PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF in Olin103

Lecture notes for Lecture 7
Chapter 3.17 of F&W

Introduction to the calculus of variations

1. **Mathematical construction**
2. **Practical use**
3. **Examples**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment	Due
1	Mon, 8/22/2022		Introduction	#1	8/26/2022
2	Wed, 8/24/2022	Chap. 1	Scattering theory		
3	Fri, 8/26/2022	Chap. 1	Scattering theory	#2	8/29/2022
4	Mon, 8/29/2022	Chap. 1	Scattering theory	#3	8/31/2022
5	Wed, 8/31/2022	Chap. 1	Summary of scattering theory	#4	9/02/2022
6	Fri, 9/02/2022	Chap. 2	Non-inertial coordinate systems	#5	9/05/2022
7	Mon, 9/05/2022	Chap. 3	Calculus of Variation	#6	9/7/2022
8	Wed, 9/07/2022	Chap. 3	Calculus of Variation		

PHY 711 -- Assignment #6

Sept. 5, 2022

Start reading Chapter 3, especially Section 17, in **Fetter & Walecka**.

1. Using calculus of variations, find the equation *y(x)* of the shortest length "curve" which passes through the points (*x=0, y=0*) and (*x=2, y=1*). What is the length of the "curve"?

Comments on the motivation and hierarchy of the Lagrangian formalism.

- I would guess that there have been lots of formulations of the equations of motion throughout the history of mathematics and science and the main ones that have survived are Newton's "laws", Lagrangian formalism, and Hamiltonian formalism. Each are mathematically sound and physical verified and useful for analysis in various context.
- Lagrangian and Hamiltonian formalisms are more advanced because the mathematics is a little bit "harder" than Newton's formulation. As long as they are correctly applied, they should describe the same physics. On the other hand, there are situations (like in quantum mechanics) where the Lagrangian and Hamiltonian formulations are preferred/needed.
- Once you become comfortable with the Lagrangian formulation, you may find that it is easier to use in analysis, particularly in complicated coordinate systems or when there are constraints on the motion.
- For now, a key motivation for the Lagrangian formalism is that it opens several powerful additional mathematical tools to the analysis of motion.
- Your textbook starts with a "derivation" of the Lagrangian, but we will first develop the abstract mathematical tools first.

