega} \equiv d\Omega\hat{\mathbf{x}}$

$$\boldsymbol{\omega} \equiv \frac{d\Omega}{dt}\hat{\mathbf{x}}$$

$$d\hat{e}_y = d\Omega\hat{e}_z$$

$$d\hat{e}_z = -d\Omega\hat{e}_y$$

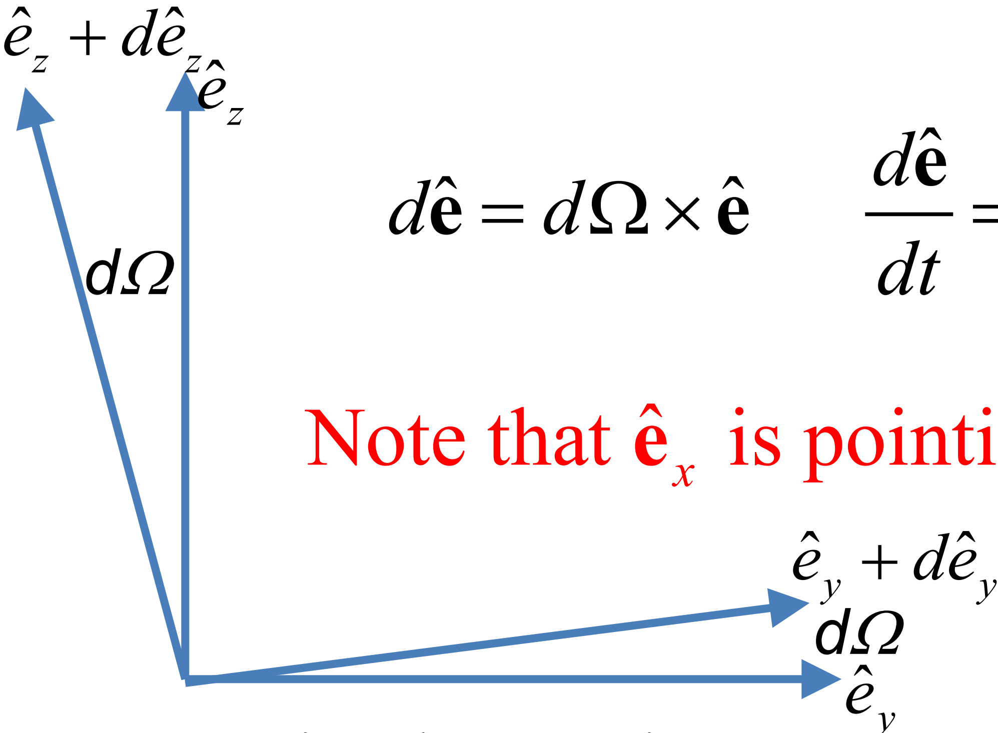
$$\Rightarrow d\hat{\mathbf{e}} = d\boldsymbol{\Omega} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \frac{d\boldsymbol{\Omega}}{dt} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Note that the coordinate $\hat{\mathbf{e}}_x$ is pointing out of the screen.

Properties of the frame motion (rotation only):



$$d\hat{\mathbf{e}} = d\Omega \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Omega}{dt} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Note that $\hat{\mathbf{e}}_x$ is pointing out of the screen.

Rotation about x-axis:

rotation matrix

$$\begin{pmatrix} e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \qquad \begin{pmatrix} e_y + de_y \\ e_z + de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega) \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) - 1 & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega) - 1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

More details

Rotation about x-axis:

$$\begin{pmatrix} e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \qquad \begin{pmatrix} e_y + de_y \\ e_z + de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega) \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega)-1 & \sin(d\Omega) \\ -\sin(d\Omega) & \cos(d\Omega)-1 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$e_y + de_y = \cos(d\Omega) e_y + \sin(d\Omega) e_z$$

