

Notes for Lecture #16

Vector potentials in magnetostatics

The vector potential corresponding to a current density distribution $\mathbf{J}(\mathbf{r})$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (1)$$

This expression is useful if the current density $\mathbf{J}(\mathbf{r})$ is confined within a finite region of space. Consider the following example corresponding to a rotating charged sphere of radius a , with ρ_0 denoting the uniform charge density within the sphere and ω denoting the angular rotation of the sphere:

$$\mathbf{J}(\mathbf{r}') = \begin{cases} \rho_0 \omega \times \mathbf{r}' & \text{for } r' \leq a \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

In order to evaluate the vector potential (1) for this problem, we can make use of the expansion:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r'^l_{<}}{r^{l+1}_{>}} Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}'). \quad (3)$$

Noting that

$$\mathbf{r}' = r' \sqrt{\frac{4\pi}{3}} \left(Y_{1-1}(\hat{\mathbf{r}}') \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}') \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}') \hat{\mathbf{z}} \right), \quad (4)$$

we see that the angular integral result takes the simple form:

$$\int d\Omega' \sum_m Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}') \mathbf{r}' = \frac{r'}{r} \mathbf{r} \delta_{l1}. \quad (5)$$

Therefore the vector potential for this system is:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \rho_0 \omega \times \mathbf{r}}{3r} \int_0^a dr' r'^3 \frac{r'_{<}}{r^2_{>}}, \quad (6)$$

which can be evaluated as:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0 \omega \times \mathbf{r}}{3} \left(\frac{a^2}{2} - \frac{3r^2}{10} \right) & \text{for } r \leq a \\ \frac{\mu_0 \rho_0 \omega \times \mathbf{r}}{3r^3} \frac{a^5}{5} & \text{for } r \geq a \end{cases}. \quad (7)$$

As another example, consider the current associated with an electron in the $|nlm = 211\rangle$ state of a H atom:

$$\mathbf{J}(\mathbf{r}') = \frac{-e\hbar}{64m\pi a^5} e^{-r'/a} \hat{\mathbf{z}} \times \mathbf{r}', \quad (8)$$

where a here denotes the Bohr radius. Using arguments similar to those above, we find that

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0\hat{\mathbf{z}} \times \mathbf{r}}{192m\pi a^5 r} \int_0^\infty dr' r'^3 e^{-r'/a} \frac{r <}{r^2}. \quad (9)$$

This expression can be integrated to give:

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0\hat{\mathbf{z}} \times \mathbf{r}}{8m\pi r^3} \left[1 - e^{-r/a} \left(1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{8a^3} \right) \right]. \quad (10)$$