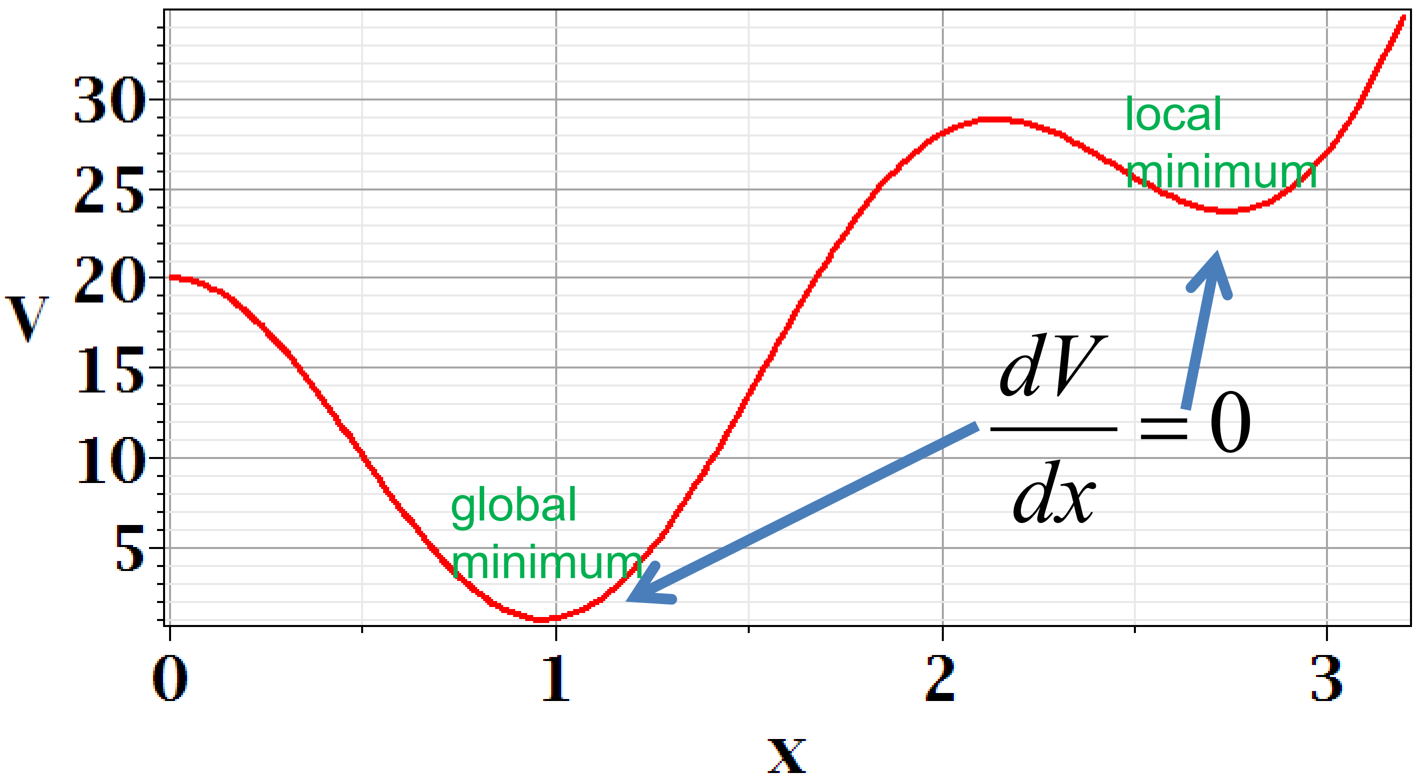
According wikipedia – **Joseph-Louis Lagrange** (born **Giuseppe Luigi Lagrangia** or **Giuseppe Ludovico De la Grange Tournier**; 25 January 1736 – 10 April 1813), also reported as **Giuseppe Luigi Lagrange** or **Lagrangia**, was an Italian mathematician and astronomer, later naturalized French. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics.

According to Wikipedia –

Leonard Euler (April 7, 1707-September 18, 1783) Swiss mathematician, physicist, astronomer, geographer, logician and engineer who founded the studies of graph theory and topology and made pioneering and influential discoveries in many other branches of mathematics such as analytic number theory, complex analysis, and infinitesimal calculus. He introduced much of modern mathematical terminology and notation, including the notion of a mathematical function. He is also known for his work in mechanics, fluid dynamics, optics, astronomy and music theory.

In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton's laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as "the calculus of variation".