$$e_z + de_z = -\sin(d\Omega) e_y + \cos(d\Omega) e_z$$

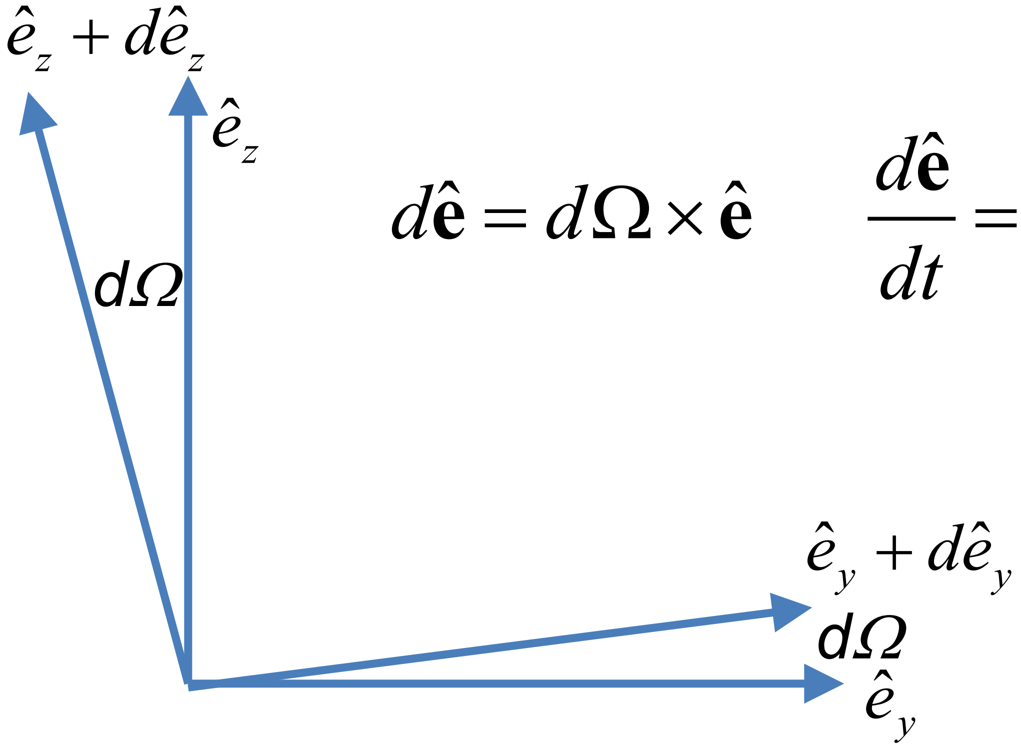
Taylor's series

$$f(x_0 + dx) = f(x_0) + dx \left.\frac{df}{dx}\right|_{x_0} + \frac{1}{2}(dx)^2 \left.\frac{d^2 f}{dx^2}\right|_{x_0} + ...$$

$$\sin(dx) = dx - \frac{1}{6}(dx)^3 .... \approx dx \qquad \cos(dx) = 1 - \frac{1}{2}(dx)^2 .... \approx 1$$

Properties of the frame motion (rotation only):

$$d\hat{\mathbf{e}} = d\Omega \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Omega}{dt} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$



Rotation about x-axis:

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} \approx \begin{pmatrix} 0 & d\Omega \\ -d\Omega & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} = d\Omega e_z \hat{\mathbf{y}} - d\Omega e_y \hat{\mathbf{z}} = d\Omega \hat{\mathbf{x}} \times \hat{\mathbf{e}}$$

Define axial vectors $\mathbf{d\Omega} \equiv d\Omega \hat{\mathbf{x}}$ also $\boldsymbol{\omega} = \omega \hat{\mathbf{x}}$

Properties of the frame motion (rotation only) -- continued

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{\mathbf{e}}_i}{dt}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{V} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega}\times\right)\mathbf{V}$$

Effects on 2nd time derivative -- acceleration (rotation only):

$$\left(\frac{d}{dt}\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega}\times\right)\left\{\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{V}\right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{V} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{V}$$

Application of Newton's laws in a coordinate system which has an angular velocity $\boldsymbol{\omega}$ and linear acceleration **a**
(Here we generalize previous case to add linear acceleration **a**.)

Newton's laws; Let **r** denote the position of particle of mass *m*:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = \mathbf{F}_{ext}$$

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = m\left(\mathbf{a} + \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} + 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}\right) = \mathbf{F}_{ext}$$

Rearranging to find the effective acceleration within the non-inertial frame --

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} = \mathbf{F}_{ext} - m\mathbf{a} - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{body} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

