February 7, 2001

Notes for Lecture #9

Finite element method

The finite element approach is based on an expansion of the unknown electrostatic potential in terms of known grid-based functions of fixed shape. In two dimensions, using the indices $\{i, j\}$ to reference the grid, we can denote the shape functions as $\{\phi_{ij}(x, y)\}$. The finite element expansion of the potential in two dimensions can take the form:

$$4\pi\varepsilon_0\Phi(x, y) = \sum_{ij} \psi_{ij}\phi_{ij}(x, y), \tag{1}$$

where ψ_{ij} represents the amplitude associated with the shape function $\phi_{ij}(x, y)$. The amplitude values can be determined for a given solution of the Poisson equation:

$$-\nabla^2 \left(4\pi\varepsilon_0\Phi(x, y)\right) = 4\pi\rho(x, y), \tag{2}$$

by solving a linear algebra problem of the form

$$\sum_{ij} M_{kl,ij}\psi_{ij} = G_{kl}, \tag{3}$$

where

$$M_{kl,ij} \equiv \int dx \int dy \nabla\phi_{kl}(x, y) \cdot \nabla\phi_{ij}(x, y) \quad \text{and} \quad G_{kl} \equiv \int dx \int dy \phi_{kl}(x, y)\ 4\pi\rho(x, y). \tag{4}$$

In obtaining this result, we have assumed that the boundary values vanish. In order for this result to be useful, we need to be able evaluate the integrals for $M_{kl,ij}$ and for G_{kl}. In the latter case, we need to know the form of the charge density. The form of $M_{kl,ij}$ only depends upon the form of the shape functions. If we take these functions to be:

$$\phi_{ij}(x, y) \equiv \mathcal{X}_i(x)\mathcal{Y}_j(y), \tag{5}$$

where

$$\mathcal{X}_i(x) \equiv \begin{cases} \left(1 - \frac{|x-x_i|}{h}\right) & \text{for } x_i - h \leq x \leq x_i + h \\ 0 & \text{otherwise} \end{cases}, \tag{6}$$

and $\mathcal{Y}_j(y)$ has a similar expression in the variable y. Then

$$M_{kl,ij} \equiv \int dx \int dy \left[\frac{d\mathcal{X}_k(x)}{dx}\frac{d\mathcal{X}_i(x)}{dx}\mathcal{Y}_l(y)\mathcal{Y}_j(y) + \mathcal{X}_k(x)\mathcal{X}_i(x)\frac{d\mathcal{Y}_l(y)}{dy}\frac{d\mathcal{Y}_j(y)}{dy}\right]. \tag{7}$$

There are four types of non-trivial contributions to these values:

$$\int_{x_i-h}^{x_i+h} \left(\mathcal{X}_i(x)\right)^2 dx = h\int_{-1}^{1}(1 - |u|)^2 du = \frac{2h}{3}, \tag{8}$$

$$\int_{x_i-h}^{x_i+h} \left(\mathcal{X}_i(x)\mathcal{X}_{i+1}(x)\right) dx = h\int_0^1 (1-u)u du = \frac{h}{6}, \tag{9}$$

$$\int_{x_i-h}^{x_i+h} \left(\frac{d\mathcal{X}_i(x)}{dx}\right)^2 dx = \frac{1}{h}\int_{-1}^1 du = \frac{2}{h}, \tag{10}$$

and

$$\int_{x_i-h}^{x_i+h} \left(\frac{d\mathcal{X}_i(x)}{dx}\frac{d\mathcal{X}_{i+1}(x)}{dx}\right) dx = -\frac{1}{h}\int_0^1 du = \frac{-1}{h}. \tag{11}$$

These basic ingredients lead to the following distinct values for the matrix:

$$M_{kl,ij} = \begin{cases} \frac{8}{3} & \text{for } k=i \text{ and } l=j \\ -\frac{1}{3} & \text{for } k-i=\pm 1 \text{ and/or } l-j=\pm 1 \\ 0 & \text{otherwise} \end{cases}. \tag{12}$$

For problems in which the boundary values are 0, Eq. 3 then can be used to find all of the interior amplitudes ψ_{ij}.

In order to use this technique to solve the boundary value problem discussed in Lecture Notes #5, we have to make one modification. The boundary value of $\Phi(x,a) = V_0$ is not consistent with the derivation of Eq. (4), however, since we are only interested in the region $0 \le y \le a$, we can extend our numerical analysis to the region $0 \le y \le a+h$ and require $\Phi(x, a+h) = 0$ in addition to $\Phi(x,a) = V_0$. Using the same indexing as in Lecture Notes #5, this means that $\psi_1 = \psi_2 = \psi_3 = V_0$. The finite element approach for this problem thus can be put into the matrix form for analysis by Maple:

$$\begin{pmatrix} 8/3 & -1/3 & -1/3 & -1/3 & 0 & 0 \\ -2/3 & 8/3 & -2/3 & -1/3 & 0 & 0 \\ -1/3 & -1/3 & 8/3 & -1/3 & -1/3 & -1/3 \\ -2/3 & -1/3 & -2/3 & 8/3 & -2/3 & -1/3 \\ 0 & 0 & -1/3 & -1/3 & 8/3 & -1/3 \\ 0 & 0 & -2/3 & -1/3 & -2/3 & 8/3 \end{pmatrix} \begin{pmatrix} \psi_5 \\ \psi_6 \\ \psi_8 \\ \psi_9 \\ \psi_{11} \\ \psi_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} V_0. \tag{13}$$

The solution to these equations and the exact results are found to be:

$$\begin{pmatrix} \psi_5 \\ \psi_6 \\ \psi_8 \\ \psi_9 \\ \psi_{11} \\ \psi_{12} \end{pmatrix} = \begin{pmatrix} 0.5070276498 \\ .5847926267 \\ 0.1928571429 \\ 0.2785714286 \\ 0.07154377880 \\ 0.1009216590 \end{pmatrix} V_0; \quad (\text{exact}) = \begin{pmatrix} .4320283318 \\ .5405292183 \\ .1820283318 \\ 0.25 \\ .06797166807 \\ .09541411792 \end{pmatrix} V_0. \tag{14}$$

We see that the results are similar to those obtained using the finite difference approach.