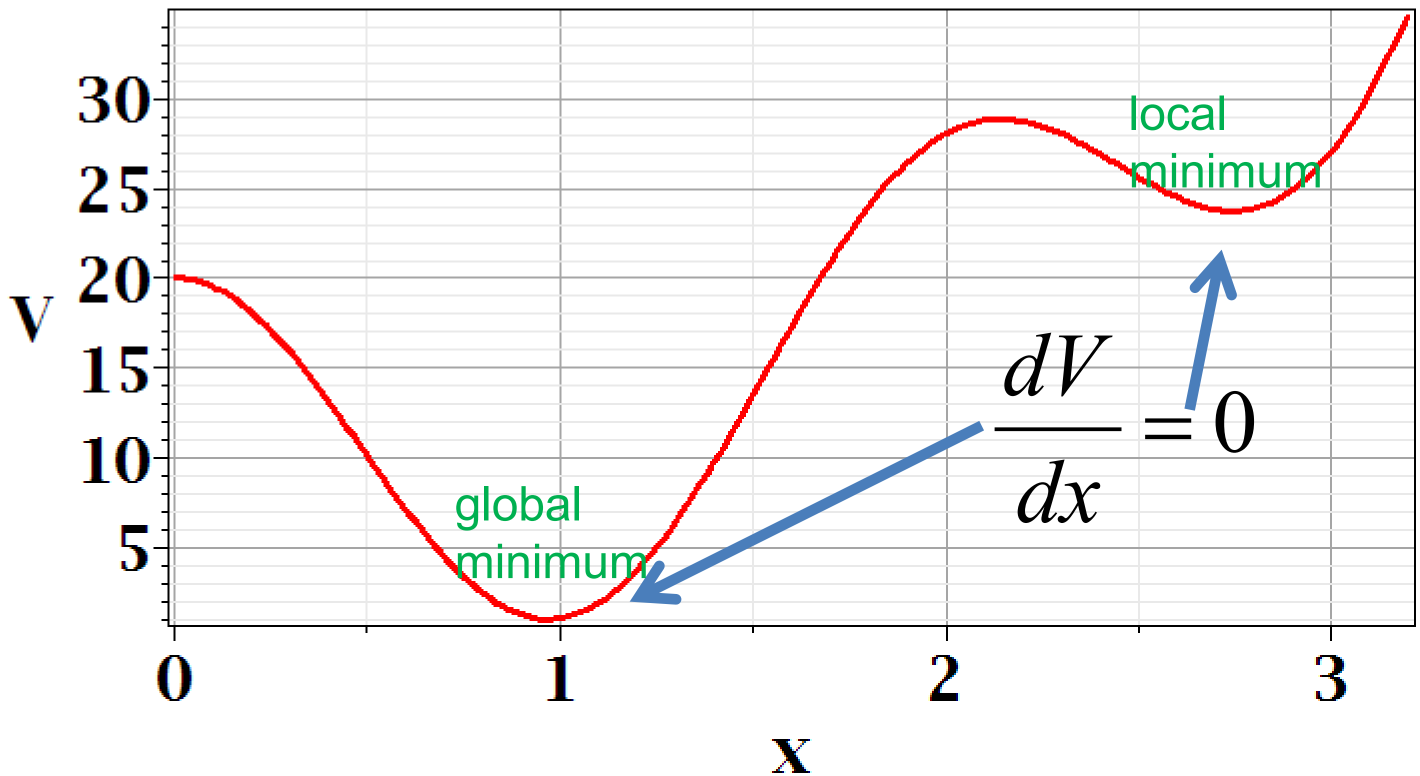
Minimization of a simple function



Minimization of a simple function

Given a function $V(x)$, find the value(s) of x for which $V(x)$ is minimized (or maximized).

Necessary condition : $\frac{dV}{dx} = 0$



Functional minimization of an integral relationship

Consider a family of functions $y(x)$, with fixed end points

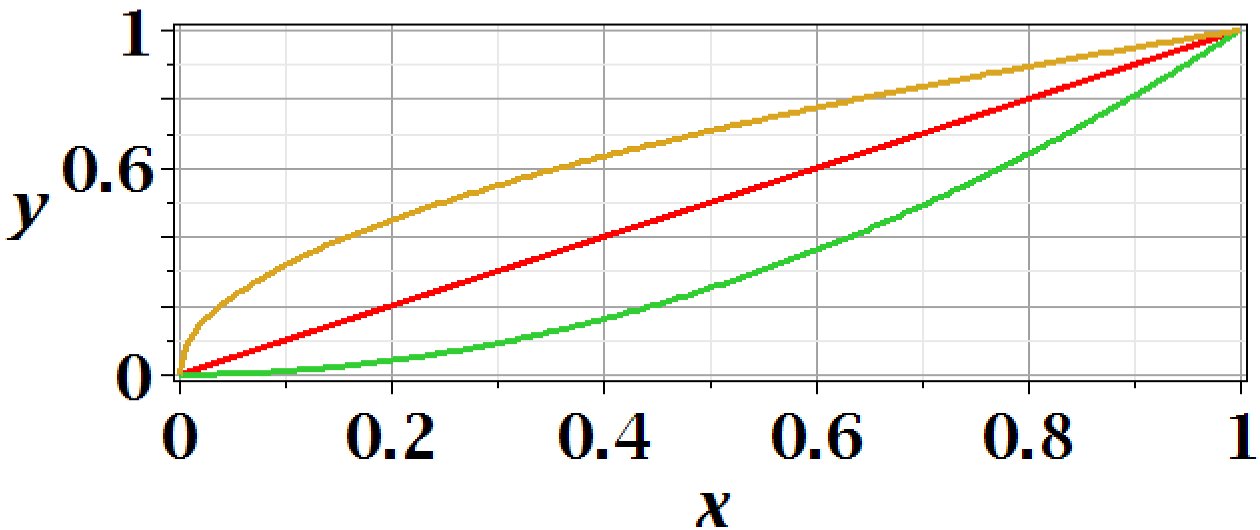
$y(x_i) = y_i$ and $y(x_f) = y_f$ and an integral form $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition: $\delta L = 0$

Example:

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$



Difference between minimization of a function V(x) and the minimization in the calculus of variation.