Coriolis force

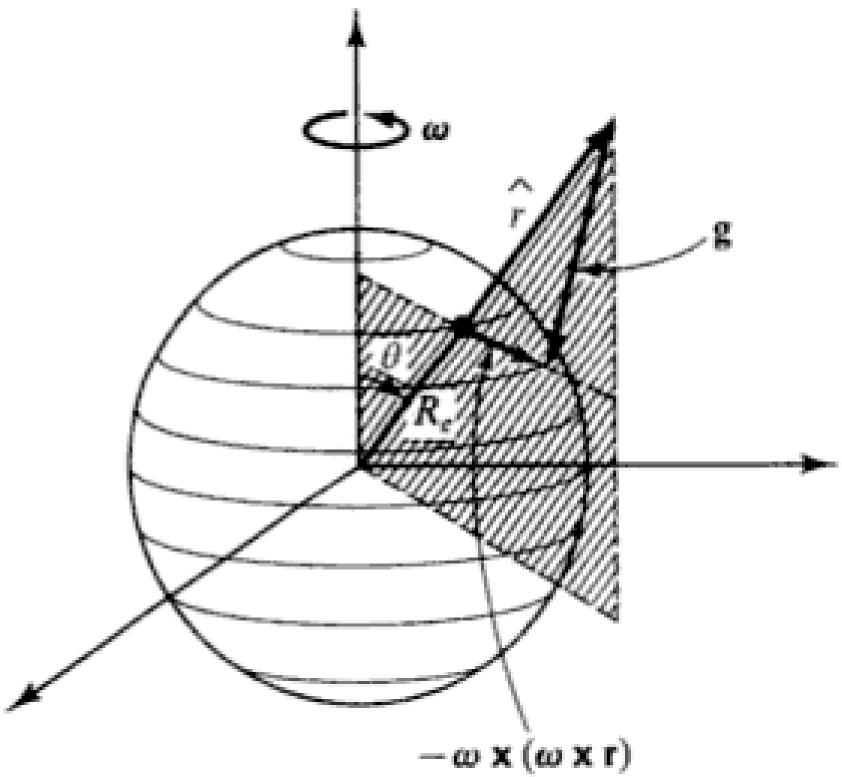
Centrifugal force

Have you ever experienced any of these “fictitious” forces?

Examples –

1. Playing on a swing
2. Playing on a merry-go-round
3. Riding on a roller coaster
4. Sitting on the surface of the earth
5. Astronaut aboard the International Space Station
6. ???

Motion on the surface of the Earth:



$$\omega = \frac{2\pi}{\tau} \approx 7.3\times 10^{-5}\,\text{rad/s}$$

$$\mathbf{F}_{ext} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}'$$

Earth's gravity

Support force

Main contributions:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

Non-inertial effects on effective gravitational "constant"

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

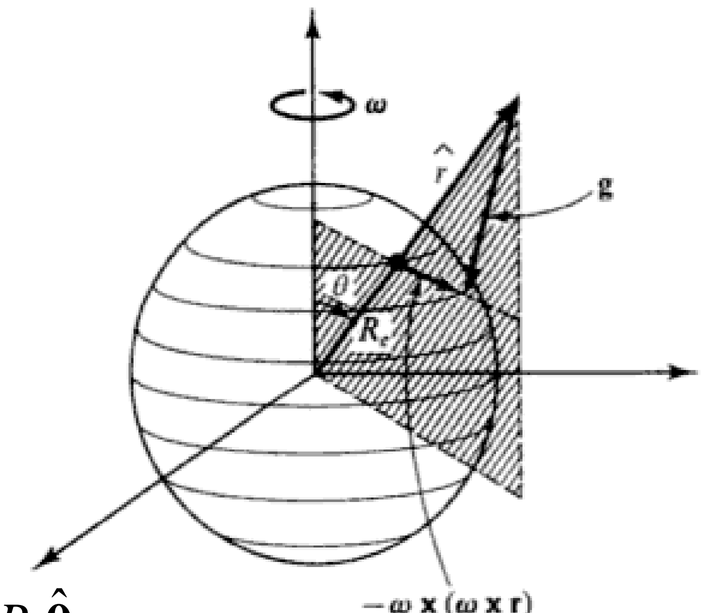
For $\left(\frac{d\mathbf{r}}{dt}\right)_{earth} = 0$ and $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = 0,$

$$0 = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

$$\mathbf{F}' = -m\mathbf{g}$$

$$\Rightarrow \mathbf{g} = -\frac{GM_e}{r^2}\hat{\mathbf{r}} - \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}\Big|_{r\approx R_e}$$

$$= \left(-\frac{GM_e}{R_e^2} + \omega^2 R_e \sin^2\theta\right)\hat{\mathbf{r}} + \sin\theta\cos\theta\,\omega^2 R_e\hat{\boldsymbol{\theta}}$$



0.03 m/s^2

9.80 m/s^2

Note that in the previous analysis we left out the term $-m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}$

Is this justified?

1. Yes
2. No

According to Google – the rate of rotation of the earth has changed during its existence….

https://www.discovermagazine.com/planet-earth/the-earths-rotation-is-gradually-slowing-down

