

## Notes for Lecture #30

### Electromagnetic wave guides

In order to understand the operation of a wave guide, we must first learn how electromagnetic waves behave in a dissipative medium. A plane wave solution to Maxwell's equations of the form:

$$\mathbf{E} = \mathbf{E}_0 e^{ik\hat{\mathbf{k}}\cdot\mathbf{r} - i\omega t} \quad \text{and} \quad \mathbf{B} = \frac{k}{\omega} \hat{\mathbf{k}} \times \mathbf{E}_0 e^{ik\hat{\mathbf{k}}\cdot\mathbf{r} - i\omega t} \quad (1)$$

for the electric and magnetic fields, with the wave vector  $k$  satisfying the relation:

$$k^2 = \omega^2 \mu \varepsilon \equiv \mathcal{R} + i\mathcal{I}. \quad (2)$$

We can determine the complex wavevector  $k_r + ik_i$  according to

$$k_r = \left( \frac{\sqrt{\mathcal{R}^2 + \mathcal{I}^2} + \mathcal{R}}{2} \right)^{1/2} \quad \text{and} \quad k_i = \left( \frac{\sqrt{\mathcal{R}^2 + \mathcal{I}^2} - \mathcal{R}}{2} \right)^{1/2} \quad (3)$$

The form of the frequency dependent constants  $\mathcal{R}$  and  $\mathcal{I}$  depend on the materials. For the Drude model at low frequency (Eq. 7.56),  $\mathcal{R} = \omega^2 \mu \varepsilon_b$  and  $\mathcal{I} = \omega \mu \sigma$ , for example. The value of  $k_i$  determines the rate of decay of the field amplitudes in the vicinity of the surface, with the skin depth given by  $\delta \equiv 1/k_i$ . In the limit that  $\mathcal{I} \gg \mathcal{R}$ , as in the case of a good conductor at low frequency,  $\delta \approx (2/(\omega \mu \sigma))^{1/2}$ .

For an "ideal" conductor  $\mathcal{I} \rightarrow \infty$ , so that the fields are confined to the surface. Because of the field continuity conditions at the surface of the conductor, this means that,  $\mathbf{B}_{\text{tangential}} \neq 0$  (because there can be a surface current),  $\mathbf{E}_{\text{normal}} \neq 0$  (because there can be a surface charge), and  $\mathbf{B}_{\text{normal}} = 0$ .

Suppose we construct a wave guide from an "ideal" conductor, designating  $\hat{\mathbf{z}}$  as the propagation direction. We will assume that the fields take the form:

$$\mathbf{E} = \mathbf{E}(x, y) e^{ikz - i\omega t} \quad \text{and} \quad \mathbf{B} = \mathbf{B}(x, y) e^{ikz - i\omega t} \quad (4)$$

inside the pipe, where now  $k$  and  $\varepsilon$  are assumed to be real. Assuming that there are no sources inside the pipe, the fields there must satisfy Maxwell's equations (8.16) which expand to the following :

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0. \quad (5)$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0. \quad (6)$$

$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x. \quad (7)$$

$$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y. \quad (8)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z. \quad (9)$$

$$\frac{\partial B_z}{\partial y} - ikB_y = -i\mu\epsilon\omega E_x. \quad (10)$$

$$ikB_x - \frac{\partial B_z}{\partial x} = -i\mu\epsilon\omega E_y. \quad (11)$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i\mu\epsilon\omega E_z. \quad (12)$$

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholtz equation:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 + \mu\epsilon\omega^2 \right) E_x(x, y) = 0, \quad (13)$$

with similar expressions for each of the other field components. For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x, y) \equiv 0 \quad \text{and} \quad B_z(x, y) = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \quad (14)$$

with  $k^2 \equiv k_{mn}^2 = \mu\epsilon\omega^2 - \left[ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]$ . From this result and Maxwell's equations, we can determine the other field components. For example Eqs. (7-8) simplify to

$$B_x = -\frac{k}{\omega} E_y \quad \text{and} \quad B_y = \frac{k}{\omega} E_x. \quad (15)$$

These results can be used in Eqs. (10-11) to solve for the fields  $E_x$  and  $E_y$  and  $B_x$  and  $B_y$ :

$$E_x = \frac{\omega}{k} B_y = \frac{-i\omega}{k^2 - \mu\epsilon\omega^2} \frac{\partial B_z}{\partial y} = \frac{-i\omega}{\left[ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]} \frac{n\pi}{b} B_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \quad (16)$$

and

$$E_y = -\frac{\omega}{k} B_x = \frac{i\omega}{k^2 - \mu\epsilon\omega^2} \frac{\partial B_z}{\partial x} = \frac{i\omega}{\left[ \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]} \frac{n\pi}{a} B_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right). \quad (17)$$

One can check this result to show that these results satisfy the boundary conditions. For example,  $\mathbf{E}_{\text{tangential}} = 0$  is satisfied since  $E_x(x, 0) = E_x(x, b) = 0$  and  $E_y(0, y) = E_y(a, y) = 0$ . This was made possible choosing  $\nabla B_z \cdot \hat{\mathbf{n}}_{\text{surface}} = 0$ , where  $\hat{\mathbf{n}}$  denotes a unit normal vector pointing out of the wave guide surface.