Minimization of a function – V(x)
➔Know V(x) ➔Find x_0 such that $V(x_0)$ is a minimum.

Calculus of variation

For $x_i \leq x \leq x_f$ want to find a function $y(x)$ that minimizes an integral that depends on $y(x)$.

The analysis involves deriving and solving a differential equation for $y(x)$.

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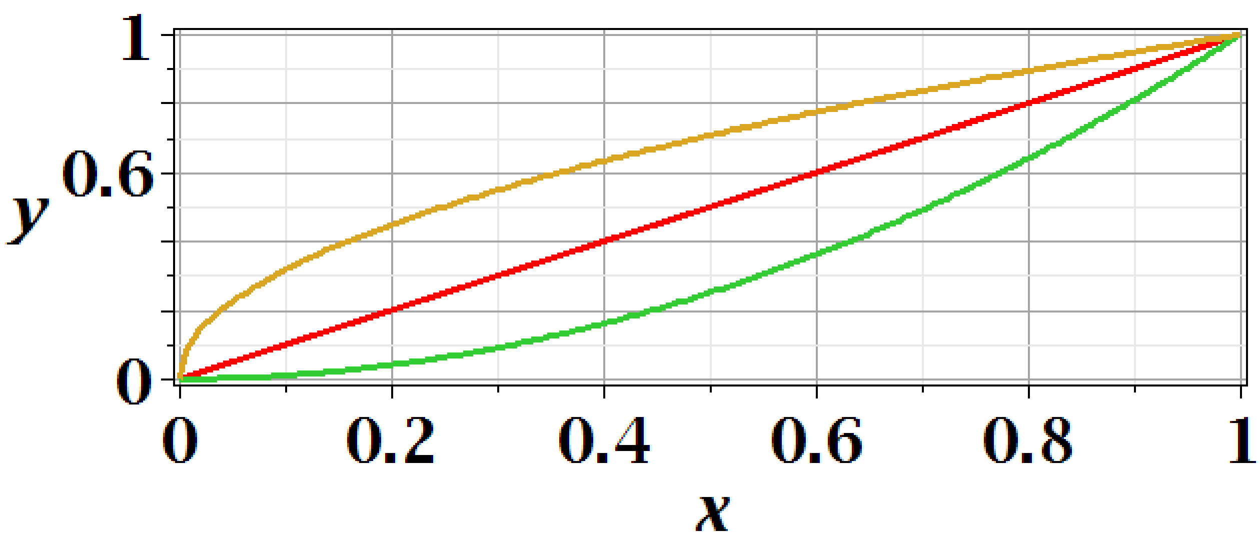
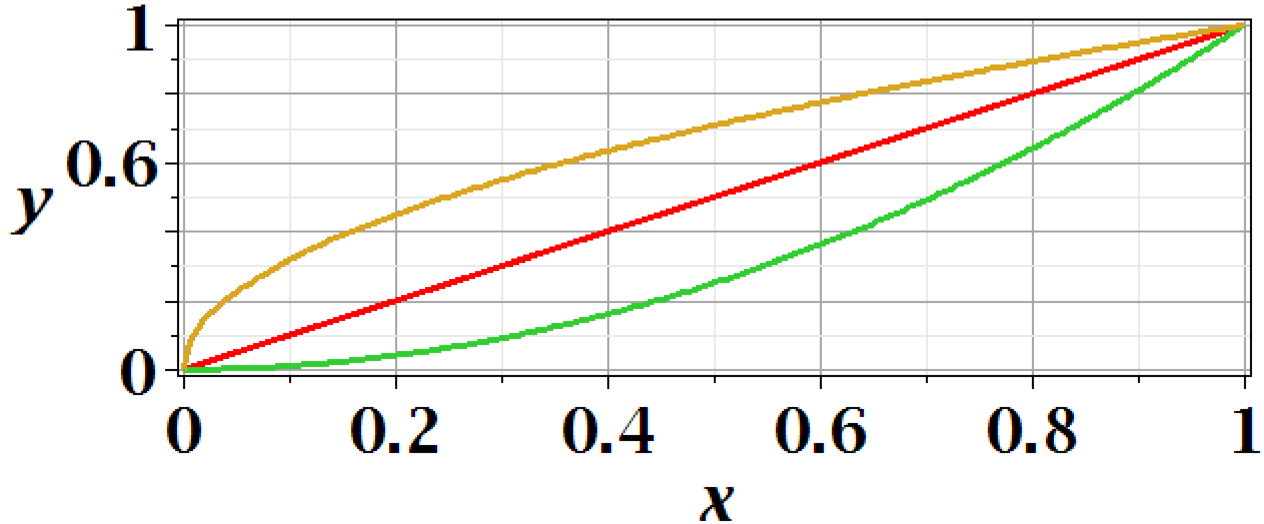