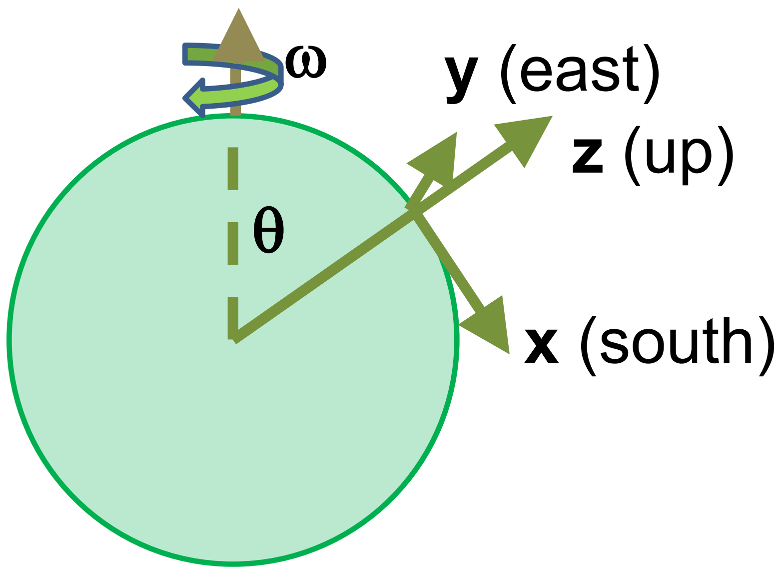
Foucault pendulum

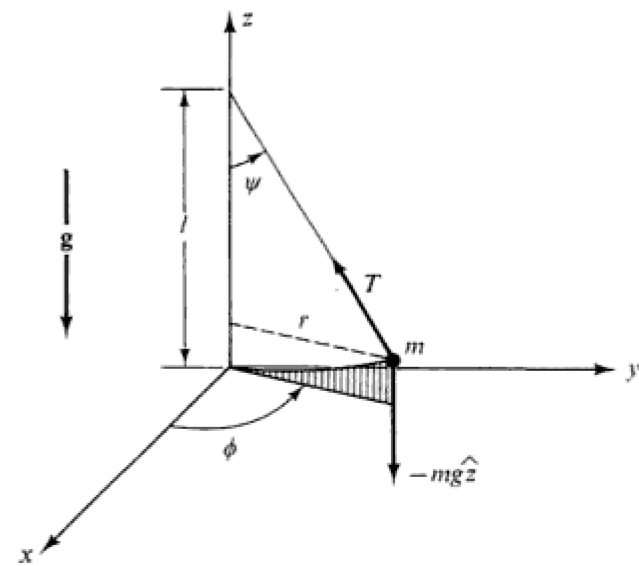
http://www.si.edu/Encyclopedia_SI/nmah/pendulum.htm

The Foucault pendulum was displayed for many years in the Smithsonian's National Museum of American History. It is named for the French physicist Jean Foucault who first used it in 1851 to demonstrate the rotation of the earth.

Equation of motion on Earth's surface

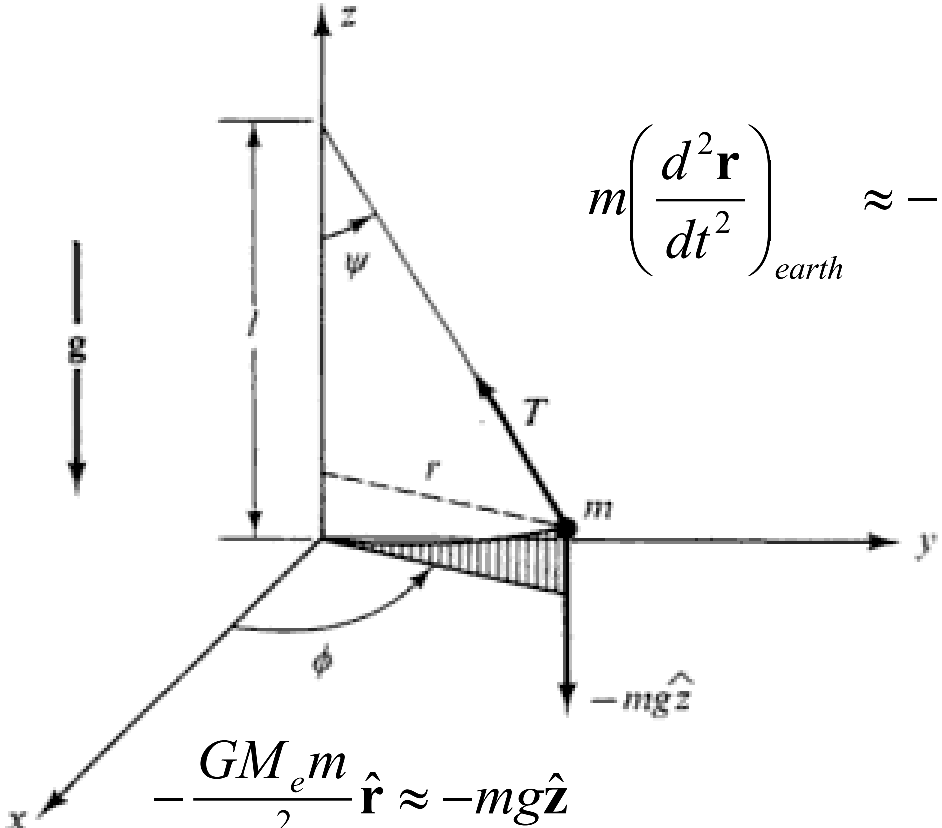
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$



