PHY 711 − Notes on Hydrodynamics − ("Solitary Waves"[1])

Basic assumptions

We assume that we have in incompressible fluid ($\rho = $ constant) a velocity potential of the form $\Phi(x, z, t)$. The surface of the fluid is described by $h + \zeta(x, t) = z$. The fluid is contained in a tank with a structureless bottom (defined by the plane $z = 0$) and is filled to a vertical height h at equilibrium. These functions satisfy the following conditions.

$$\text{Poisson equation: } \frac{\partial^2 \Phi(x, z, t)}{\partial x^2} + \frac{\partial^2 \Phi(x, z, t)}{\partial z^2} = 0 \tag{1}$$

$$\text{Zero vertical velocity at bottom of the tank: } \frac{\partial \Phi(x, 0, t)}{\partial z} = 0 \tag{2}$$

$$\text{Bernoulli's equation: } \left. -\frac{\partial \Phi(x, z, t)}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi(x, z, t)}{\partial x}\right)^2 + g\zeta(x, t) \right|_{z = h + \zeta} = 0 \tag{3}$$

$$\text{Surface equation: } \left. -\frac{\partial \Phi(x, z, t)}{\partial z} + \frac{\partial \Phi(x, z, t)}{\partial x}\frac{\partial \zeta(x, t)}{\partial x} - \frac{\partial \zeta(x, t)}{\partial t} \right|_{z = h + \zeta} = 0 \tag{4}$$

In this treatment, we assume seek the form of surface waves traveling along the $x-$ direction and assume that the effective wavelength is much larger than the height of the surface h. This allows us to approximate the $z-$ dependence of $\Phi(x, z, t)$ by means of a Taylor series expansion:

$$\Phi(x, z, t) \approx \Phi(x, 0, t) + z\frac{\partial \Phi}{\partial z}(x, 0, t) + \frac{z^2}{2}\frac{\partial^2 \Phi}{\partial z^2}(x, 0, t) + \frac{z^3}{3!}\frac{\partial^3 \Phi}{\partial z^3}(x, 0, t)\cdots \tag{5}$$

and

$$\frac{\partial \Phi(x, z, t)}{\partial z} \approx +\frac{\partial \Phi}{\partial z}(x, 0, t) + z\frac{\partial^2 \Phi}{\partial z^2}(x, 0, t) + \frac{z^2}{2}\frac{\partial^3 \Phi}{\partial z^3}(x, 0, t) + \frac{z^3}{3!}\frac{\partial^4 \Phi}{\partial z^4}(x, 0, t)\cdots. \tag{6}$$

These expansions can be simplified because of the bottom boundary condition (2) and the Poisson equation (1). In particular, (2) and (1) ensure that all odd derivatives $\frac{\partial^n \Phi}{\partial z^n}(x, 0, t)$ vanish from the Taylor expansions (5) and (6). Taking the time derivative of 5 and evaluating it at the surface, $z = h + \zeta(x, t)$, and using the Poisson equation (1) to convert all derivatives with respect to z to derivatives with respect to x, we find:

$$\frac{\partial \Phi(x, z, t)}{\partial t} \approx \frac{\partial \Phi(x, 0, t)}{\partial t} - \frac{\partial}{\partial t}\left(\frac{(h + \zeta(x, t))^2}{2}\frac{\partial^2 \Phi}{\partial x^2}(x, 0, t)\right)\cdots, \tag{7}$$

and simplifying (6), we find:

$$\frac{\partial \Phi(x, z, t)}{\partial z} \approx -(h + \zeta(x, t))\frac{\partial^2 \Phi}{\partial x^2}(x, 0, t) + \frac{(h + \zeta(x, t))^3}{3!}\frac{\partial^4 \Phi}{\partial x^4}(x, 0, t)\cdots. \tag{8}$$

We can now use these results to approximate the boundary conditions implied by Bernoulli's equation (3 and the time derivative the of the surface equation (4). Keeping all terms up to leading order in non-linearity and up to fourth order derivatives in the linear terms we find that the surface definition equation becomes:

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial \Phi(x,0,t)}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \Phi(x,0,t)}{\partial x^4} - \frac{\partial \zeta(x,t)}{\partial t} = 0, \qquad (9)$$

and the Bernoulli's equation becomes:

$$-\frac{\partial \Phi(x,0,t)}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \Phi(x,0,t)}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \Phi(x,0,t)}{\partial x}\right)^2 + g\zeta(x,t) = 0. \qquad (10)$$

We would like to solve Eqs. (9-10) for a traveling wave of the form:

$$\Phi(x,0,t) = \phi(x-ct) \text{ and } \zeta(x,t) = \psi(x-ct), \qquad (11)$$

where the speed of the wave c will be determined. Letting $u \equiv x - ct$, Eqs. (9 and 10) become:

$$\frac{d}{du}\left((h+\psi(u))\frac{d\phi(u)}{du}\right) - \frac{h^3}{6}\frac{d^4\phi(u)}{du^4} + c\frac{d\psi(u)}{du} = 0, \qquad (12)$$

and

$$c\frac{d\phi(u)}{du} - \frac{ch^2}{2}\frac{d^3\phi(u)}{du^3} + \frac{1}{2}\left(\frac{d\phi(u)}{du}\right)^2 + g\psi(u) = 0. \qquad (13)$$

The modified surface equation (12) can be integrated once with respect to u, choosing the constant of integration to be zero and giving the new form for the surface condition:

$$(h+\psi)\phi' - \frac{h^3}{6}\phi''' + c\psi = 0, \qquad (14)$$

where we have abreviated derivatives with respect to u with the "'" symbol. This equation, and the modified Bernoulli equation (13) are now two coupled non-linear equations. In order to solve them, we use, the modified Bernoulli equation to approximate $\phi'(u)$ and its higher drivatives in terms of the surface function $\psi(u)$. Equation (13) becomes approximately:

$$\phi' = -\frac{g}{c}\psi + \frac{h^2}{2}\phi''' - \frac{1}{2c}(\phi')^2 \approx -\frac{g}{c}\psi - \frac{h^2 g}{2c}\psi'' - \frac{g^2}{2c^3}\psi^2. \qquad (15)$$

Using similar approximations, we can eliminate $\phi'(u)$ and its higher drivatives from the surface equation (14):

$$(h+\psi)\left(-\frac{g}{c}\psi - \frac{h^2 g}{2c}\psi'' - \frac{g^2}{2c^3}\psi^2\right) + \frac{h^3 g}{6c}\psi'' + c\psi = 0, \qquad (16)$$

where some terms involving non-linearity of higher than 2 or involving higher order derivatives have been discarded. Collecting the leading terms, we obtain:

$$\left(1 - \frac{gh}{c^2}\right)\psi - \frac{gh^3}{3c^2}\psi'' - \frac{g}{c^2}\left(1 + \frac{gh}{2c^2}\right)\psi^2 = 0. \qquad (17)$$

For the second two terms, *Fetter and Walecka* argue that it is consistent to approximate $gh \approx c^2$, which reduces (17) to

$$\left(1 - \frac{hg}{c^2}\right)\psi(u) - \frac{h^2}{3}\psi''(u) - \frac{3}{2h}\left[\psi(u)\right]^2 = 0. \tag{18}$$

Your text shows that a solution to Eq. (18), with the initial condition $\psi(0) = \psi_0$ and $\psi'(0) = 0$, is the solitary wave form:

$$\zeta(x,t) = \psi(x - ct) = \psi_0 \operatorname{sech}^2\left(\sqrt{\frac{3\psi_0}{h}} \, \frac{x - ct}{2h}\right), \tag{19}$$

with

$$c = \sqrt{\frac{gh}{1 - \psi_0/h}} \approx \sqrt{gh}\left(1 + \frac{\psi_0}{2h}\right). \tag{20}$$

References

[1] Alexander L. Fetter and John Dirk Walecka, **Theoretical Mechanics of Particles and Continua**, (McGraw Hill, 1980), Chapt. 10.