Notes for Lecture #20

Magnetic field due to electrons in the vicinity of a nucleus

According to the Biot-Savart law, the magnetic field produced by a current density $\mathbf{J}(\mathbf{r}')$ is given by:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{1}$$

In this case, we assume that the current density is due to an electron in a bound atomic state with quantum numbers $|nlm_l\rangle$, as described by a wavefunction $\psi_{nlm_l}(\mathbf{r})$, where the azimuthal quantum number m_l is associated with a factor of the form $\mathrm{e}^{im_l\phi}$. For such a wavefunction the quantum mechanical current density operator can be evaluated:

$$\mathbf{J}(\mathbf{r}') = \frac{-e\hbar}{2mi} \left(\psi^*_{nlm_l} \nabla' \psi_{nlm_l} - \psi_{nlm_l} \nabla' \psi^*_{nlm_l} \right). \tag{2}$$

Since the only complex part of this wavefunction is associated with the azimuthal quantum number, this can be written:

$$\mathbf{J}(\mathbf{r}') = \frac{-e\hbar}{2mir' \sin\theta'} \left(\psi^*_{nlm_l} \frac{\partial}{\partial\phi'} \psi_{nlm_l} - \psi_{nlm_l} \frac{\partial}{\partial\phi'} \psi^*_{nlm_l} \right) \hat{\phi}' = \frac{-e\hbar m_l \hat{\phi}'}{mr' \sin\theta'} |\psi_{nlm_l}|^2 . \tag{3}$$

We need to use this current density in the Biot-Savart law and evaluate the field at the nucleus ($\mathbf{r} = \mathbf{0}$). The vector cross product in the numerator can be evaluated in spherical polar coordinates as:

$$\hat{\phi}' \times (-\mathbf{r}') = r' \left(-\hat{\mathbf{x}} \cos\theta' \cos\phi' - \hat{\mathbf{y}} \cos\theta' \sin\phi' + \hat{\mathbf{z}} \sin\theta' \right) \tag{4}$$

Thus the magnetic field evaluated at the nucleus is given by the integral:

$$\mathbf{B}(\mathbf{0}) = -\frac{\mu_0 e\hbar m_l}{4\pi m} \int d^3r' \, |\psi_{nlm_l}|^2 \, \frac{r' \left(-\hat{\mathbf{x}} \cos\theta' \cos\phi' - \hat{\mathbf{y}} \cos\theta' \sin\phi' + \hat{\mathbf{z}} \sin\theta' \right)}{r' \sin\theta' \, r'^3}. \tag{5}$$

In evaluating the integration over the azimuthal variable ϕ', the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components vanish leaving the simple result:

$$\mathbf{B}(\mathbf{0}) = -\frac{\mu_0 e\hbar m_l \hat{\mathbf{z}}}{4\pi m} \int d^3r' \, |\psi_{nlm_l}|^2 \, \frac{1}{r'^3} \equiv -\frac{\mu_0 e}{4\pi m} \mathbf{L} \left\langle \frac{1}{r'^3} \right\rangle. \tag{6}$$