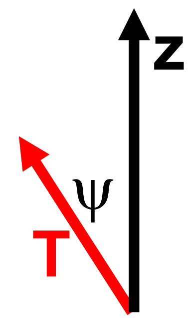


$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

Foucault pendulum continued – keeping leading terms:



$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$



$$-\frac{GM_e m}{r^2}\hat{\mathbf{r}} \approx -mg\hat{\mathbf{z}}$$

$$\mathbf{F}' \approx -T\sin\psi\cos\phi\hat{\mathbf{x}} - T\sin\psi\sin\phi\hat{\mathbf{y}} + T\cos\psi\hat{\mathbf{z}}$$

$$\boldsymbol{\omega} \approx -\omega\sin\theta\hat{\mathbf{x}} + \omega\cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} \approx \omega\left(-\dot{y}\cos\theta\hat{\mathbf{x}} + \left(\dot{x}\cos\theta + \dot{z}\sin\theta\right)\hat{\mathbf{y}} - \dot{y}\sin\theta\hat{\mathbf{z}}\right)$$

Some details --

$$\boldsymbol{\omega} \approx -\omega \sin\theta \hat{\mathbf{x}} + \omega \cos\theta \hat{\mathbf{z}}$$

$$\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{earth} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -\omega\sin\theta & 0 & \omega\cos\theta \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$$

$$\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{earth} \approx \omega\left(-\dot{y}\cos\theta \hat{\mathbf{x}} + \left(\dot{x}\cos\theta + \dot{z}\sin\theta\right)\hat{\mathbf{y}} - \dot{y}\sin\theta \hat{\mathbf{z}}\right)$$

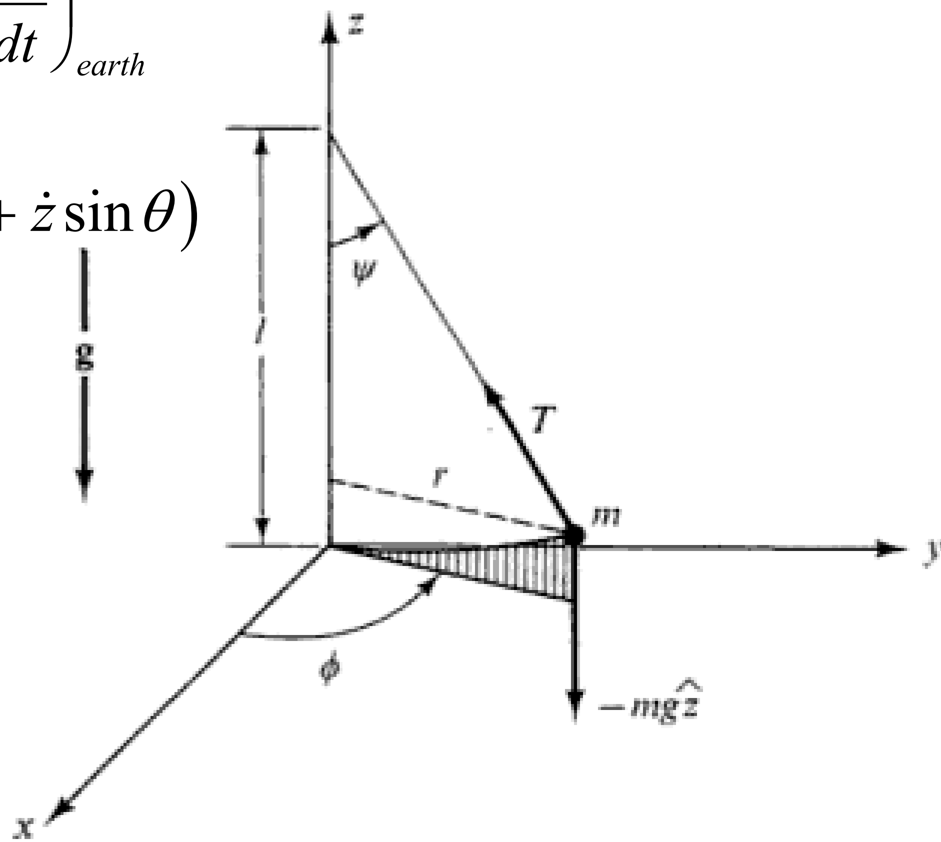
Foucault pendulum continued – keeping leading terms:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} \approx -\frac{GM_e m}{R_e^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$m\ddot{x} \approx -T\sin\psi\cos\varphi + 2m\omega\dot{y}\cos\theta$$

$$m\ddot{y} \approx -T\sin\psi\sin\varphi - 2m\omega\left(\dot{x}\cos\theta + \dot{z}\sin\theta\right)$$

$$m\ddot{z} \approx T\cos\psi - mg + 2m\omega\dot{y}\sin\theta$$



Further approximation :