Example:



$$L = \int_{(0,0)}^{(1,1)} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Sample functions :

$y_1(x) = \sqrt{x}$ $\qquad L = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx = 1.4789$

$y_2(x) = x$ $\qquad L = \int_0^1 \sqrt{1+1}\, dx = \sqrt{2} = 1.4142$

$y_2(x) = x^2$ $\qquad L = \int_0^1 \sqrt{1 + 4x^2}\, dx = 1.4789$

Calculus of variation example for a pure integral functions

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$

where $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx.$

Necessary condition : $\delta L = 0$

At any x, let $y(x) \rightarrow y(x) + \delta y(x)$

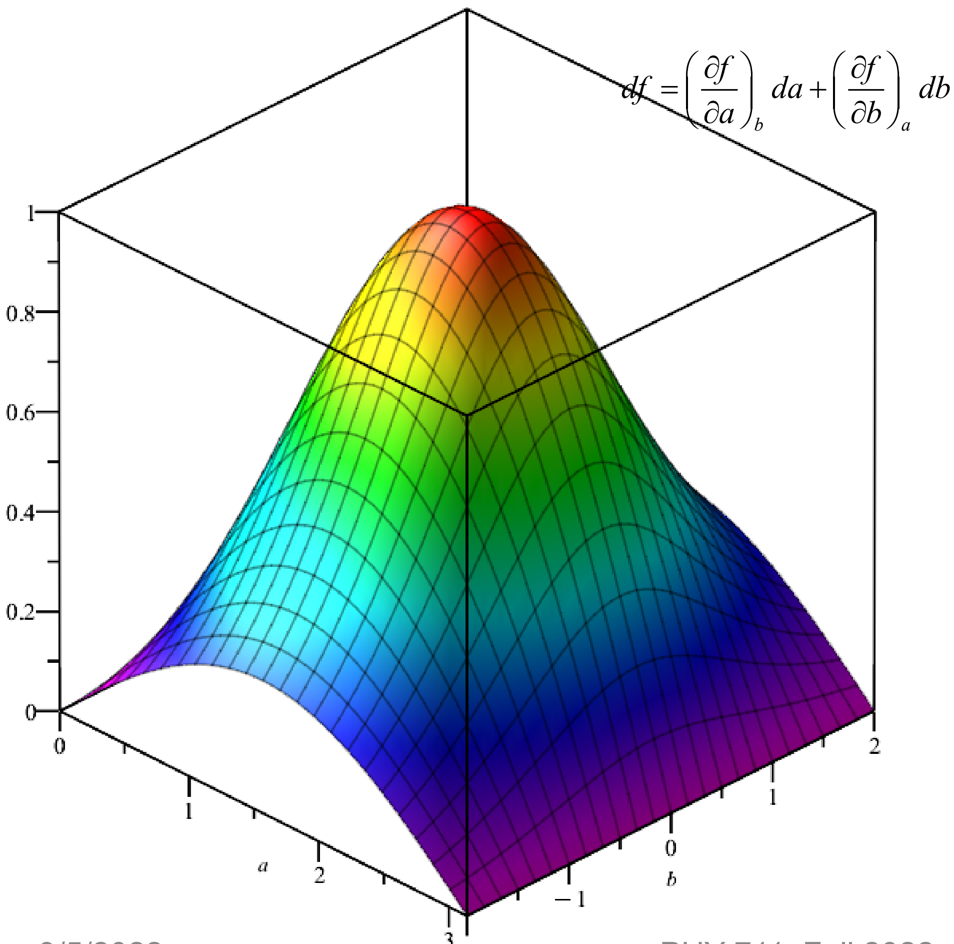
$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta \frac{dy(x)}{dx}$$

