Notes for Lecture #19

Magnetic dipole field

These notes are very similar to the notes for Lecture #13 on the electric dipole field.

The magnetic dipole moment is defined by

$$\mathbf{m} = \frac{1}{2}\int d^3r'\mathbf{r}' \times \mathbf{J}(\mathbf{r}'), \tag{1}$$

with the corresponding potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}, \tag{2}$$

and magnetostatic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\frac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}}) - \mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\delta^3(\mathbf{r})\right\}. \tag{3}$$

The last term of the field expression follows from the following derivation. We note that Eq. (3) is poorly defined as $r \to 0$, and consider the value of a small integral of $\mathbf{B}(\mathbf{r})$ about zero. (For this purpose, we are supposing that the dipole $\mathbf{m}$ is located at $\mathbf{r} = \mathbf{0}$.) In this case we will approximate

$$\mathbf{B}(\mathbf{r} \approx \mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{B}(\mathbf{r})\mathbf{d^3r}\right)\delta^{\mathbf{3}}(\mathbf{r}). \tag{4}$$

First we note that

$$\int_{r\leq R}\mathbf{B}(\mathbf{r})d^3r = R^2\int_{r=R}\hat{\mathbf{r}}\times\mathbf{A}(\mathbf{r})\,d\Omega. \tag{5}$$

This result follows from the divergence theorm:

$$\int_{\text{vol}}\nabla\cdot\mathcal{V}\mathbf{d^3r} = \int_{\text{surface}}\mathcal{V}\cdot\mathbf{dA}. \tag{6}$$

In our case, this theorem can be used to prove Eq. (5) for each cartesian coordinate of $\nabla\times\mathbf{A}$ since $\nabla\times\mathbf{A} = \hat{\mathbf{x}}\,(\hat{\mathbf{x}}\cdot(\nabla\times\mathbf{A})) + \hat{\mathbf{y}}\,(\hat{\mathbf{y}}\cdot(\nabla\times\mathbf{A})) + \hat{\mathbf{z}}\,(\hat{\mathbf{z}}\cdot(\nabla\times\mathbf{A}))$. Note that $\hat{\mathbf{x}}\cdot(\nabla\times\mathbf{A}) = -\nabla\cdot(\hat{\mathbf{x}}\times\mathbf{A})$ and that we can use the Divergence theorem with $\mathcal{V}\equiv\hat{\mathbf{x}}\times\mathbf{A}(\mathbf{r})$ for the $x-$ component for example:

$$\int_{\text{vol}}\nabla\cdot(\hat{\mathbf{x}}\times\mathbf{A})d^3r = \int_{\text{surface}}(\hat{\mathbf{x}}\times\mathbf{A})\cdot\hat{\mathbf{r}}dA = \int_{\text{surface}}(\mathbf{A}\times\hat{\mathbf{r}})\cdot\hat{\mathbf{x}}dA. \tag{7}$$

Therefore,

$$\int_{r\leq R}(\nabla\times\mathbf{A})d^3r = -\int_{r=R}(\mathbf{A}\times\hat{\mathbf{r}})\cdot(\hat{\mathbf{x}}\hat{\mathbf{x}}+\hat{\mathbf{y}}\hat{\mathbf{y}}+\hat{\mathbf{z}}\hat{\mathbf{z}})dA = R^2\int_{r=R}(\hat{\mathbf{r}}\times\mathbf{A})d\Omega \tag{8}$$

which is identical to Eq. (5). Now, expressing the vector potential in terms of the current density:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \tag{9}$$

we can use the identity,

$$\int d\Omega \frac{\hat{\mathbf{r}}}{|\mathbf{r} - \mathbf{r}'|} = \frac{4\pi}{3} \frac{r_<}{r_>^2} \hat{\mathbf{r}'}. \tag{10}$$

Therefore,

$$R^2 \int_{r=R} (\hat{\mathbf{r}} \times \mathbf{A}) d\Omega = \frac{4\pi R^2}{3} \int d^3 r' \, \frac{r_<}{r_>^2} \, \hat{\mathbf{r}'} \times \mathbf{J}(\mathbf{r}'). \tag{11}$$

If the sphere R contains the entire current distribution, then $r_> = R$ and $r_< = r'$ so that (11) becomes

$$R^2 \int_{r=R} (\hat{\mathbf{r}} \times \mathbf{A}) d\Omega = \frac{4\pi}{3} \int d^3 r' \, \mathbf{r}' \times \mathbf{J}(\mathbf{r}') \equiv \frac{8\pi}{3} \mathbf{m}. \tag{12}$$