$\psi << 1; \quad \ddot{z} \approx 0; \quad T \approx mg$

$m\ddot{x} \approx -mg\sin\psi\cos\phi + 2m\omega\dot{y}\cos\theta$

$m\ddot{y} \approx -mg\sin\psi\sin\phi - 2m\omega\dot{x}\cos\theta$

Also note that :

$x \approx \ell\sin\psi\cos\phi$

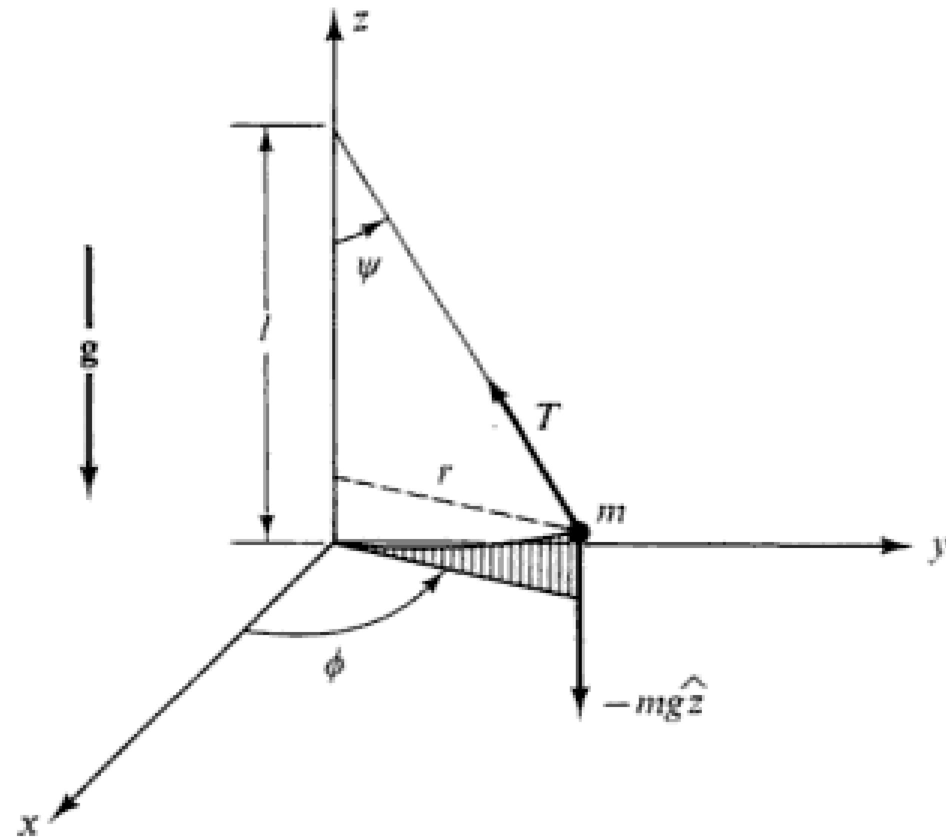
$y \approx \ell\sin\psi\sin\phi$

ℓ denotes the length of the rope/wire

Foucault pendulum continued – coupled equations:

$$\ddot{x} \approx -\frac{g}{\ell} x + 2\omega\cos\theta\dot{y}$$

$$\ddot{y} \approx -\frac{g}{\ell} y - 2\omega\cos\theta\dot{x}$$



Try to find a solution of the form :

$$x(t) = Xe^{-iqt} \qquad y(t) = Ye^{-iqt}$$

Denote $\omega_\perp \equiv \omega\cos\theta$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non - trivial solutions :

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp^{\ 2} + \frac{g}{\ell}}$$

Foucault pendulum continued – coupled equations:

Solution continued:

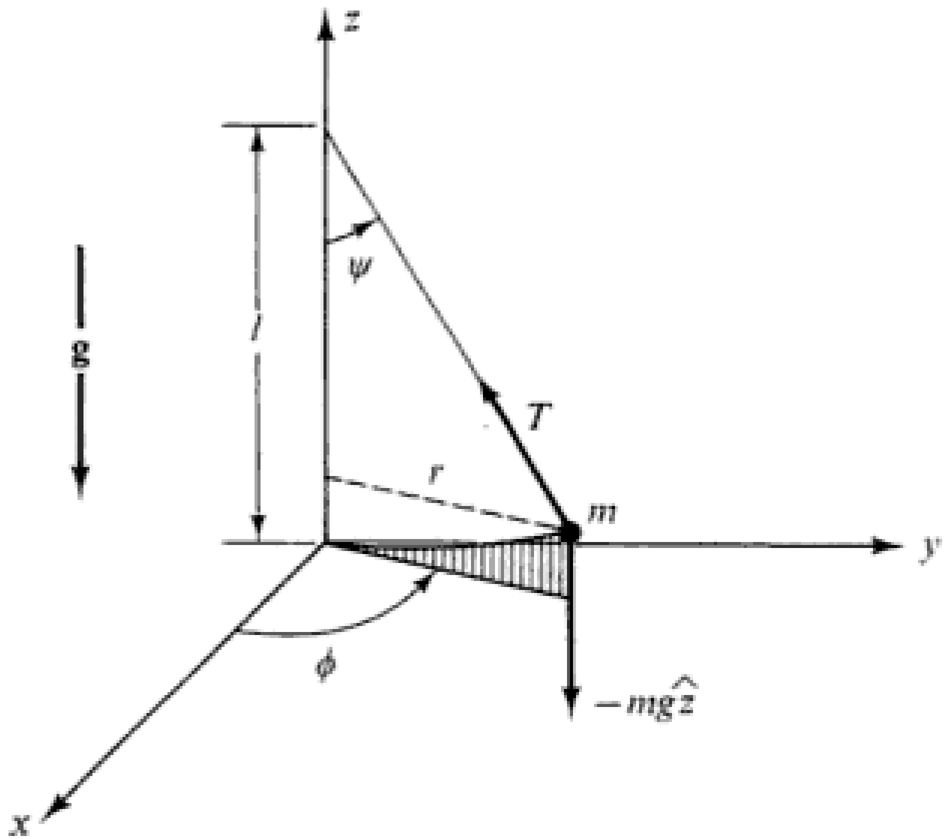
$$x(t) = Xe^{-iqt} \qquad y(t) = Ye^{-iqt}$$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non-trivial solutions:

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp^{\ 2} + \frac{g}{\ell}}$$

Amplitude relationship: $X = iY$



General solution with complex amplitudes C and D:

$$x(t) = \mathrm{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$$

$$y(t) = \mathrm{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$$

General solution with complex amplitudes C and D:

$$x(t) = \mathrm{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$$

$$y(t) = \mathrm{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$$

$$q_{\pm} = \alpha \pm \beta \equiv \omega_{\perp} \pm \sqrt{\omega_{\perp}^{\;2} + \frac{g}{\ell}} \approx \omega_{\perp} \pm \sqrt{\frac{g}{\ell}}$$

since $\quad \omega_{\perp} \approx 7\times 10^{-5}\cos\theta \;\; rad/s << \sqrt{\frac{g}{\ell}}$

Suppose: $\quad x(0) = X_0 \qquad y(0) = 0$

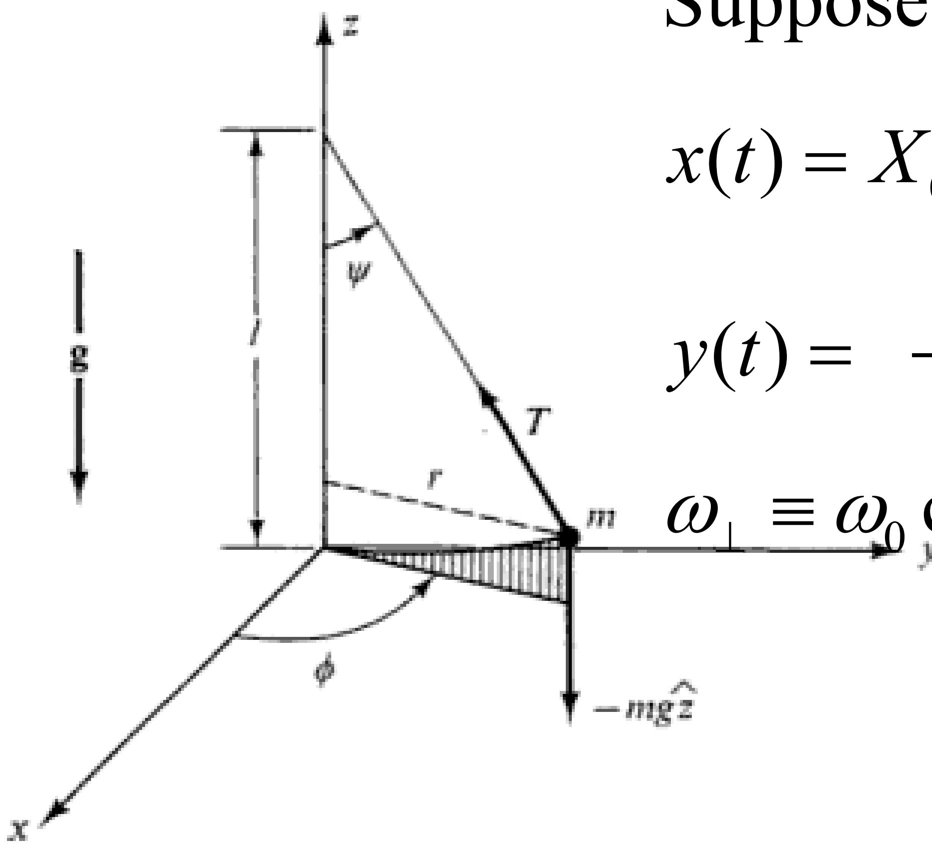
$$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\cos(\omega_{\perp} t)$$

