January 19, 2000

Notes for Lecture #2

"Proof" of the identity (Eq. (1.31))

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi \delta^3(\mathbf{r} - \mathbf{r}'). \tag{1}$$

Noting that

$$\int_{\substack{\text{small sphere} \\ \text{about } \mathbf{r}'}} d^3r \; \delta^3(\mathbf{r} - \mathbf{r}') f(\mathbf{r}) = f(\mathbf{r}'), \tag{2}$$

we see that we must show that

$$\int_{\substack{\text{small sphere} \\ \text{about } \mathbf{r}'}} d^3r \; \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) f(\mathbf{r}) = -4\pi f(\mathbf{r}'). \tag{3}$$

We introduce a small radius a such that:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \lim_{a \to 0} \frac{1}{\sqrt{|\mathbf{r} - \mathbf{r}'|^2 + a^2}}. \tag{4}$$

For a fixed value of a,

$$\nabla^2 \frac{1}{\sqrt{|\mathbf{r} - \mathbf{r}'|^2 + a^2}} = \frac{-3a^2}{(|\mathbf{r} - \mathbf{r}'|^2 + a^2)^{5/2}}. \tag{5}$$

If the function $f(\mathbf{r})$ is continuous, we can make a Tayor expansion about the point $\mathbf{r} = \mathbf{r}'$. Jackson's text shows that it is necessary to keep only the leading term. The integral over the small sphere about $\mathbf{r}'$ can be carried out analytically, by changing to a coordinate system centered at $\mathbf{r}'$;

$$\mathbf{u} = \mathbf{r} - \mathbf{r}', \tag{6}$$

so that

$$\int_{\substack{\text{small sphere} \\ \text{about } \mathbf{r}'}} d^3r \; \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) f(\mathbf{r}) \approx f(\mathbf{r}') \int_{u<R} d^3u \frac{-3a^2}{(u^2 + a^2)^{5/2}}. \tag{7}$$

We note that

$$\int_{u<R} d^3u \frac{-3a^2}{(u^2 + a^2)^{5/2}} = 4\pi \int_0^R du \; \frac{-3a^2 u^2}{(u^2 + a^2)^{5/2}} = 4\pi \frac{-R^3}{(R^2 + a^2)^{3/2}}. \tag{8}$$

If the infinitesimal value a is $a \ll R$, then $(R^2 + a^2)^{3/2} \approx R^3$ and the right hand side of Eq. 8 is -4π. Therefore, Eq. 7 becomes,

$$\int_{\substack{\text{small sphere}\\ \text{about } \mathbf{r}'}} d^3r\, \nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) f(\mathbf{r}) \approx f(\mathbf{r}')(-4\pi), \tag{9}$$

which is consistent with Eq. 3.

Examples of solutions of the one-dimensional Poisson equation

Consider the following one dimensional charge distribution:

$$\rho(x) = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0 & \text{for } -a < x < 0 \\ +\rho_0 & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases} \tag{10}$$

We want to find the electrostatic potential such that

$$\frac{d^2\Phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}, \tag{11}$$

with the boundary condition $\Phi(-\infty) = 0$.

In class, we showed that the solution is given by:

$$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ (\rho_0/(2\varepsilon_0))(x+a)^2 & \text{for } -a < x < 0 \\ -(\rho_0/(2\varepsilon_0))(x-a)^2 + (\rho_0 a^2)/\varepsilon_0 & \text{for } 0 < x < a \\ (\rho_0 a^2)/\varepsilon_0 & \text{for } x > a \end{cases}. \tag{12}$$

The electrostatic field is given by:

$$E(x) = \begin{cases} 0 & \text{for } x < -a \\ -(\rho_0/\varepsilon_0)(x+a) & \text{for } -a < x < 0 \\ (\rho_0/\varepsilon_0)(x-a) & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}. \tag{13}$$

The electrostatic potential can be determined by piecewise solution within each of the four regions or by use of the Green's function $G(x, x') = x_<$, where,

$$\Phi(x) = \frac{1}{\varepsilon_0}\int_{-\infty}^{\infty} G(x, x')\rho(x')dx'. \tag{14}$$

In the expression for $G(x, x')$, $x_<$ should be taken as the smaller of x and x'. It can be shown that Eq. 14 gives the identical result for $\Phi(x)$ as given in Eq. 12.