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Scattering amplitudes and phase shifts

Suppose we have a particle with momentum $\hbar\mathbf{k}$ and energy $E = \frac{\hbar^2 k^2}{2m}$. Apart from normalization, its wavefunction is given by

$$\Psi^0(\mathbf{r}) = \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}. \tag{1}$$

If this particle is scattered, far from the target, the wavefunction take the form

$$\Psi(\mathbf{r}) = \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta)\frac{\mathrm{e}^{ikr}}{r}, \tag{2}$$

where the scattering amplitude $f(\theta)$ will be discussed further below. The angle θ measures the angle of the scattered particle relative to its incident direction. It turns out that scattering cross section for this process can be determined in terms of the scattering amplitude according to:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2. \tag{3}$$

As we will see, it is convenient to determine $f(\theta)$ in terms of scattering "phase shifts" δ_l for each angular momentum quantum number l:

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\mathrm{e}^{i\delta_l}\sin\delta_l P_l(\cos\theta). \tag{4}$$

In this expression, $P_l(\cos\theta)$ denotes a Legendre polynomial. The total diffential cross section can also be analyzed by integrating over all angles. Because of the orthogonal properties of the Legendre polynomials, the result simplifies:

$$\sigma = \int \frac{d\sigma}{d\Omega}d\Omega = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l. \tag{5}$$

The scattering equations can be directly derived from a Green's function analysis. A simplier argument goes as follows. First, we note that a plane wave can be expressed in spherical polar coordinates centered at the target as a some of "partial wave" contributions:

$$\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta). \tag{6}$$

Here $j_l(kr)$ denotes a spherical Bessel function:

$$j_0(kr) = \frac{\sin(kr)}{kr} \qquad j_1(kr) = \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr} \qquad \ldots \tag{7}$$

which has the asymptotic form:

$$j_l(kr) \stackrel{kr\gg 1}{=} \frac{\sin(kr - \frac{\pi l}{2})}{kr}. \tag{8}$$

Now consider a general solution to the Schrödinger equation for the particle in the potential of the target. Outside the range R of the potential the Schrödinger equation is that of a free particle in spherical polar coordinates with the general solution form:

$$\Psi(\mathbf{r}) = \sum_{l=0}^{\infty} A_l(\cos\delta_l j_l(kr) - \sin\delta_l n_l(kr)) P_l(\cos\theta). \tag{9}$$

Here the amplitude A_l and the "phase shift" δ_l are to be determined. The function $n_l(kr)$ denotes a "Neumann" function:

$$n_0(kr) = -\frac{\cos(kr)}{kr} \quad n_1(kr) = -\frac{\cos(kr)}{(kr)^2} - \frac{\sin(kr)}{kr} \quad \ldots \tag{10}$$

which has the asymptotic form:

$$n_l(kr) \overset{kr\gg 1}{=} -\frac{\cos(kr - \frac{\pi l}{2})}{kr}. \tag{11}$$

The phase shift "δ" depends on the form of the target potential, while the amplitude is determined by requiring that the wave function take the asympotic form (2). Evaluating the asymptotic form of the general solution (9):

$$\Psi(\mathbf{r}) \overset{kr\gg 1}{=} \sum_{l=0}^{\infty} A_l \frac{\sin(kr - \frac{\pi l}{2} + \delta_l)}{kr} P_l(\cos\theta), \tag{12}$$

or

$$\Psi(\mathbf{r}) \overset{kr\gg 1}{=} \sum_{l=0}^{\infty} A_l \frac{\mathrm{e}^{i(kr - \frac{\pi l}{2} + \delta_l)} - \mathrm{e}^{-i(kr - \frac{\pi l}{2} + \delta_l)}}{2ikr} P_l(\cos\theta). \tag{13}$$

From this expression, we note that if we choose the amplitude

$$A_l = (2l+1) i^l \mathrm{e}^{i\delta_l}, \tag{14}$$

then the expansion becomes

$$\Psi(\mathbf{r}) \overset{kr\gg 1}{=} \sum_{l=0}^{\infty} (2l+1) i^l \frac{\mathrm{e}^{i(kr - \frac{\pi l}{2} + 2\delta_l)} - \mathrm{e}^{-i(kr - \frac{\pi l}{2})}}{2ikr} P_l(\cos\theta), \tag{15}$$

which can be rearranged to from

$$\Psi(\mathbf{r}) \overset{kr\gg 1}{=} \sum_{l=0}^{\infty} (2l+1) i^l \left\{ \frac{\sin(kr - \frac{\pi l}{2})}{kr} + \frac{\mathrm{e}^{2i\delta_l} - 1}{2ik} \frac{1}{i^l} \frac{\mathrm{e}^{ikr}}{r} \right\} P_l(\cos\theta), \tag{16}$$

or

$$\Psi(\mathbf{r}) \overset{kr\gg 1}{=} \quad \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} + \sum_{l=0}^{\infty} (2l+1) \frac{\mathrm{e}^{2i\delta_l} - 1}{2ik} P_l(\cos\theta) \quad \frac{\mathrm{e}^{ikr}}{r}, \tag{17}$$

which determines the form of the scattering amplitude $f(\theta)$ according to Eq. (4).