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Summary of angular momentum formalisms

Coordinate representation of orbital angular momentum

In spherical polar coordinates, the operator representing the squared angular momentum $\mathbf{L}^2$ takes the form:

$$\mathbf{L}^2 = -\hbar^2 \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right\}, \tag{1}$$

while the operator representing $z-$component of angular momentum takes the form:

$$L_z = -i\hbar \frac{\partial}{\partial\phi}. \tag{2}$$

The spherical harmonic functions Y_{lm} are eigenfunctions of both $\mathbf{L}^2$ and L_z with

$$\mathbf{L}^2 Y_{lm} = \hbar^2 l(l+1) \tag{3}$$

and

$$L_z Y_{lm} = \hbar m. \tag{4}$$

Some of these spherical harmonic functions are:

$$Y_{00} = \frac{1}{4\pi} \tag{5}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \tag{6}$$

$$Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta \mathrm{e}^{\pm i\phi} \tag{7}$$

In the process of evaluating the differential eigenvalue equations, we find that the “quantum numbers” l must be postive integers ($l = 0, 1, 2, \ldots$), and m is restricted to the integer values between $-l \le m \le l$.

Operator representation of general angular momentum

The following derivation follows the discussion of Shankar’s text (*Principles of Quantum Mechanics*, 2nd edition, Chapter 12). It turns out that a very similar eigenvalue structure can be derived in an operator formalism. In this operator formalism, we will see that additional half-integer solutions for the angular momentum quantum numbers are also possible. For this generalization we will use $\mathbf{J}^2$ and J_z to represent the square and $z-$ components of the angular momentum, respectively. Furthermore, we will assume that we can find the eigenvalues of these operators which we will denote by a and b for the moment:

$$\mathbf{J}^2|ab\rangle = a|ab\rangle \tag{8}$$

$$J_z|ab\rangle = b|ab\rangle. \tag{9}$$

We can now introduce 2 other operators which will prove to be very helpful:

$$J_\pm \equiv J_x \pm J_y. \tag{10}$$

We can show that these operators have the effect of incrementing or decrementing the b eigenvalue of $|ab\rangle$ by one.

First we note the following commutation relations:

$$[J_z, J_\pm] = \pm\hbar J_\pm \tag{11}$$

and

$$[\mathbf{J}^2, J_\pm] = 0. \tag{12}$$

Later, we will also need to use the result

$$[J_-, J_+] = -2\hbar J_z, \tag{13}$$

which follows from the identity

$$[J_x, J_y] = i\hbar J_z. \tag{14}$$

We can then show that the function $(J_\pm|ab\rangle)$ has eigenvalues a and $b \pm \hbar$ of $\mathbf{J}^2$ and J_z, respectively. Acting on $(J_\pm|ab\rangle)$ with $\mathbf{J}^2$:

$$\mathbf{J}^2(J_\pm|ab\rangle) = J_\pm\mathbf{J}^2|ab\rangle = J_\pm a|ab\rangle = a(J_\pm|ab\rangle). \tag{15}$$

Acting on $(J_\pm|ab\rangle)$ with J_z:

$$J_z(J_\pm|ab\rangle) = \pm\hbar|ab\rangle + J_\pm J_z|ab\rangle = \pm\hbar|ab\rangle + J_\pm b|ab\rangle = (\pm\hbar + b)(J_\pm|ab\rangle). \tag{16}$$

This mean that we can write the function $(J_\pm|ab\rangle)$ as $\mathcal{N}|a(b\pm\hbar)\rangle$, where $\mathcal{N}$ is a normalization constant determined from:

$$\mathcal{N}^2\langle a(b\pm\hbar)|a(b\pm\hbar)\rangle = \langle ab|J_\pm^\dagger J_\pm|ab\rangle = \langle ab|(\mathbf{J}^2 - J_z^2 \mp \hbar J_z|ab\rangle = a - b^2 \mp \hbar b, \tag{17}$$

assuming that $\langle ab||ab\rangle = 1$. This result means that

$$\mathcal{N} = \sqrt{a - b^2 \mp \hbar b}. \tag{18}$$

In order to make further progress, we notice that since the normalization cannot be negative, for a given value of a, there are restrictions on the value of b. In particular, we can safely assume that there is a maximum value of b which we will denote by b_{max}. From the behavior of a maximum value, we know that

$$J_+|ab_{\text{max}}\rangle = 0. \tag{19}$$

Now multiplying the above equation by J_-, we find

$$J_-J_+|ab_{\rm max}\rangle = 0 = (J_x^2 + J_y^2 + i[J_x, J_y])|ab_{\rm max}\rangle = (\mathbf{J}^2 - J_z^2 - \hbar J_z)|ab_{\rm max}\rangle = a - b_{\rm max}^2 - \hbar b_{\rm max}. \tag{20}$$

This defines the eigenvalue a in terms of $b_{\rm max}$ to be

$$a = b_{\rm max}(b_{\rm max} + \hbar). \tag{21}$$

We can also use Eq. (18) to argue that b has a minimum value $b_{\rm min}$ and analyzing the properties of $|ab_{\rm min}\rangle$ using similar steps as above, we can also show that

$$a = b_{\rm min}(b_{\rm min} - \hbar). \tag{22}$$

Comparing Eqs. (21) and (22), it is apparent that

$$b_{\rm min} = -b_{\rm max}. \tag{23}$$

It is now convenient to define $b_{\rm max} \equiv \hbar j$ so that the eigenvalue a can be written

$$a = \hbar^2 j(j+1). \tag{24}$$

This analysis then suggests that if we define a general value of the eigenvalue b to take the form

$$b \equiv \hbar m_j, \tag{25}$$

the results tell us that m_j can take the values $-j \leq m_j \leq j$, ($2j+1$ different values in all for a given j). With these definitions, the normalized increment or decrement operation can be written:

$$J_\pm|jm_j\rangle = \hbar\sqrt{j(j+1) - m_j(m_j \pm 1)}|j(m_j \pm 1)\rangle. \tag{26}$$

This structure of the eigenvalues jm_j is very similar to the eigenvalues of orbital angular moment lm. There is one new "wrinkle", however. The above arguments tell us that we can get from the maximum value of $m_j = j$ to the minimum value $m_j = -j$ in a number of applications of the operator J_-. Suppose that that number of applications is U. This means that the sequence of values of the eigenvalue m_j is

$$j, j-1, j-2, \ldots j-U, \tag{27}$$

so that

$$j - U = -j \tag{28}$$

or

$$j = \frac{U}{2}. \tag{29}$$

Since U must be an integer, j can be an integer if U is even, but *can also be a half-integer* if U is odd!! This means that we can use this formalism to describe orbital, spin, *and* total angular momentum.