Formally:

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta\left(\frac{dy}{dx} \right) \right] \right] dx.$$

Comment on partial derivatives -- function $f(a,b)$

$$\frac{\partial f}{\partial a} \equiv \lim_{da \to 0}\left(\frac{f(a+da,b)-f(a,b)}{da}\right) \equiv \left.\frac{\partial f}{\partial a}\right|_b$$



More comments about notation concerning functional dependence and partial derivatives – 3 dimensional example --

Suppose x, y, z represent independent variables that determine a function f :
We write $f(x, y, z)$. A partial derivative with respect to x implies that we hold y, z fixed and infinitessimally change x

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}\right)$$

After some derivations, we find

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y \, dx = 0 \quad \text{for all } x_i \le x \le x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Note that this is a
“total” derivative

“Some” derivations --

Consider the term

$$\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]dx:$$

If $y(x)$ is a well-defined function, then $\delta\left(\frac{dy}{dx}\right)=\frac{d}{dx}\delta y$ *

$$\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]dx=\int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\frac{d}{dx}\delta y\right]dx$$

$$=\int_{x_i}^{x_f}\left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]-\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

Note that the δy notation is meant to imply a general infinitessimal variation of the function $y(x)$

*Clarification -- what is the meaning of the following statement:

$$\delta\left(\frac{dy}{dx}\right)=\frac{d}{dx}\delta y$$

Up to now, the operator δ is not well defined and meant to be general.

Now let us suppose that it implies an infinitessimal difference to its function.

As an example, suppose that $y(x,\eta)$ where x and η are independent such as

$y(x,\eta)=x^{\eta}$ For $\eta>0$, and $0\leq x\leq 1$

assume $\eta>0$

$$\frac{d}{d\eta}\frac{d}{dx}y(x,\eta)=\frac{d}{dx}\frac{d}{d\eta}y(x,\eta)=\left(1+\eta\ln(x)\right)x^{\eta-1}$$

Note that the construction of this system is that

$y(x_i,\eta)$ has the same value for all η and

$y(x_f,\eta)$ has the same value for all η.

Example $y(x,\eta)=x^{\eta}$ for $x_i=0$ and $x_f=1$

$y_i=y(0,\eta)=0$ and $y_f=y(1,\eta)=1$

"Some" derivations (continued)--

$$\int_{x_i}^{x_f} \left[\frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] - \frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx$$

$$= \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right]_{x_i}^{x_f} - \int_{x_i}^{x_f} \left[\frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx$$

$$= \quad 0 \quad - \int_{x_i}^{x_f} \left[\frac{d}{dx} \left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta y \right] dx$$

Euler-Lagrange equation:

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Clarfication – Why does this term go to zero?

$$\int_{x_i}^{x_f}\left[\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]-\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]_{x_i}^{x_f}-\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

$$=\qquad 0 \qquad -\int_{x_i}^{x_f}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta y\right]dx$$

Answer --

By construction $\delta y(x_i)=\delta y(x_f)=0$

Recap --

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \delta \left(\frac{dy}{dx} \right) \right] \right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] \right] \delta y dx = 0 \quad \text{for all } x_i \leq x \leq x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y} \right)_{x, \frac{dy}{dx}} - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right)_{x,y} \right] = 0 \quad \text{for all } x_i \leq x \leq x_f$$

Here we conclude that the integrand has to vanish at every argument in order for the integral to be zero

a. Necessary?
b. Overkill?

Example: End points -- $y(0)=0; y(1)=1$

$$L=\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

Solution:

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=K \qquad \frac{dy}{dx}=K'\equiv\frac{K}{\sqrt{1-K^2}}$$

$$\Rightarrow y(x)=K'x+C \qquad y(x)=x$$