Notes on quantum treatment of spin transitions

Reference: Charles P. Slichter, **Principles of Magnetic Resonance**, Harper & Row, 1963.

In the following, we will use the following Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

In terms of these matrices, the electron magnetic moment can be written:

$$\vec{\mu} = -\frac{g\mu_B}{2}\vec{\sigma} \equiv \mu_e(\sigma_x\hat{\mathbf{x}} + \sigma_y\hat{\mathbf{y}} + \sigma_z\hat{\mathbf{z}}), \tag{2}$$

where g denotes the electron g-factor, $\mu_B = 5.788 \times 10^{-5}$ eV/T is the Bohr magneton, and $\mu_e = -5.795 \times 10^{-5}$ eV/T is the electron magnetic moment. The Hamiltonian which represents the interaction between the electron magnetic moment and a magnetic field **B** is given by

$$\mathcal{H} = -\vec{\mu} \cdot \mathbf{B}. \tag{3}$$

In magnetic resonance experiments, the magnetic field is generally composed of a constant component (B_0) and a rotating component (B_1) in the perpendicular direction. Suppose that the rotation frequence is denoted by Ω, the magnetic field can be written:

$$\mathbf{B} = B_1(\cos(\Omega t)\hat{\mathbf{x}} + \sin(\Omega t)\hat{\mathbf{y}}) + B_0\hat{\mathbf{z}}, \tag{4}$$

where it is generally assumed that $B_0 >> B_1$. For this field, the interaction Hamiltonian can be written:

$$\mathcal{H} = -\mu_e\mathbf{B} \cdot \vec{\sigma} \equiv -\mu_e \begin{pmatrix} B_0 & B_1 \mathrm{e}^{-i\Omega t} \\ B_1\mathrm{e}^{i\Omega t} & -B_0 \end{pmatrix}. \tag{5}$$

We would like to solve the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\Psi(t)}{\partial t} = \mathcal{H}\Psi(t). \tag{6}$$

In order simplify the mathematics, we notice that

$$\begin{pmatrix} B_0 & B_1 \mathrm{e}^{-i\Omega t} \\ B_1\mathrm{e}^{i\Omega t} & -B_0 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-i\Omega t/2} & 0 \\ 0 & \mathrm{e}^{i\Omega t/2} \end{pmatrix} \begin{pmatrix} B_0 & B_1 \\ B_1 & -B_0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\Omega t/2} & 0 \\ 0 & \mathrm{e}^{-i\Omega t/2} \end{pmatrix}. \tag{7}$$

Similarly, we can write,

$$i\hbar\frac{\partial\Psi(t)}{\partial t} = \begin{pmatrix} \mathrm{e}^{-i\Omega t/2} & 0 \\ 0 & \mathrm{e}^{i\Omega t/2} \end{pmatrix} \left\{ i\hbar\frac{\partial}{\partial t} - \begin{pmatrix} \frac{-\hbar\Omega}{2} & 0 \\ 0 & \frac{\hbar\Omega}{2} \end{pmatrix} \right\} \begin{pmatrix} \mathrm{e}^{i\Omega t/2} & 0 \\ 0 & \mathrm{e}^{-i\Omega t/2} \end{pmatrix} \Psi(t). \tag{8}$$

Defining a transformed wavefunction Ψ'

$$\Psi' \equiv \begin{pmatrix} e^{i\Omega t/2} & 0 \\ 0 & e^{-i\Omega t/2} \end{pmatrix} \Psi(t), \tag{9}$$

the differential equation that must be solved to find the solutions to the Schrödinger equation can then be written:

$$i\hbar \frac{\partial \Psi'(t)}{\partial t} = \begin{pmatrix} -\mu_e B_0 - \frac{\hbar\Omega}{2} & -\mu_e B_1 \\ -\mu_e B_1 & -(-\mu_e B_0 - \frac{\hbar\Omega}{2}) \end{pmatrix} \Psi'(t) \equiv \mathcal{H}_{\text{eff}} \Psi'(t). \tag{10}$$

The transformed wavefunction Ψ' can be interpreted as representing the spin in a rotating coordinate system in which the effective Hamiltonian is now independent of time. Solving the differential equation 10, we find

$$\Psi'(t) = e^{-i\mathcal{H}_{\text{eff}}/\hbar t} \Psi'(0), \tag{11}$$

where $\Psi'(0)$ denotes the initial value of the transformed wavefunction and where the expontial function must be evaluated by taking its Taylor series expansion. Consider a general 2×2 matrix of the form

$$m \equiv \begin{pmatrix} a & b \\ b & -a \end{pmatrix}. \tag{12}$$

The exponential of m can be evaluated:

$$e^{-im} \equiv 1 - im - \frac{1}{2!} m^2 - \frac{1}{3!} m^2(-im) + \frac{1}{4!} (m^2)^2 \cdots \tag{13}$$

For our form of m, all even terms are diagonal,

$$m^2 = \begin{pmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix} \equiv (a^2 + b^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{14}$$

and all odd terms are proportial to m itself, so that we can simplify the expansion by summing the odd and even terms separately:

$$e^{-im} = \cos(\sqrt{a^2 + b^2}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \frac{\sin(\sqrt{a^2 + b^2})}{\sqrt{a^2 + b^2}} \begin{pmatrix} a & b \\ b & -a \end{pmatrix}. \tag{15}$$

Defining the simplifying notation, $\Omega_0 \equiv -\mu_e B_0/\hbar$, $\Omega_1 \equiv -\mu_e B_1/\hbar$, $\Omega_T \equiv \sqrt{(\Omega_0 - \Omega/2)^2 + \Omega_1^2}$, $\cos(\theta_0) \equiv (\Omega_0 - \Omega/2)/\Omega_T$, and $\sin(\theta_0) \equiv \Omega_1/\Omega_T$, we can write the full solution to the Schrödinger equation in the form

$$\Psi(t) = \begin{pmatrix} e^{-i\Omega t/2} & 0 \\ 0 & e^{i\Omega t/2} \end{pmatrix} \left\{ \cos(\Omega_T t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin(\Omega_T t) \begin{pmatrix} \cos(\theta_0) & \sin(\theta_0) \\ \sin(\theta_0) & -\cos(\theta_0) \end{pmatrix} \right\} \Psi(0). \tag{16}$$

For the special value of the rotational frequency $\Omega = 2\Omega_0$, the general result 16 simplifies to

$$\Psi(t) = \begin{pmatrix} e^{-i\Omega_0 t} & 0 \\ 0 & e^{i\Omega_0 t} \end{pmatrix} \begin{pmatrix} \cos(\Omega_1 t) & -i\sin(\Omega_1 t) \\ -i\sin(\Omega_1 t) & \cos(\Omega_1 t) \end{pmatrix} \Psi(0). \tag{17}$$