Notes for Lecture #2

Examples of solutions of the one-dimensional Poisson equation

Consider the following one-dimensional charge distribution:

\[
\rho(x) = \begin{cases} 
0 & \text{for } x < -a \\
-p_0 & \text{for } -a < x < 0 \\
+p_0 & \text{for } 0 < x < a \\
0 & \text{for } x > a 
\end{cases}
\] (1)

We want to find the electrostatic potential such that

\[
\frac{d^2\Phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_0},
\] (2)

with the boundary condition \(\Phi(-\infty) = 0\).

The solution to the differential equation is given by:

\[
\Phi(x) = \begin{cases} 
0 & \text{for } x < -a \\
\frac{p_0}{2\varepsilon_0}(x + a)^2 & \text{for } -a < x < 0 \\
-\frac{p_0}{2\varepsilon_0}(x - a)^2 + \frac{(p_0 a^2)}{\varepsilon_0} & \text{for } 0 < x < a \\
\frac{p_0}{\varepsilon_0}a^2 & \text{for } x > a 
\end{cases}
\] (3)

The electrostatic field is given by:

\[
E(x) = \begin{cases} 
0 & \text{for } x < -a \\
-\frac{p_0}{\varepsilon_0}(x + a) & \text{for } -a < x < 0 \\
\frac{p_0}{\varepsilon_0}(x - a) & \text{for } 0 < x < a \\
0 & \text{for } x > a 
\end{cases}
\] (4)

The electrostatic potential can be determined by piecewise solution within each of the four regions or by use of the Green’s function \(G(x, x') = 4\pi x_\beta\), where,

\[
\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} G(x, x')\rho(x')dx'.
\] (5)

In the expression for \(G(x, x')\), \(x_\beta\) should be taken as the smaller of \(x\) and \(x'\). It can be shown that Eq. 5 gives the identical result for \(\Phi(x)\) as given in Eq. 3.
Notes on the one-dimensional Green’s functions

The Green’s function for the Poisson equation can be defined as a solution to the equation:
\[
\nabla^2 G(x, x') = -4\pi \delta(x - x').
\]

(6)

Here the factor of $4\pi$ is not really necessary, but ensures consistency with your text’s treatment of the 3-dimensional case. The meaning of this expression is that $x'$ is held fixed while taking the derivative with respect to $x$. It is easily shown that with this definition of the Green’s function (6), Eq. (5) finds the electrostatic potential $\Phi(x)$ for an arbitrary charge density $\rho(x)$. In order to find the Green’s function which satisfies Eq. (6), we notice that we can use two independent solutions to the homogeneous equation
\[
\nabla^2 \phi_i(x) = 0,
\]

(7)

where $i = 1$ or 2, to form
\[
G(x, x') = \frac{4\pi}{W} \phi_1(x_<)\phi_2(x_>).
\]

(8)

This notation means that $x_<$ should be taken as the smaller of $x$ and $x'$ and $x_>$ should be taken as the larger. In this expression $W$ is the “Wronskian”:
\[
W \equiv \frac{d\phi_1(x)}{dx} \phi_2(x) - \phi_1(x) \frac{d\phi_2(x)}{dx}.
\]

(9)

We can check that this “recipe” works by noting that for $x \neq x'$, Eq. (8) satisfies the defining equation 6 by virtue of the fact that it is equal to a product of solutions to the homogeneous equation 7. The defining equation is singular at $x = x'$, but integrating 6 over $x$ in the neighborhood of $x'$ ($x' - \epsilon < x < x' + \epsilon$), gives the result:
\[
\frac{dG(x, x')}{dx}|_{x=x'+\epsilon} - \frac{dG(x, x')}{dx}|_{x=x'-\epsilon} = -4\pi.
\]

(10)

In our present case, we can choose $\phi_1(x) = x$ and $\phi_2(x) = 1$, so that $W = 1$, and the Green’s function is as given above. For this piecewise continuous form of the Green’s function, the integration 5 can be evaluated:
\[
\Phi(x) = \frac{1}{4\pi \varepsilon_0} \left\{ \int_{-\infty}^{x} G(x, x')\rho(x')dx' + \int_{x}^{\infty} G(x, x')\rho(x')dx' \right\},
\]

(11)

which becomes
\[
\Phi(x) = \frac{1}{\varepsilon_0} \left\{ \int_{-\infty}^{x} x'\rho(x')dx' + x \int_{x}^{\infty} \rho(x')dx' \right\}.
\]

(12)

Evaluating this expression, we find that we obtain the same result as given in Eq. (3).

In general, the Green’s function $G(x, x')$ solution (5) depends upon the boundary conditions of the problem as well as on the charge density $\rho(x)$. In this example, the solution is valid for all neutral charge densities, that is $\int_{-\infty}^{\infty} \rho(x)dx = 0$. 
