

## Notes for Lecture #3

### The “mean value theorem” for solutions to the Laplace equation

Consider an electrostatic field  $\Phi(\mathbf{r})$  in a charge-free region so that it satisfies the Laplace equation:

$$\nabla^2\Phi(\mathbf{r}) = 0. \quad (1)$$

The “mean value theorem” value theorem states that the value of  $\Phi(\mathbf{r})$  at the arbitrary (charge-free) point  $\mathbf{r}$  is equal to the average of  $\Phi(\mathbf{r}')$  over the surface of any sphere centered on the point  $\mathbf{r}$  (see Jackson problem #1.10). One way to prove this theorem is the following. Consider a point  $\mathbf{r}' = \mathbf{r} + \mathbf{u}$ , where  $\mathbf{u}$  will describe a sphere of radius  $R$  about the fixed point  $\mathbf{r}$ . We can make a Taylor series expansion of the electrostatic potential  $\Phi(\mathbf{r}')$  about the fixed point  $\mathbf{r}$ :

$$\Phi(\mathbf{r} + \mathbf{u}) = \Phi(\mathbf{r}) + \mathbf{u} \cdot \nabla\Phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{u} \cdot \nabla)^2\Phi(\mathbf{r}) + \frac{1}{3!}(\mathbf{u} \cdot \nabla)^3\Phi(\mathbf{r}) + \frac{1}{4!}(\mathbf{u} \cdot \nabla)^4\Phi(\mathbf{r}) + \dots \quad (2)$$

According to the premise of the theorem, we want to integrate both sides of the equation 2 over a sphere of radius  $R$  in the variable  $\mathbf{u}$ :

$$\int_{\text{sphere}} dS_u = R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u). \quad (3)$$

We note that

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) 1 = 4\pi R^2, \quad (4)$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \mathbf{u} \cdot \nabla = 0, \quad (5)$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^2 = \frac{4\pi R^4}{3} \nabla^2, \quad (6)$$

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^3 = 0, \quad (7)$$

and

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) (\mathbf{u} \cdot \nabla)^4 = \frac{4\pi R^4}{5} \nabla^4. \quad (8)$$

Since  $\nabla^2\Phi(\mathbf{r}) = 0$ , the only non-zero term of the average is thus the first term:

$$R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r} + \mathbf{u}) = 4\pi R^2 \Phi(\mathbf{r}), \quad (9)$$

or

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2} R^2 \int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) \Phi(\mathbf{r} + \mathbf{u}). \quad (10)$$

Since this result is independent of the radius  $R$ , we see that we have proven the theorem.