$$y(t) = -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\sin(\omega_{\perp} t)$$

Note that

$$\omega = \frac{2\pi}{24\cdot 3600 \text{ s}} = 7\times 10^{-5} \text{ rad/sec}$$

Summary of approximate solution for Foucault pendulum:

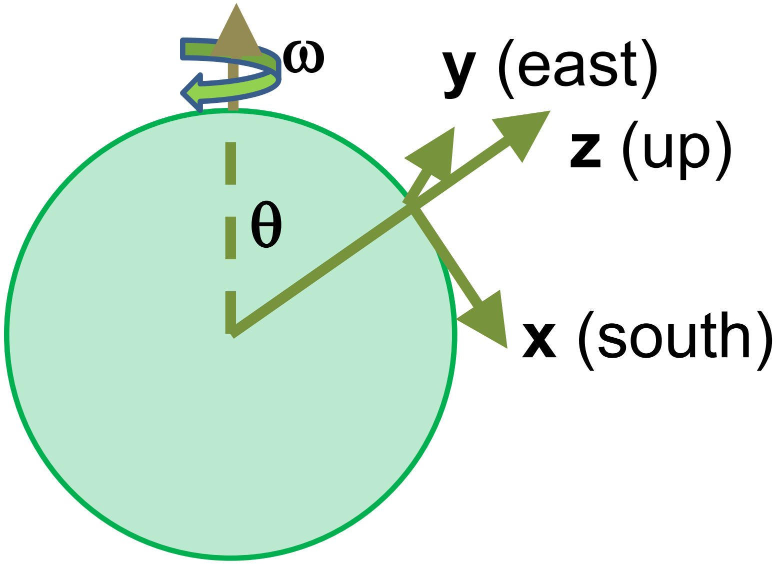


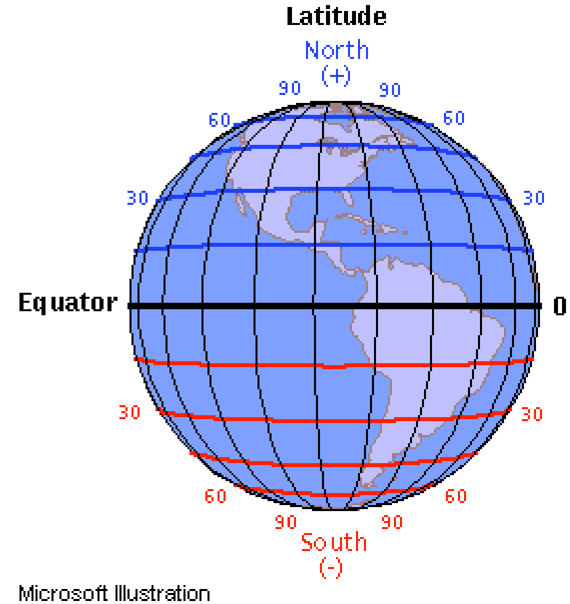
Suppose: $x(0) = X_0 \qquad y(0) = 0$

$$x(t) = X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}\,t\right)\cos\left(\omega_\perp t\right)$$

$$y(t) = -X_0 \cos\left(\sqrt{\tfrac{g}{\ell}}\,t\right)\sin\left(\omega_\perp t\right)$$

$$\omega_\perp \equiv \omega_0 \cos\theta$$





$$\omega_{\perp} \equiv \omega_0 \cos\theta$$

$$x(t) = X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\cos\left(\omega_{\perp} t\right)$$

$$y(t) = \ -X_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right)\sin\left(\omega_{\perp} t\right)$$

Latitude and Longitude
https://www.latlong.net/

Latitude and Longitude Finder

Latitude and Longitude are the units that represent the *coordinates at geographic coordinate system*. To make a search, use the name of a place, city, state, or address, or click the location on the map to **find lat long coordinates**.

Place Name

Winston-Salem, NC, USA

Find

Add the country code for better results. Ex: London, UK

Latitude

36.096260

Longitude

-80.243736

Note that $\theta = 90^o$ - Latitude